# OpenStax Calculus, Volume 1

*A solutions manual for exercises*

**Author:** Amir Koutahi

## Attribution and License

## Contents

## Preface

My engagement with university-level mathematics began during high school, when I started studying advanced textbooks independently. While this approach allowed me to move quickly through material, it also meant that I occasionally skipped foundational prerequisites, leaving gaps in my understanding that only became apparent later.

Approximately one month before beginning my undergraduate studies at the University of Toronto, I decided to work through *OpenStax Calculus, Volume 1* as a preparatory text for the upcoming course MATA30. During this period, I completed the exercises and problems as a form of structured prereading, with the goal of strengthening my understanding of fundamental concepts before the start of the academic term. While doing so, I found myself repeatedly wishing for a solutions manual not to replace independent problem-solving, but to confirm answers, diagnose misunderstandings, and learn from alternative solution strategies. At the time, such a resource was not readily available to students.

This solutions manual was developed in response to that need. It is intended primarily for students encountering calculus for the first time, particularly those preparing for or enrolled in first-year university mathematics courses.

I would like to express my sincere gratitude to Professor Michael Cavers whose teaching sparked my first sustained exposure to pure mathematics. His lectures, together with conversations during office hours, I developed a deeperappreciation for the theoretical foundations, which significantly shaped both my approach to the subject and the preparation of this manual.

This document is an independent solutions manual for *OpenStax Calculus, Volume 1*. It is not an official OpenStax publication, and no endorsement by OpenStax or Rice University is implied.

Despite careful preparation, this manual may contain typographical errors or mathematical inaccuracies. Readers who identify mistakes or wish to suggest improvements are encouraged to contact me by email at amir.koutahi@mail.utoronto.ca. I would be happy to acknowledge contributors by name in future revisions of this document.

## Contributors

The following individuals have contributed corrections, suggestions, or improvements to this manual:

- —

# Chapter 1

*Functions and Their Graphs*

## Chapter contents

**Review of Functions**

## Exercise 1

Ordered pairs:

$$(-3,9),(-2,4),(-1,1),(0,0),(1,1),(2,4),(3,9)$$

$$\text{Domain} = \{-3,-2,-1,0,1,2,3\}, \quad \text{Range} = \{0,1,4,9\}$$

Each $x$ has exactly one corresponding $y$, so the relation is a function.

**Final answer:** Domain $\{-3,-2,-1,0,1,2,3\}$; Range $\{0,1,4,9\}$; Function: Yes.

—

## Exercise 2

Ordered pairs:

$$(-3,-2),(-2,-8),(-1,-1),(0,0),(1,1),(2,8),(3,-2)$$

$$\text{Domain} = \{-3,-2,-1,0,1,2,3\}, \quad \text{Range} = \{-8,-2,-1,0,1,8\}$$

Each input appears once, so the relation is a function.

**Final answer:** Domain $\{-3,-2,-1,0,1,2,3\}$; Range $\{-8,-2,-1,0,1,8\}$; Function: Yes.

—

## Exercise 3

Ordered pairs:

$$(1,-3),(1,1),(2,-2),(2,2),(3,-1),(3,3),(0,0)$$

$$\text{Domain} = \{0,1,2,3\}, \quad \text{Range} = \{-3,-2,-1,0,1,2,3\}$$

Since $x = 1,2,3$ each correspond to two different $y$-values, the relation is not a function.

**Final answer:** Domain $\{0,1,2,3\}$; Range $\{-3,-2,-1,0,1,2,3\}$; Function: No.

—

## Exercise 4

Ordered pairs:

$$(1, 1), (2, 1), (3, 1), (4, 1), (5, 1), (6, 1), (7, 1)$$

$$\text{Domain} = \{1, 2, 3, 4, 5, 6, 7\}, \quad \text{Range} = \{1\}$$

Every $x$ maps to exactly one $y$.

**Final answer:** Domain $\{1, 2, 3, 4, 5, 6, 7\}$; Range $\{1\}$; Function: Yes.

—

## Exercise 5

Ordered pairs:

$$(3, 3), (5, 2), (8, 1), (10, 0), (15, 1), (21, 2), (33, 3)$$

$$\text{Domain} = \{3, 5, 8, 10, 15, 21, 33\}, \quad \text{Range} = \{0, 1, 2, 3\}$$

Each input has one output, so the relation is a function.

**Final answer:** Domain $\{3, 5, 8, 10, 15, 21, 33\}$; Range $\{0, 1, 2, 3\}$; Function: Yes.

—

## Exercise 6

Ordered pairs:

$$(-7, 11), (-2, 5), (-2, 1), (0, -1), (1, -2), (3, 4), (6, 11)$$

$$\text{Domain} = \{-7, -2, 0, 1, 3, 6\}, \quad \text{Range} = \{-2, -1, 1, 4, 5, 11\}$$

Since $x = -2$ corresponds to two different values, the relation is not a function.

**Final answer:** Domain $\{-7, -2, 0, 1, 3, 6\}$; Range $\{-2, -1, 1, 4, 5, 11\}$; Function: No.

—

## Exercise 7

Given

$$f(x) = 5x - 2$$

$$f(0) = 5(0) - 2 = -2$$

$$f(1) = 5(1) - 2 = 3$$

$$f(3) = 5(3) - 2 = 13$$

$$f(-x) = 5(-x) - 2 = -5x - 2$$

$$f(a) = 5a - 2$$

$$f(a + h) = 5(a + h) - 2 = 5a + 5h - 2$$

## Exercise 8

Given

$$f(x) = 4x^2 - 3x + 1$$

$$f(0) = 1$$

$$f(1) = 4 - 3 + 1 = 2$$

$$f(3) = 36 - 9 + 1 = 28$$

$$f(-x) = 4(-x)^2 - 3(-x) + 1 = 4x^2 + 3x + 1$$

$$f(a) = 4a^2 - 3a + 1$$

$$f(a+h) = 4(a+h)^2 - 3(a+h) + 1$$
$$= 4a^2 + 8ah + 4h^2 - 3a - 3h + 1$$

**Exercise 9**

Given

$$f(x) = \frac{2}{x}$$

$$f(0) \text{ is undefined}$$

$$f(1) = 2$$

$$f(3) = \frac{2}{3}$$

$$f(-x) = \frac{2}{-x} = -\frac{2}{x}$$

$$f(a) = \frac{2}{a}$$

$$f(a+h) = \frac{2}{a+h}$$

**Exercise 10**

Given

$$f(x) = |x-7| + 8$$

$$f(0) = |-7| + 8 = 15$$

$$f(1) = |-6| + 8 = 14$$

$$f(3) = |-4| + 8 = 12$$

$$f(-x) = |-x-7| + 8 = |x+7| + 8$$

$$f(a) = |a-7| + 8$$

$$f(a+h) = |a+h-7| + 8$$

**Exercise 11**

Given

$$f(x) = \sqrt{6x+5}$$

$$f(0) = \sqrt{5}$$

$$f(1) = \sqrt{11}$$

$$f(3) = \sqrt{23}$$

$$f(-x) = \sqrt{5-6x}$$

$$f(a) = \sqrt{6a+5}$$

$$f(a+h) = \sqrt{6a+6h+5}$$

**Exercise 12**

Given

$$f(x) = \frac{x-2}{3x+7}$$

$$f(0) = -\frac{2}{7}$$

$$f(1) = -\frac{1}{10}$$

$$f(3) = \frac{1}{16}$$

$$f(-x) = \frac{-x-2}{-3x+7}$$

$$f(a) = \frac{a-2}{3a+7}$$

$$f(a+h) = \frac{a+h-2}{3a+3h+7}$$

**Exercise 13**

Given

$$f(x) = 9$$

$$f(0) = 9$$

$$f(1) = 9$$

$$f(3) = 9$$

$$f(-x) = 9$$

$$f(a) = 9$$

$$f(a+h) = 9$$

**Exercise 14**

Given

$$f(x) = \frac{x}{x^2-16}$$

$$x^2 - 16 \neq 0 \Rightarrow x \neq \pm 4$$

$$\text{Domain} = (-\infty, -4) \cup (-4, 4) \cup (4, \infty)$$

Zero:

$$\frac{x}{x^2 - 16} = 0 \Rightarrow x = 0$$

Range: all real numbers (horizontal asymptote $y = 0$ is crossed).

**Final answer:** Domain $= \mathbb{R} \setminus \{\pm 4\}$, Range $= \mathbb{R}$, Zero at $x = 0$

## Exercise 15

Given

$$g(x) = \sqrt{8x - 1}$$

$$8x - 1 \geq 0 \Rightarrow x \geq \tfrac{1}{8}$$

$$\text{Range} = [0, \infty)$$

Zero:

$$\sqrt{8x - 1} = 0 \Rightarrow x = \tfrac{1}{8}$$

**Final answer:** Domain $= [\frac{1}{8}, \infty)$, Range $= [0, \infty)$, Zero at $x = \frac{1}{8}$

## Exercise 16

Given

$$h(x) = \frac{3}{x^2 + 4}$$

$$x^2 + 4 > 0 \ \forall x \Rightarrow \text{Domain} = \mathbb{R}$$

$$h(x) > 0, \quad \max h = \frac{3}{4} \text{ at } x = 0$$

$$\text{Range} = (0, \tfrac{3}{4}]$$

No zeros.

**Final answer:** Domain $= \mathbb{R}$, Range $= (0, \frac{3}{4}]$, No zeros

## Exercise 17

Given

$$f(x) = -1 + \sqrt{x+2}$$

$$x + 2 \geq 0 \Rightarrow x \geq -2$$

$$\text{Range} = [-1, \infty)$$

Zero:

$$-1 + \sqrt{x+2} = 0 \Rightarrow x = -1$$

**Final answer:** Domain = $[-2, \infty)$, Range = $[-1, \infty)$, Zero at $x = -1$

## Exercise 18

Given

$$f(x) = \frac{1}{\sqrt{x-9}}$$

$$x - 9 > 0 \Rightarrow x > 9$$

$$f(x) > 0, \quad f(x) \to \infty \text{ as } x \to 9^+$$

$$\text{Range} = (0, \infty)$$

No zeros.

**Final answer:** Domain = $(9, \infty)$, Range = $(0, \infty)$, No zeros

## Exercise 19

Given

$$g(x) = \frac{3}{x-4}$$

$$x \neq 4$$

$$\text{Domain} = \mathbb{R} \setminus \{4\}$$

$$\text{Range} = \mathbb{R} \setminus \{0\}$$

No zeros.

**Final answer:** Domain $= \mathbb{R} \setminus \{4\}$, Range $= \mathbb{R} \setminus \{0\}$, No zeros

### Exercise 20

Given

$$f(x) = 4|x + 5|$$

$$\text{Domain} = \mathbb{R}$$

$$\text{Range} = [0, \infty)$$

Zero:

$$|x + 5| = 0 \Rightarrow x = -5$$

**Final answer:** Domain $= \mathbb{R}$, Range $= [0, \infty)$, Zero at $x = -5$

### Exercise 21

Given

$$g(x) = \sqrt{\frac{7}{x - 5}}$$

$$\frac{7}{x - 5} \geq 0 \Rightarrow x > 5$$

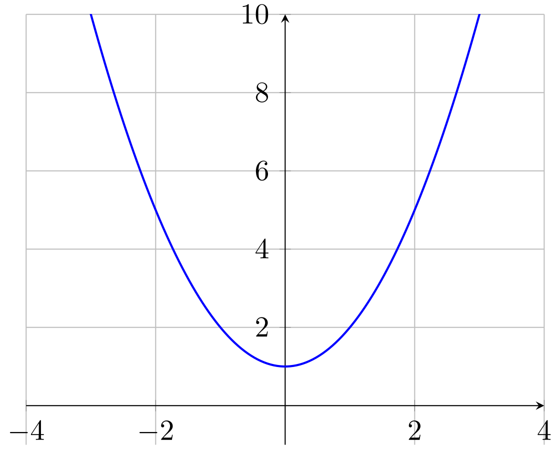
$$\text{Range} = (0, \infty)$$

No zeros.

**Final answer:** Domain $= (5, \infty)$, Range $= (0, \infty)$, No zeros

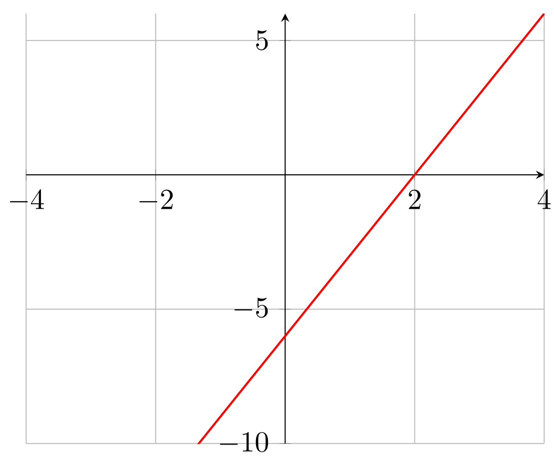
**Exercise 22**

$$f(x) = x^2 + 1$$



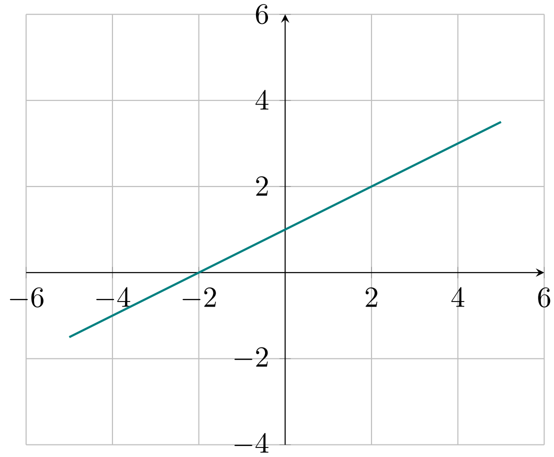

**Exercise 23**

$$f(x) = 3x - 6$$



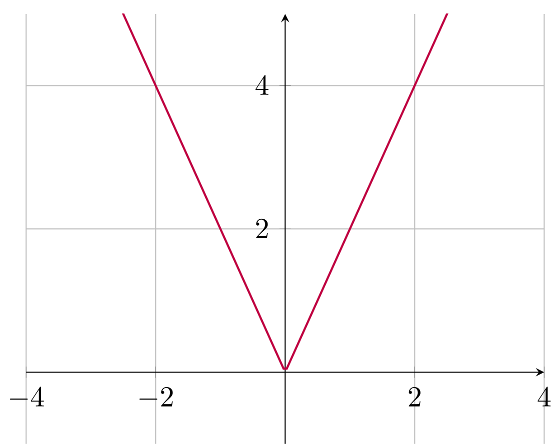

**Exercise 24**

$$f(x) = \frac{1}{2}x + 1$$



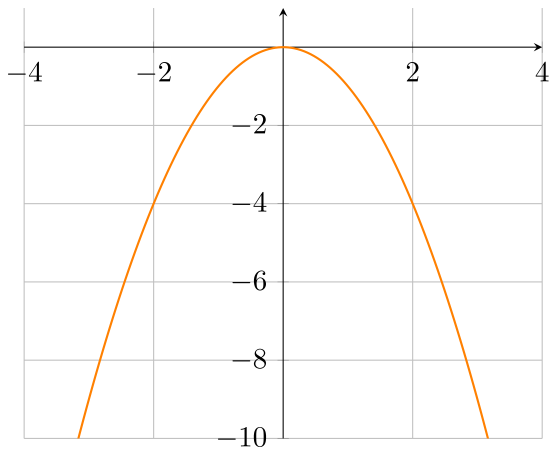

**Exercise 25**

$$f(x) = 2|x|$$



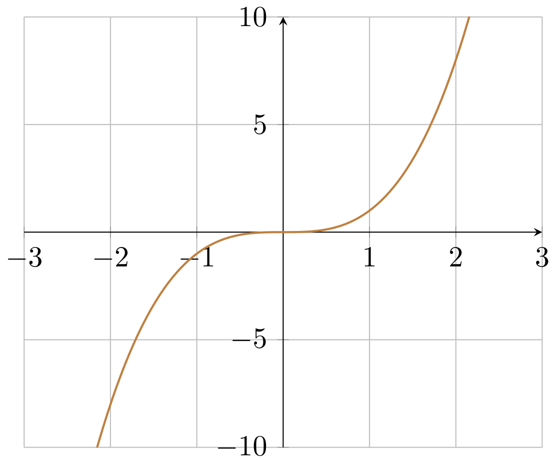
## Exercise 26

$$f(x) = -x^2$$




## Exercise 27

$$f(x) = x^3$$




## Exercise 28

This graph does *not* represent a function (fails the vertical line test).

**Reason:** Some vertical lines intersect the graph at two points.

**Final answer:** Not a function.

## Exercise 29

This graph represents a function.

Domain: $(-\infty, \infty)$

Range: $[0, \infty)$

$x$-intercepts: $x = -1,\ 0,\ 1$

$y$-intercept: $(0, 0)$

Increasing on: $(-1, 0) \cup (1, \infty)$

Decreasing on: $(-\infty, -1) \cup (0, 1)$

Constant on: none

Symmetry: symmetric about the $y$-axis

Even/Odd: even

**Final answer:** Function with the properties listed above.

## Exercise 30

This graph represents a function.

Domain: $(-\infty, \infty)$

Range: $(-\infty, 3]$

$x$-intercepts: $x \approx 0.5,\ 3.5$

$y$-intercept: $(0, -1)$

Increasing on: $(-\infty, 2)$

Decreasing on: $(2, \infty)$

Constant on: none

Symmetry: none

Even/Odd: neither

**Final answer:** Function with maximum at $x = 2$.

## Exercise 31

This graph represents a function.

Domain: $(-\infty, \infty)$

Range: approximately $(-2, 1)$

$x$-intercept: $x \approx 0$

$y$-intercept: $(0, 0)$

Increasing on: $(-\infty, \infty)$

Decreasing on: none

Constant on: none

Symmetry: none

Even/Odd: neither

**Final answer:** Increasing function for all $x$.

### Exercise 32

This graph does *not* represent a function (fails the vertical line test).

**Reason:** Some $x$-values correspond to two $y$-values.

**Final answer:** Not a function.

### Exercise 33

This graph represents a function.

Domain: $(-\infty, \infty)$

Range: $[-2, 2]$

$x$-intercept: $x = 0$

$y$-intercept: $(0, 0)$

Increasing on: $(-1, 2)$

Decreasing on: none

Constant on: $(-\infty, -1] \cup [2, \infty)$

Symmetry: none

Even/Odd: neither

**Final answer:** Piecewise function with increasing middle interval.

### Exercise 34

This graph represents a function.

Domain: $[0, \infty)$

Range: $[0, \infty)$

$x$-intercept: $(0, 0)$

$y$-intercept: $(0, 0)$

Increasing on: $(0, \infty)$

Decreasing on: none

Constant on: none

Symmetry: none

Even/Odd: neither

**Final answer:** Linear increasing function for $x \geq 0$.

### Exercise 35

This graph represents a function.

Domain: $[-4, 4]$

Range: $[-4, 4]$

$x$-intercept: $x \approx 1$

$y$-intercept: $(0, 4)$

Increasing on: none

Decreasing on: $(0, 4)$

Constant on: $[-4, 0]$

Symmetry: none

Even/Odd: neither

**Final answer:** Function with constant then decreasing behavior.

### Exercise 36

Given $f(x) = 3x + 4$ and $g(x) = x - 2$.

$$(f+g)(x) = (3x+4) + (x-2) = 4x+2, \qquad \text{Dom} = \mathbb{R}$$
$$(f-g)(x) = (3x+4) - (x-2) = 2x+6, \qquad \text{Dom} = \mathbb{R}$$
$$(f \cdot g)(x) = (3x+4)(x-2) = 3x^2 - 2x - 8, \qquad \text{Dom} = \mathbb{R}$$
$$\left(\frac{f}{g}\right)(x) = \frac{3x+4}{x-2}, \qquad \text{Dom} = \mathbb{R} \setminus \{2\}$$

**Final answer:**

$$f+g = 4x+2,\ \mathbb{R}$$
$$f-g = 2x+6,\ \mathbb{R}$$
$$f \cdot g = 3x^2 - 2x - 8,\ \mathbb{R}$$
$$\frac{f}{g} = \frac{3x+4}{x-2},\ \mathbb{R} \setminus \{2\}$$

## Exercise 37

Given $f(x) = x - 8$ and $g(x) = 5x^2$.

$$(f+g)(x) = 5x^2 + x - 8, \qquad \text{Dom} = \mathbb{R}$$
$$(f-g)(x) = x - 8 - 5x^2, \qquad \text{Dom} = \mathbb{R}$$
$$(f \cdot g)(x) = 5x^3 - 40x^2, \qquad \text{Dom} = \mathbb{R}$$
$$\left(\frac{f}{g}\right)(x) = \frac{x-8}{5x^2}, \qquad \text{Dom} = \mathbb{R} \setminus \{0\}$$

**Final answer:**

$$f+g = 5x^2 + x - 8,\ \mathbb{R}$$
$$f-g = -5x^2 + x - 8,\ \mathbb{R}$$
$$f \cdot g = 5x^3 - 40x^2,\ \mathbb{R}$$
$$\frac{f}{g} = \frac{x-8}{5x^2},\ \mathbb{R} \setminus \{0\}$$

## Exercise 38

Given $f(x) = 3x^2 + 4x + 1$ and $g(x) = x + 1$.

$$(f+g)(x) = 3x^2 + 5x + 2, \qquad \text{Dom} = \mathbb{R}$$

$$(f-g)(x) = 3x^2 + 3x, \qquad \text{Dom} = \mathbb{R}$$

$$(f \cdot g)(x) = (3x^2+4x+1)(x+1) = 3x^3 + 7x^2 + 5x + 1$$

$$\left(\frac{f}{g}\right)(x) = \frac{3x^2+4x+1}{x+1}, \qquad \text{Dom} = \mathbb{R} \setminus \{-1\}$$

**Final answer:**

$$f+g = 3x^2+5x+2,\ \mathbb{R}$$
$$f-g = 3x^2+3x,\ \mathbb{R}$$
$$f \cdot g = 3x^3+7x^2+5x+1,\ \mathbb{R}$$
$$\frac{f}{g} = \frac{3x^2+4x+1}{x+1},\ \mathbb{R} \setminus \{-1\}$$

## Exercise 39

Given $f(x) = 9 - x^2$ and $g(x) = x^2 - 2x - 3$.

$$(f+g)(x) = 6 - 2x, \qquad \text{Dom} = \mathbb{R}$$

$$(f-g)(x) = 12 - 2x^2 + 2x, \qquad \text{Dom} = \mathbb{R}$$

$$(f \cdot g)(x) = (9-x^2)(x^2-2x-3)$$

$$\left(\frac{f}{g}\right)(x) = \frac{9-x^2}{x^2-2x-3}, \quad \text{Dom} = \mathbb{R} \setminus \{-1, 3\}$$

**Final answer:**

$$f+g = 6-2x,\ \mathbb{R}$$
$$f-g = 12-2x^2+2x,\ \mathbb{R}$$
$$f \cdot g = (9-x^2)(x^2-2x-3),\ \mathbb{R}$$
$$\frac{f}{g} = \frac{9-x^2}{x^2-2x-3},\ \mathbb{R} \setminus \{-1, 3\}$$

## Exercise 40

Given $f(x) = \sqrt{x}$ and $g(x) = x - 2$.

$$(f+g)(x) = \sqrt{x} + x - 2, \qquad \text{Dom} = [0, \infty)$$

$$(f-g)(x) = \sqrt{x} - x + 2, \qquad \text{Dom} = [0, \infty)$$

$$(f \cdot g)(x) = \sqrt{x}(x-2), \qquad \text{Dom} = [0, \infty)$$

$$\left(\frac{f}{g}\right)(x) = \frac{\sqrt{x}}{x-2}, \quad \text{Dom} = [0, \infty) \setminus \{2\}$$

**Final answer:**

$$\begin{aligned} f + g &= \sqrt{x} + x - 2, \ [0, \infty) \\ f - g &= \sqrt{x} - x + 2, \ [0, \infty) \\ f \cdot g &= \sqrt{x}(x-2), \ [0, \infty) \\ \frac{f}{g} &= \frac{\sqrt{x}}{x-2}, \ [0, \infty) \setminus \{2\} \end{aligned}$$

## Exercise 41

Given $f(x) = 6 + \frac{1}{x}$ and $g(x) = \frac{1}{x}$.

$$(f+g)(x) = 6 + \frac{2}{x}, \qquad \text{Dom} = \mathbb{R} \setminus \{0\}$$

$$(f-g)(x) = 6, \qquad \text{Dom} = \mathbb{R} \setminus \{0\}$$

$$(f \cdot g)(x) = \frac{6}{x} + \frac{1}{x^2}, \qquad \text{Dom} = \mathbb{R} \setminus \{0\}$$

$$\left(\frac{f}{g}\right)(x) = 6x + 1, \qquad \text{Dom} = \mathbb{R} \setminus \{0\}$$

**Final answer:**

$$\begin{aligned} f + g &= 6 + \frac{2}{x}, \ \mathbb{R} \setminus \{0\} \\ f - g &= 6, \ \mathbb{R} \setminus \{0\} \\ f \cdot g &= \frac{6}{x} + \frac{1}{x^2}, \ \mathbb{R} \setminus \{0\} \\ \frac{f}{g} &= 6x + 1, \ \mathbb{R} \setminus \{0\} \end{aligned}$$

## Exercise 42

Given $f(x) = 3x$ and $g(x) = x + 5$:

$$(f \circ g)(x) = f(g(x)) = f(x+5) = 3(x+5) = 3x + 15$$

Domain: $(-\infty, \infty)$.

$$(g \circ f)(x) = g(f(x)) = g(3x) = 3x + 5$$

Domain: $(-\infty, \infty)$.

**Final answer:** $(f \circ g)(x) = 3x + 15,\ \text{Dom} = \mathbb{R}; \quad (g \circ f)(x) = 3x + 5,\ \text{Dom} = \mathbb{R}.$

### Exercise 43

Given $f(x) = x + 4$ and $g(x) = 4x - 1$:

$$(f \circ g)(x) = f(4x - 1) = (4x - 1) + 4 = 4x + 3$$

Domain: $(-\infty, \infty)$.

$$(g \circ f)(x) = g(x + 4) = 4(x + 4) - 1 = 4x + 15$$

Domain: $(-\infty, \infty)$.

**Final answer:** $(f \circ g)(x) = 4x + 3,\ \text{Dom} = \mathbb{R}; \quad (g \circ f)(x) = 4x + 15,\ \text{Dom} = \mathbb{R}.$

### Exercise 44

Given $f(x) = 2x + 4$ and $g(x) = x^2 - 2$:

$$(f \circ g)(x) = f(x^2 - 2) = 2(x^2 - 2) + 4 = 2x^2$$

Domain: $(-\infty, \infty)$.

$$(g \circ f)(x) = g(2x + 4) = (2x + 4)^2 - 2 = 4x^2 + 16x + 14$$

Domain: $(-\infty, \infty)$.

**Final answer:** $(f \circ g)(x) = 2x^2,\ \text{Dom} = \mathbb{R}; \quad (g \circ f)(x) = 4x^2 + 16x + 14,\ \text{Dom} = \mathbb{R}.$

### Exercise 45

Given $f(x) = x^2 + 7$ and $g(x) = x^2 - 3$:

$$(f \circ g)(x) = f(x^2 - 3) = (x^2 - 3)^2 + 7 = x^4 - 6x^2 + 16$$

Domain: $(-\infty, \infty)$.

$$(g \circ f)(x) = g(x^2+7) = (x^2+7)^2 - 3 = x^4 + 14x^2 + 46$$

Domain: $(-\infty, \infty)$.

**Final answer:** $(f \circ g)(x) = x^4 - 6x^2 + 16,\ \text{Dom} = \mathbb{R}; \quad (g \circ f)(x) = x^4 + 14x^2 + 46,\ \text{Dom} = \mathbb{R}.$

### Exercise 46

Given $f(x) = \sqrt{x}$ and $g(x) = x + 9$:

$$(f \circ g)(x) = f(x+9) = \sqrt{x+9}$$

Domain: $x + 9 \geq 0 \Rightarrow x \geq -9$.

$$(g \circ f)(x) = g(\sqrt{x}) = \sqrt{x} + 9$$

Domain: $x \geq 0$.

**Final answer:** $(f \circ g)(x) = \sqrt{x+9},\ \text{Dom} = [-9, \infty); \quad (g \circ f)(x) = \sqrt{x} + 9,\ \text{Dom} = [0, \infty).$

### Exercise 47

Given $f(x) = \dfrac{3}{2x+1}$ and $g(x) = \dfrac{2}{x}$:

$$(f \circ g)(x) = f\left(\frac{2}{x}\right) = \frac{3}{\frac{4}{x} + 1} = \frac{3x}{x+4}$$

Domain: $x \neq 0, -4$.

$$(g \circ f)(x) = g\left(\frac{3}{2x+1}\right) = \frac{2(2x+1)}{3} = \frac{4x+2}{3}$$

Domain: $2x + 1 \neq 0 \Rightarrow x \neq -\frac{1}{2}$.

**Final answer:** $(f \circ g)(x) = \dfrac{3x}{x+4},\ \text{Dom} = \mathbb{R} \setminus \{0, -4\}; \quad (g \circ f)(x) = \dfrac{4x+2}{3},\ \text{Dom} = \mathbb{R} \setminus \{-\frac{1}{2}\}.$

### Exercise 48

Given $f(x) = |x+1|$ and $g(x) = x^2 + x - 4$:

$$(f \circ g)(x) = f(x^2 + x - 4) = |x^2 + x - 3|$$

Domain: $(-\infty, \infty)$.

$$(g \circ f)(x) = g(|x+1|) = |x+1|^2 + |x+1| - 4 = (x+1)^2 + |x+1| - 4$$

Domain: $(-\infty, \infty)$.

**Final answer:** $(f \circ g)(x) = |x^2 + x - 3|,\ \text{Dom} = \mathbb{R}; \quad (g \circ f)(x) = (x+1)^2 + |x+1| - 4,\ \text{Dom} = \mathbb{R}.$

### Exercise 49

The table lists NBA championship winners from 2001 to 2012.

**(a)** Consider the relation where the domain is the set of years $\{2001, \ldots, 2012\}$ and the range is the corresponding championship winner.

Each year is associated with exactly one winner. Therefore, every element of the domain maps to exactly one element in the range.

**Final answer:** Yes, this relation *is a function* because each year has exactly one corresponding winner.

**(b)** Now consider the relation where the domain is the set of winners and the range is the set of years.

Some teams (e.g., LA Lakers, San Antonio Spurs, Miami Heat) correspond to multiple years. Thus, a single input (winner) maps to more than one output (year).

**Final answer:** No, this relation *is not a function* because one winner corresponds to multiple years.

### Exercise 50

The area $A$ of a square depends on the length of its side $s$.

**(a)** The area of a square with side length $s$ is

$$A(s) = s^2.$$

**Final answer:** $A(s) = s^2$.

**(b)** Evaluate $A(6.5)$:

$$A(6.5) = (6.5)^2 = 42.25.$$

This represents the area of a square whose side length is 6.5 units.

**Final answer:** $A(6.5) = 42.25$ square units.

**(c)** To find the side length of a square with area 56, solve

$$s^2 = 56 \quad \Rightarrow \quad s = \sqrt{56}.$$

$$\sqrt{56} \approx 7.48.$$

Rounded to two significant digits:

$$s \approx 7.5.$$

**Final answer:** $s \approx 7.5$ units.

### Exercise 51

The volume of a cube with side length $s$ is

$$V(s) = s^3.$$

Evaluate:

$$V(11.8) = 11.8^3 = 1643.432.$$

**Final answer:**

$$V(s) = s^3, \qquad V(11.8) = 1643.432$$

### Exercise 52

The total cost is a flat fee plus an hourly charge.

$$C(t) = 20 + 10.25t$$

$$2 \text{ days } 7 \text{ hours } = 55 \text{ hours}$$

$$C(55) = 20 + 10.25(55) = 583.75$$

Solve for time when $C(t) = 432.73$:

$$20 + 10.25t = 432.73$$

$$t = \frac{412.73}{10.25} \approx 40.27$$

**Final answer:**

$$C(t) = 20 + 10.25t, \quad C(55) = \$583.75, \quad t \approx 40.27 \text{ hr}$$

## Exercise 53

Miles driven depend on gallons of gas:

$$N(x) = 15x$$

Full tank:

$$N(20) = 300$$

Three-quarters tank:

$$N(15) = 225$$

Domain:

$$0 \leq x \leq 20$$

Range:

$$0 \leq N \leq 300$$

Stops needed for 578 miles:

$$\frac{578}{300} \approx 1.93 \Rightarrow 1 \text{ stop}$$

**Final answer:**

$$N(x) = 15x, \quad N(20) = 300, \quad N(15) = 225, \quad 1 \text{ stop}$$

## Exercise 54

Volume of Earth using mean radius:

$$V = \frac{4}{3}\pi r^3$$

$$r = 6.371 \times 10^6$$

$$V = \frac{4}{3}\pi(6.371 \times 10^6)^3 \approx 1.083 \times 10^{21} \text{ m}^3$$

**Final answer:**

$$V \approx 1.083 \times 10^{21} \text{ m}^3$$

**Exercise 55**

Radius:

$$r(t) = 6 - \frac{5}{t^2 + 1}$$

Area:

$$A(t) = \pi r(t)^2 = \pi \left(6 - \frac{5}{t^2 + 1}\right)^2$$

At $t = 3$:

$$r(3) = 6 - \frac{5}{10} = 5.5$$

$$A(3) = \pi(5.5)^2 = 30.25\pi \approx 95.03$$

Circumference:

$$C(t) = 2\pi r(t)$$

$$C(3) = 11\pi \approx 34.56$$

**Final answer:**

$$A(t) = \pi \left(6 - \frac{5}{t^2 + 1}\right)^2, \quad A(3) = 30.25\pi$$
$$C(t) = 2\pi r(t), \quad C(3) = 11\pi$$

**Exercise 56**

Euro conversion:

$$E(x) = 0.79x$$

Back to dollars:

$$D(E(x)) = 1.245(0.79x) = 0.98355x$$

Loss occurs since coefficient $< 1$.

Extra \$200:

$$200(0.98355) = 196.71$$

**Final answer:**

$$D(E(x)) = 0.98355x, \quad \text{Loss occurs}, \quad \$196.71$$

## Exercise 57

Salary function:

$$S(x) = 750 + 8.5x$$

$$S(25) = 962.50, \quad S(40) = 1090, \quad S(55) = 1217.50$$

Solve:

$$750 + 8.5x = 1400$$

$$x = \frac{650}{8.5} \approx 76.47$$

**Final answer:**

$$S(x) = 750 + 8.5x, \quad x \approx 76.47$$

## Exercise 58

The equation is

$$y = \sqrt{25 - (x-4)^2}.$$

**x-intercepts:** Set $y = 0$:

$$25 - (x-4)^2 = 0$$

$$(x-4)^2 = 25$$

$$x - 4 = \pm 5$$

$$x = -1,\ 9$$

So the $x$-intercepts are $(-1, 0)$ and $(9, 0)$.

**y-intercept:** Set $x = 0$:

$$y = \sqrt{25 - (0-4)^2} = \sqrt{25 - 16} = \sqrt{9} = 3$$

So the $y$-intercept is $(0, 3)$.

**Final answer:**

$$x\text{-intercepts: } (-1, 0) \text{ and } (9, 0), \qquad y\text{-intercept: } (0, 3)$$

**Basic Classes of Functions**

## Exercise 59

Points $(-2, 4)$ and $(1, 1)$.

$$m = \frac{1-4}{1-(-2)} = \frac{-3}{3} = -1$$

**Final answer:** slope $= -1$, decreasing.

## Exercise 60

Points $(-1, 4)$ and $(3, -1)$.

$$m = \frac{-1-4}{3-(-1)} = \frac{-5}{4}$$

**Final answer:** slope $= -\frac{5}{4}$, decreasing.

## Exercise 61

Points $(3, 5)$ and $(-1, 2)$.

$$m = \frac{2-5}{-1-3} = \frac{-3}{-4} = \frac{3}{4}$$

**Final answer:** slope $= \frac{3}{4}$, increasing.

## Exercise 62

Points $(6, 4)$ and $(4, -3)$.

$$m = \frac{-3-4}{4-6} = \frac{-7}{-2} = \frac{7}{2}$$

**Final answer:** slope $= \frac{7}{2}$, increasing.

## Exercise 63

Points $(2, 3)$ and $(5, 7)$.

$$m = \frac{7-3}{5-2} = \frac{4}{3}$$

**Final answer:** slope $= \frac{4}{3}$, increasing.

**Exercise 64**

Points $(1, 9)$ and $(-8, 5)$.

$$m = \frac{5-9}{-8-1} = \frac{-4}{-9} = \frac{4}{9}$$

**Final answer:** slope $= \frac{4}{9}$, increasing.

**Exercise 65**

Points $(2, 4)$ and $(1, 4)$.

$$m = \frac{4-4}{1-2} = 0$$

**Final answer:** slope $= 0$, horizontal.

**Exercise 66**

Points $(1, 4)$ and $(1, 0)$.

$$m = \frac{0-4}{1-1} \text{ undefined}$$

**Final answer:** vertical line.

**Exercise 67**

Slope $m = -6$, through $(1, 3)$.

$$y - 3 = -6(x-1) \Rightarrow y = -6x + 9$$

**Final answer:** $y = -6x + 9$.

**Exercise 68**

Slope $m = 3$, through $(-3, 2)$.

$$y - 2 = 3(x+3) \Rightarrow y = 3x + 11$$

**Final answer:** $y = 3x + 11$.

**Exercise 69**

Slope $m = \frac{1}{3}$, through $(0, 4)$.

$$y = \frac{1}{3}x + 4$$

**Final answer:** $y = \frac{1}{3}x + 4$.

**Exercise 70**

Slope $m = \frac{2}{5}$, $x$-intercept $= 8$ gives point $(8, 0)$.

$$y = \frac{2}{5}(x - 8) \Rightarrow y = \frac{2}{5}x - \frac{16}{5}$$

**Final answer:** $y = \frac{2}{5}x - \frac{16}{5}$.

**Exercise 71**

Through $(2, 1)$ and $(-2, -1)$.

$$m = \frac{-1 - 1}{-2 - 2} = \frac{1}{2} \Rightarrow y = \frac{1}{2}x$$

**Final answer:** $y = \frac{1}{2}x$.

**Exercise 72**

Through $(-3, 7)$ and $(1, 2)$.

$$m = \frac{2 - 7}{1 + 3} = -\frac{5}{4} \Rightarrow y = -\frac{5}{4}x + \frac{13}{4}$$

**Final answer:** $y = -\frac{5}{4}x + \frac{13}{4}$.

**Exercise 73**

$x$-intercept 5, $y$-intercept $-3$.

$$m = \frac{-3 - 0}{0 - 5} = \frac{3}{5} \Rightarrow y = \frac{3}{5}x - 3$$

**Final answer:** $y = \frac{3}{5}x - 3$.

## Exercise 74

$x$-intercept $-6$, $y$-intercept 9.

$$m = \frac{9-0}{0+6} = \frac{3}{2} \Rightarrow y = \frac{3}{2}x + 9$$

**Final answer:** $y = \frac{3}{2}x + 9$.

## Exercise 75

$y = 2x - 3$. **Final answer:** $m = 2$, $b = -3$.

## Exercise 76

$y = -\frac{1}{7}x + 1$. **Final answer:** $m = -\frac{1}{7}$, $b = 1$.

## Exercise 77

$f(x) = -6x$. **Final answer:** $m = -6$, $b = 0$.

## Exercise 78

$f(x) = -5x + 4$. **Final answer:** $m = -5$, $b = 4$.

## Exercise 79

$4y + 24 = 0 \Rightarrow y = -6$. **Final answer:** horizontal line $y = -6$.

## Exercise 80

$8x - 4 = 0 \Rightarrow x = \frac{1}{2}$. **Final answer:** vertical line $x = \frac{1}{2}$.

## Exercise 81

$2x + 3y = 6 \Rightarrow y = -\frac{2}{3}x + 2$. **Final answer:** $m = -\frac{2}{3}$, $b = 2$.

## Exercise 82

$6x - 5y + 15 = 0 \Rightarrow y = \frac{6}{5}x + 3$. **Final answer:** $m = \frac{6}{5}$, $b = 3$.

**Exercise 83**

$f(x) = 2x^2 - 3x - 5$. Zeros from quadratic formula: $x = \frac{3\pm7}{4}$. **Final answer:** degree 2, zeros $x = -1,\ \frac{5}{2}$, $y$-intercept $-5$, neither even nor odd.

**Exercise 84**

$f(x) = -3x^2 + 6x = -3x(x-2)$. **Final answer:** degree 2, zeros $0, 2$, $y$-intercept 0, neither.

**Exercise 85**

$f(x) = \frac{1}{2}x^2 - 1$. **Final answer:** degree 2, zeros $\pm\sqrt{2}$, $y$-intercept $-1$, even.

**Exercise 86**

$f(x) = x^3 + 3x^2 - x - 3 = (x+3)(x^2-1)$. **Final answer:** degree 3, zeros $-3, -1, 1$, $y$-intercept $-3$, neither.

**Exercise 87**

$f(x) = 3x - x^3$. **Final answer:** degree 3, zeros $0, \pm\sqrt{3}$, $y$-intercept 0, odd.

**Exercise 88**

$g(x) = x^2 - 1$. **Final answer:** shift down 1.

**Exercise 89**

$g(x) = (x+3)^2 + 1$. **Final answer:** left 3, up 1.

**Exercise 90**

$g(x) = \sqrt{x+2}$. **Final answer:** left 2.

**Exercise 91**

$g(x) = -\sqrt{x} - 1$. **Final answer:** reflect over $x$-axis, down 1.

### Exercise 92

Given $g(x) = f(x) + 1$.

This is a vertical shift of $f$ upward by 1 unit.

**Final answer:** $g(x)$ is $f(x)$ shifted up by 1.

### Exercise 93

Given $g(x) = f(x-1) + 2$.

This is a horizontal shift right by 1 and vertical shift up by 2.

**Final answer:** shift right 1, up 2.

### Exercise 94

$$f(x) = \begin{cases} 4x+3, & x \leq 0 \\ -x+1, & x > 0 \end{cases}$$

$$f(-3) = 4(-3) + 3 = -9, \quad f(0) = 3, \quad f(2) = -2 + 1 = -1$$

**Final answer:** $f(-3) = -9,\ f(0) = 3,\ f(2) = -1$.

### Exercise 95

$$f(x) = \begin{cases} x^2 - 3, & x < 0 \\ 4x - 3, & x \geq 0 \end{cases}$$

$$f(-4) = 13, \quad f(0) = -3, \quad f(2) = 5$$

**Final answer:** $f(-4) = 13,\ f(0) = -3,\ f(2) = 5$.

### Exercise 96

$$h(x) = \begin{cases} x+1, & x \leq 5 \\ 4, & x > 5 \end{cases}$$

$$h(0) = 1, \quad h(\pi) = \pi + 1, \quad h(5) = 6$$

**Final answer:** $h(0) = 1,\ h(\pi) = \pi + 1,\ h(5) = 6.$

### Exercise 97

$$g(x) = \begin{cases} \dfrac{3}{x-2}, & x \neq 2 \\ 4, & x = 2 \end{cases}$$

$$g(0) = -\tfrac{3}{2}, \quad g(-4) = -\tfrac{1}{2}, \quad g(2) = 4$$

**Final answer:** $g(0) = -\frac{3}{2},\ g(-4) = -\frac{1}{2},\ g(2) = 4.$

### Exercise 98

$f(x) = \dfrac{4x+1}{7x-2}$ is rational, not transcendental.

**Final answer:** False.

### Exercise 99

$g(x) = \sqrt[3]{x}$ satisfies $g(-x) = -g(x)$.

**Final answer:** True.

### Exercise 100

Logarithmic functions are transcendental.

**Final answer:** False.

### Exercise 101

$f(x) = x^b$ is a power function, not exponential.

**Final answer:** False.

### Exercise 102

Even root functions require the radicand to be nonnegative.

**Final answer:** False.

### Exercise 103

Linear depreciation from $(0, 20500)$ to $(3, 12300)$.

$$m = \frac{12300 - 20500}{3} = -2733.\overline{3}$$

$$V(t) = 20500 - 2733.\overline{3}t$$

$$V(5) = 20500 - 13666.\overline{6} = 6833.\overline{3}$$

Solve $V(t) = 3000$:

$$t \approx 6.4$$

**Final answer:** $V(t) = 20500 - 2733.\overline{3}t$, $x$-intercept $\approx 6.4$ years, $V(5) \approx \$6833$.

### Exercise 104

Slope:

$$m = \frac{48.1 - 42.3}{1} = 5.8$$

$$S(t) = 42.3 + 5.8t$$

Solve $S(t) = 60$:

$$t \approx 3.05$$

**Final answer:** $S(t) = 42.3 + 5.8t$, reaches \$60B in about 2015.

### Exercise 105

$$C(x) = 125 + 0.75x$$

$$C(160) = 245$$

Revenue $R(x) = 1.50x$. Solve $R = C$:

$$x = 166.7$$

**Final answer:** $C(x) = 125 + 0.75x$, $C(160) = 245$, break-even $\approx 167$ cupcakes.

### Exercise 106

$$m = \frac{500000 - 250000}{18} = 13888.9$$
$$P(t) = 250000 + 13888.9t$$
$$P(15) \approx 458333$$

**Final answer:** $P(t) = 250000 + 13888.9t$, $P(15) \approx \$458{,}333$.

### Exercise 107

$$V(t) = 26000 - 1500t$$
$$V(4) = 20000$$

**Final answer:** $V(t) = 26000 - 1500t$, $V(4) = \$20{,}000$.

### Exercise 108

$$m = \frac{60500 - 432000}{35} = -10642.9$$

**Final answer:** depreciation rate $\approx$ \$10,643/year.

### Exercise 109

$$C(175) = 10.50(175) + 28500 = 30337.5$$

**Final answer:** \$30,337.50.

### Exercise 110

We are given two data points relating time $t$ (in minutes) to number of pages typed $N$:

$$(45, 4), \quad (90, 9)$$

#### (a)

Compute the slope:
$$m = \frac{9-4}{90-45} = \frac{5}{45} = \frac{1}{9}$$

Using point $(45, 4)$:

$$N(t) = \frac{1}{9}t + b, \quad 4 = \frac{45}{9} + b \Rightarrow b = -1$$

**Final answer:**

$$N(t) = \frac{1}{9}t - 1$$

### (b)

Two hours is 120 minutes:

$$N(120) = \frac{120}{9} - 1 = \frac{111}{9} \approx 12.33$$

**Final answer:** Approximately 12.3 pages.

### (c)

Solve $N(t) = 20$:

$$20 = \frac{1}{9}t - 1 \Rightarrow t = 189$$

**Final answer:** 189 minutes (about 3.15 hours).

## Exercise 111

Given:

$$P(t) = 1.8576t + 68.052$$

The year 2015 corresponds to $t = 15$.

$$P(15) = 1.8576(15) + 68.052 = 95.916$$

**Final answer:** Approximately 95.9% output in 2015.

## Exercise 112

### (a)

Since 65% of admitted students enroll:

$$N(x) = 0.65x$$

**Final answer:**

$$N(x) = 0.65x$$

## (b)

Solve $0.65x = 1350$:

$$x = \frac{1350}{0.65} \approx 2077$$

**Final answer:** Approximately 2077 students should be admitted.

### Trigonometric Functions

## Exercise 113

$$240^\circ = \frac{240\pi}{180} = \frac{4\pi}{3}$$

**Final answer:** $\frac{4\pi}{3}$

## Exercise 114

$$15^\circ = \frac{15\pi}{180} = \frac{\pi}{12}$$

**Final answer:** $\frac{\pi}{12}$

## Exercise 115

$$-60^\circ = \frac{-60\pi}{180} = -\frac{\pi}{3}$$

**Final answer:** $-\frac{\pi}{3}$

## Exercise 116

$$-225^\circ = \frac{-225\pi}{180} = -\frac{5\pi}{4}$$

**Final answer:** $-\frac{5\pi}{4}$

**Exercise 117**

$$330^\circ = \frac{330\pi}{180} = \frac{11\pi}{6}$$

**Final answer:** $\frac{11\pi}{6}$

**Exercise 118**

$$\frac{\pi}{2}\text{ rad} = 90^\circ$$

**Final answer:** $90^\circ$

**Exercise 119**

$$\frac{7\pi}{6}\text{ rad} = 210^\circ$$

**Final answer:** $210^\circ$

**Exercise 120**

$$\frac{11\pi}{2}\text{ rad} = 990^\circ$$

**Final answer:** $990^\circ$

**Exercise 121**

$$-3\pi\text{ rad} = -540^\circ$$

**Final answer:** $-540^\circ$

**Exercise 122**

$$\frac{5\pi}{12}\text{ rad} = 75^\circ$$

**Final answer:** $75^\circ$

**Exercise 123**

$$\cos\left(\frac{4\pi}{3}\right) = -\frac{1}{2}$$

**Final answer:** $-\frac{1}{2}$

**Exercise 124**

$$\tan\left(\frac{19\pi}{4}\right) = \tan\left(\frac{3\pi}{4}\right) = -1$$

**Final answer:** $-1$

**Exercise 125**

$$\sin\left(-\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

**Final answer:** $-\frac{\sqrt{2}}{2}$

**Exercise 126**

$$\sec\left(\frac{\pi}{6}\right) = \frac{1}{\cos(\pi/6)} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

**Final answer:** $\dfrac{2\sqrt{3}}{3}$

**Exercise 127**

$$\sin\left(\frac{\pi}{12}\right) = \sin(15^\circ) = \frac{\sqrt{6}-\sqrt{2}}{4}$$

**Final answer:** $\dfrac{\sqrt{6}-\sqrt{2}}{4}$

**Exercise 128**

$$\cos\left(\frac{5\pi}{12}\right) = \cos(75^\circ) = \frac{\sqrt{6}-\sqrt{2}}{4}$$

**Final answer:** $\dfrac{\sqrt{6}-\sqrt{2}}{4}$

**Exercise 129**

$$b = \sqrt{c^2 - a^2} = \sqrt{49-16} = \sqrt{33} \approx 5.7$$

Trig values at $A$:

$$\sin A = \frac{4}{7}, \quad \cos A = \frac{\sqrt{33}}{7}, \quad \tan A = \frac{4}{\sqrt{33}}$$

**Final answer:** $b \approx 5.7$

### Exercise 130

$$b = \sqrt{29^2 - 21^2} = 20$$

**Final answer:** $b = 20$

### Exercise 131

$$c = \sqrt{85.3^2 + 125.5^2} \approx 151.8$$

**Final answer:** $c \approx 151.8$

### Exercise 132

$$a = \sqrt{41^2 - 40^2} = 9$$

**Final answer:** $a = 9$

### Exercise 133

$$c = \sqrt{84^2 + 13^2} = 85$$

**Final answer:** $c = 85$

### Exercise 134

$$a = \sqrt{35^2 - 28^2} = 21$$

**Final answer:** $a = 21$

### Exercise 135

$$x^2 + y^2 = 1 \Rightarrow y = \frac{24}{25}$$

**Final answer:** $y = \frac{24}{25}$

**Exercise 136**

$$y = -\frac{8}{17}$$

**Final answer:** $-\frac{8}{17}$

**Exercise 137**

$$x = -\frac{4}{3}$$

**Final answer:** $-\frac{4}{3}$

**Exercise 138**

$$x = \frac{1}{4}$$

**Final answer:** $\frac{1}{4}$

**Exercise 139**

$$\tan^2 x + \sin x \csc x = \tan^2 x + 1 = \sec^2 x$$

**Final answer:** $\sec^2 x$

**Exercise 140**

$$\sec x \sin x \cot x = \sec x \cos x = 1$$

**Final answer:** 1

**Exercise 141**

$$\frac{\tan^2 x}{\sec^2 x} = \sin^2 x$$

**Final answer:** $\sin^2 x$

**Exercise 142**

$$\sec x - \cos x = \frac{1-\cos^2 x}{\cos x} = \tan x \sin x$$

**Final answer:** $\tan x \sin x$

**Exercise 143**

$$(1+\tan\theta)^2 - 2\tan\theta = 1 + \tan^2\theta = \sec^2\theta$$

**Final answer:** $\sec^2\theta$

**Exercise 144**

$$\sin x(\csc x - \sin x) = \sin x \csc x - \sin^2 x = 1 - \sin^2 x = \cos^2 x$$

**Final answer:** $\cos^2 x$

**Exercise 145**

$$\frac{\cos t}{\sin t} + \frac{\sin t}{1+\cos t} = \cot t + \frac{\sin t(1-\cos t)}{(1+\cos t)(1-\cos t)}$$
$$= \cot t + \frac{1-\cos^2 t}{\sin t(1+\cos t)} = \cot t + \frac{\sin t}{1+\cos t}$$

**Final answer:** $\cot t + \dfrac{\sin t}{1+\cos t}$

**Exercise 146**

$$\frac{1+\tan^2\alpha}{1+\cot^2\alpha} = \frac{\sec^2\alpha}{\csc^2\alpha} = \frac{1/\cos^2\alpha}{1/\sin^2\alpha} = \tan^2\alpha$$

**Final answer:** $\tan^2\alpha$

**Exercise 147**

$$\frac{\tan\theta\cot\theta}{\csc\theta} = \frac{1}{\csc\theta} = \sin\theta$$

**Final answer:** Identity verified

**Exercise 148**

$$\frac{\sec^2\theta}{\tan\theta} = \frac{1/\cos^2\theta}{\sin\theta/\cos\theta} = \frac{1}{\sin\theta\cos\theta} = \sec\theta\csc\theta$$

**Final answer:** Identity verified

**Exercise 149**

$$\frac{\sin t}{\csc t} + \frac{\cos t}{\sec t} = \sin^2 t + \cos^2 t = 1$$

**Final answer:** Identity verified

**Exercise 150**

$$\frac{\sin x}{\cos x + 1} + \frac{\cos x - 1}{\sin x} = \frac{\sin^2 x + (\cos x - 1)(\cos x + 1)}{\sin x(\cos x + 1)}$$
$$= \frac{\sin^2 x + \cos^2 x - 1}{\sin x(\cos x + 1)} = 0$$

**Final answer:** Identity verified

**Exercise 151**

$$\cot y + \tan y = \frac{\cos y}{\sin y} + \frac{\sin y}{\cos y} = \frac{1}{\sin y\cos y} = \sec y\csc y$$

**Final answer:** Identity verified

**Exercise 152**

$$\sin^2\beta + \tan^2\beta + \cos^2\beta = 1 + \tan^2\beta = \sec^2\beta$$

**Final answer:** Identity verified

**Exercise 153**

$$\frac{1}{1 - \sin\alpha} + \frac{1}{1 + \sin\alpha} = \frac{2}{1 - \sin^2\alpha} = \frac{2}{\cos^2\alpha} = 2\sec^2\alpha$$

**Final answer:** Identity verified

**Exercise 154**

$$\frac{\tan\theta - \cot\theta}{\sin\theta\cos\theta} = \frac{\frac{\sin^2\theta - \cos^2\theta}{\sin\theta\cos\theta}}{\sin\theta\cos\theta} = \frac{\sin^2\theta - \cos^2\theta}{\sin^2\theta\cos^2\theta}$$
$$= \sec^2\theta - \csc^2\theta$$

**Final answer:** Identity verified

**Exercise 155**

$$2\sin\theta - 1 = 0 \Rightarrow \sin\theta = \frac{1}{2}$$

**Final answer:** $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

**Exercise 156**

$$1 + \cos\theta = \frac{1}{2} \Rightarrow \cos\theta = -\frac{1}{2}$$

**Final answer:** $\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$

**Exercise 157**

$$2\tan^2\theta = 2 \Rightarrow \tan\theta = \pm 1$$

**Final answer:** $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

**Exercise 158**

$$4\sin^2\theta - 2 = 0 \Rightarrow \sin^2\theta = \frac{1}{2}$$

**Final answer:** $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

**Exercise 159**

$$\sqrt{3}\cot\theta + 1 = 0 \Rightarrow \tan\theta = -\sqrt{3}$$

**Final answer:** $\theta = \dfrac{2\pi}{3}, \dfrac{5\pi}{3}$

### Exercise 160

$$3\sec\theta - 2\sqrt{3} = 0 \Rightarrow \cos\theta = \frac{\sqrt{3}}{2}$$

**Final answer:** $\theta = \dfrac{\pi}{6}, \dfrac{11\pi}{6}$

### Exercise 161

$$2\cos\theta\sin\theta = \sin\theta \Rightarrow \sin\theta(2\cos\theta - 1) = 0$$

**Final answer:** $\theta = 0, \pi, \dfrac{\pi}{3}, \dfrac{5\pi}{3}$

### Exercise 162

$$\csc^2\theta + 2\csc\theta + 1 = 0 \Rightarrow (\csc\theta + 1)^2 = 0$$

**Final answer:** $\theta = \dfrac{3\pi}{2}$

### Exercise 163

From the graph:

- The maximum value is approximately 4 and the minimum value is approximately $-4$, so the amplitude is
$$A = \frac{4 - (-4)}{2} = 4.$$
- The midline is $y = 0$, so there is no vertical shift.
- One full cycle occurs from $x = -4$ to $x = 4$, so the period is $T = 8$.
- Hence the angular frequency is
$$B = \frac{2\pi}{T} = \frac{\pi}{4}.$$
- The graph passes through the origin and initially increases, which matches a sine function with no phase shift.

A suitable model is

$$f(x) = 4\sin\left(\frac{\pi}{4}x\right).$$

**Final answer:**

$$f(x) = 4\sin\left(\frac{\pi}{4}x\right)$$

—

## Exercise 164

From the graph:

- The maximum is about 2 and the minimum is about $-2$, so the amplitude is

$$A = 2.$$

- The midline is $y = 0$, so there is no vertical shift.
- Consecutive peaks are spaced 2 units apart, so the period is $T = 2$.
- The angular frequency is

$$B = \frac{2\pi}{2} = \pi.$$

- The graph has a minimum at $x = 0$, which corresponds to a negative cosine.

A suitable model is

$$f(x) = -2\cos(\pi x).$$

**Final answer:**

$$f(x) = -2\cos(\pi x)$$

—

## Exercise 165

From the graph:

- The maximum is about 1 and the minimum is about $-1$, so the amplitude is

$$A = 1.$$

- The midline is $y = 0$, so there is no vertical shift.

- There are approximately 8 full cycles from $x = -4$ to $x = 4$, giving a period

$$T = 1.$$

- The angular frequency is

$$B = \frac{2\pi}{1} = 2\pi.$$

- The graph crosses the origin and initially increases, which matches a sine function.

A suitable model is

$$f(x) = \sin(2\pi x).$$

**Final answer:**

$$f(x) = \sin(2\pi x)$$

—

**Exercise 166**

From the graph:

- The maximum is about 1 and the minimum is about $-1$, so the amplitude is

$$A = 1.$$

- The midline is $y = 0$, so there is no vertical shift.
- There are approximately 12 oscillations from $x = -4$ to $x = 4$, so the period is

$$T = \frac{8}{12} = \frac{2}{3}.$$

- The angular frequency is

$$B = \frac{2\pi}{T} = 3\pi.$$

- The graph has a maximum at $x = 0$, which corresponds to a cosine function.

A suitable model is

$$f(x) = \cos(3\pi x).$$

**Final answer:**

$$f(x) = \cos(3\pi x)$$

**Exercise 167**

Given
$$y = \sin\left(x - \frac{\pi}{4}\right)$$

Comparing with the standard form $y = A\sin(Bx - C)$:
$$A = 1, \quad B = 1, \quad C = \frac{\pi}{4}.$$

Amplitude:
$$|A| = 1.$$

Period:
$$T = \frac{2\pi}{B} = 2\pi.$$

Phase shift:
$$\frac{C}{B} = \frac{\pi}{4},$$

to the **right**.

**Final answer:** Amplitude $= 1$; Period $= 2\pi$; Phase shift $= \dfrac{\pi}{4}$ right.

—

**Exercise 168**

Given
$$y = 3\cos(2x + 3)$$

Rewrite:
$$y = 3\cos(2x - (-3)).$$

Thus
$$A = 3, \quad B = 2, \quad C = -3.$$

Amplitude:
$$|A| = 3.$$

Period:
$$T = \frac{2\pi}{2} = \pi.$$

Phase shift:
$$\frac{C}{B} = -\frac{3}{2},$$

to the **left**.

**Final answer:** Amplitude $= 3$; Period $= \pi$; Phase shift $= \dfrac{3}{2}$ left.

—

**Exercise 169**

Given

$$y = -\frac{1}{2}\sin\left(\frac{1}{4}x\right)$$

Here

$$A = -\frac{1}{2}, \quad B = \frac{1}{4}, \quad C = 0.$$

Amplitude:

$$|A| = \frac{1}{2}.$$

Period:

$$T = \frac{2\pi}{1/4} = 8\pi.$$

Phase shift:

$$\frac{C}{B} = 0.$$

(The negative sign reflects the graph vertically.)

**Final answer:** Amplitude $= \dfrac{1}{2}$; Period $= 8\pi$; Phase shift $= 0$.

—

**Exercise 170**

Given

$$y = 2\cos\left(x - \frac{\pi}{3}\right)$$

Thus

$$A = 2, \quad B = 1, \quad C = \frac{\pi}{3}.$$

Amplitude:

$$|A| = 2.$$

Period:

$$T = 2\pi.$$

Phase shift:

$$\frac{C}{B} = \frac{\pi}{3},$$

to the **right**.

**Final answer:** Amplitude = 2; Period = $2\pi$; Phase shift = $\frac{\pi}{3}$ right.

—

## Exercise 171

Given

$$y = -3\sin(\pi x + 2)$$

Rewrite:

$$y = -3\sin(\pi x - (-2)).$$

Thus

$$A = -3, \quad B = \pi, \quad C = -2.$$

Amplitude:

$$|A| = 3.$$

Period:

$$T = \frac{2\pi}{\pi} = 2.$$

Phase shift:

$$\frac{C}{B} = -\frac{2}{\pi},$$

to the **left**.

**Final answer:** Amplitude = 3; Period = 2; Phase shift = $\frac{2}{\pi}$ left.

—

## Exercise 172

Given

$$y = 4\cos\left(2x - \frac{\pi}{2}\right)$$

Thus

$$A = 4, \quad B = 2, \quad C = \frac{\pi}{2}.$$

Amplitude:

$$|A| = 4.$$

Period:

$$T = \frac{2\pi}{2} = \pi.$$

Phase shift:

$$\frac{C}{B} = \frac{\pi}{4},$$

to the **right**.

**Final answer:** Amplitude = 4; Period = $\pi$; Phase shift = $\frac{\pi}{4}$ right.

—

### Exercise 173

The diameter is 40 in, so the radius is

$$r = 20 \text{ in}.$$

Convert 120° to radians:

$$120° = \frac{2\pi}{3}.$$

Arc length:

$$s = r\theta = 20 \cdot \frac{2\pi}{3} = \frac{40\pi}{3} \approx 41.89.$$

**Final answer:** The wheel moves approximately 42 inches.

—

### Exercise 174

Use the arc length formula

$$s = r\theta,$$

with $\theta$ in radians.

#### (a)

$$s = 12.8 \cdot \frac{5\pi}{6} \approx 33.51$$

#### (b)

$$s = 4.378 \cdot \frac{7\pi}{6} \approx 16.03$$

**(c)**

$$50^\circ = \frac{5\pi}{18}, \quad s = 0.964 \cdot \frac{5\pi}{18} \approx 0.84$$

**(d)**

$$325^\circ = \frac{65\pi}{36}, \quad s = 8.55 \cdot \frac{65\pi}{36} \approx 48.52$$

**Final answer:** (a) 33.51 cm (b) 16.03 cm (c) 0.84 cm (d) 48.52 cm

### Exercise 175

Given $\omega = \frac{\theta}{t}$.

a) $\theta = \frac{7\pi}{4}$ rad, $t = 10$ s

$$\omega = \frac{7\pi/4}{10} = \frac{7\pi}{40}$$

**Final answer:** $\omega \approx 0.550$ rad/s

b) $\theta = \frac{3\pi}{5}$ rad, $t = 8$ s

$$\omega = \frac{3\pi}{40}$$

**Final answer:** $\omega \approx 0.236$ rad/s

c) $\theta = \frac{2\pi}{9}$ rad, $t = 1$ min

$$\omega = \frac{2\pi}{9}$$

**Final answer:** $\omega \approx 0.698$ rad/min

d) $\theta = 23.76$ rad, $t = 14$ min

$$\omega = \frac{23.76}{14}$$

**Final answer:** $\omega \approx 1.697$ rad/min

### Exercise 176

Total area $A = 250{,}000$ m$^2$.

a) For a circle, $A = \pi r^2$:

$$r = \sqrt{\frac{250000}{\pi}}$$

**Final answer:** $r \approx 282.1$ m

b) A $45° = \frac{\pi}{4}$ sector. Arc length $s = r\theta$:

$$s = 282.1 \cdot \frac{\pi}{4}$$

**Final answer:** $s \approx 221.6$ m

## Exercise 177

$$A = \frac{1}{2}x^2 \sin\theta, \quad x = 8,\ \theta = \frac{5\pi}{12}$$
$$A = \frac{1}{2}(64)\sin\left(\frac{5\pi}{12}\right)$$

**Final answer:** $A \approx 30.9\ \text{in}^2$

## Exercise 178

$$\omega(t) = 9|\cos(\pi t - \pi/12)|$$

At $t = 9$:

$$\omega = 9\left|\cos\left(\frac{107\pi}{12}\right)\right|$$

**Final answer:** $\omega \approx 8.70$ rad/s

## Exercise 179

$V(t) = 150\cos(368t)$

a) Period:

$$T = \frac{2\pi}{368}$$

**Final answer:** $T \approx 0.0171$ s

b) Number of periods in 1 s:

$$\frac{1}{T}$$

**Final answer:** $\approx 58.6$ periods

## Exercise 180

$$N(t) = 12 + 3\sin\left(\frac{2\pi}{365}(t - 79)\right)$$

a) Amplitude $= 3$, period $= 365$ **Final answer:** $A = 3,\ T = 365$

b) Longest day:

$$12 + 3 = 15$$

**Final answer:** 15 hours

c) Shortest day:

$$12 - 3 = 9$$

**Final answer:** 9 hours

d) At $t = 90$:

$$N(90) = 12 + 3\sin\left(\frac{2\pi}{365}(11)\right)$$

**Final answer:** $\approx 12.57$ hours

## Exercise 181

$$T(t) = 50 + 10\sin\left(\frac{\pi}{12}(t - 8)\right)$$

a) Amplitude = 10, period:

$$T = \frac{2\pi}{\pi/12} = 24$$

**Final answer:** $A = 10,\ T = 24$

b) At $t = 7$:

$$T = 50 + 10\sin\left(-\frac{\pi}{12}\right)$$

**Final answer:** $\approx 47.4°\text{F}$

c) $T = 60$:

$$\sin\left(\frac{\pi}{12}(t - 8)\right) = 1$$

**Final answer:** $t = 14$

## Exercise 182

$$H(t) = 8\sin\left(\frac{\pi t}{6}\right)$$

a) Amplitude = 8, period:

$$T = \frac{2\pi}{\pi/6} = 12$$

**Final answer:** $A = 8,\ T = 12$

b) One period: $0 \leq t \leq 12$

c) 4:30 a.m. $\Rightarrow t = 4.5$:

$$H(4.5) = 8\sin\left(\frac{3\pi}{4}\right)$$

**Final answer:** $H \approx 5.66$ ft

**Inverse Functions**

### Exercise 183

The graph is a V-shape (similar to $y = |x|$). A horizontal line drawn above the vertex intersects the graph at two points.

**Conclusion:** The graph is **not one-to-one** because it fails the horizontal line test.

### Exercise 184

The graph is increasing for all $x$ in its domain. Every horizontal line intersects the graph at exactly one point.

**Conclusion:** The graph **is one-to-one** since it passes the horizontal line test.

### Exercise 185

The graph is a semicircle. Many horizontal lines intersect the graph at two points.

**Conclusion:** The graph is **not one-to-one** because it fails the horizontal line test.

### Exercise 186

The graph decreases and then increases (has a local minimum). Some horizontal lines intersect the graph more than once.

**Conclusion:** The graph is **not one-to-one**.

### Exercise 187

The graph is strictly increasing over its entire domain. Each horizontal line intersects the graph at most once.

**Conclusion:** The graph **is one-to-one**.

## Exercise 188

The graph has a horizontal (constant) segment. Any horizontal line at that $y$-value intersects the graph infinitely many times.

**Conclusion:** The graph is **not one-to-one**.

## Exercise 189

Given

$$f(x) = x^2 - 4, \quad x \geq 0$$

Set $y = x^2 - 4$. Then

$$x^2 = y + 4 \quad \Rightarrow \quad x = \sqrt{y+4}$$

(since $x \geq 0$).

$$f^{-1}(x) = \sqrt{x+4}$$

**Final answer:**

$$f^{-1}(x) = \sqrt{x+4}, \quad \text{Domain } [-4, \infty), \text{ Range } [0, \infty)$$

—

## Exercise 190

Given

$$f(x) = \sqrt[3]{x-4}$$

Set $y = \sqrt[3]{x-4}$. Then

$$y^3 = x - 4 \quad \Rightarrow \quad x = y^3 + 4$$

$$f^{-1}(x) = x^3 + 4$$

**Final answer:**

$$f^{-1}(x) = x^3 + 4, \quad \text{Domain } (-\infty, \infty), \text{ Range } (-\infty, \infty)$$

—

**Exercise 191**

Given

$$f(x) = x^3 + 1$$

Set $y = x^3 + 1$. Then

$$x^3 = y - 1 \quad \Rightarrow \quad x = \sqrt[3]{y - 1}$$

$$f^{-1}(x) = \sqrt[3]{x - 1}$$

**Final answer:**

$$f^{-1}(x) = \sqrt[3]{x - 1}, \quad \text{Domain } (-\infty, \infty), \text{ Range } (-\infty, \infty)$$

—

**Exercise 192**

Given

$$f(x) = (x - 1)^2, \quad x \leq 1$$

Set $y = (x - 1)^2$. Then

$$x - 1 = -\sqrt{y} \quad \Rightarrow \quad x = 1 - \sqrt{y}$$

(negative root chosen because $x \leq 1$).

$$f^{-1}(x) = 1 - \sqrt{x}$$

**Final answer:**

$$f^{-1}(x) = 1 - \sqrt{x}, \quad \text{Domain } [0, \infty), \text{ Range } (-\infty, 1]$$

—

**Exercise 193**

Given

$$f(x) = \sqrt{x - 1}$$

Set $y = \sqrt{x - 1}$. Then

$$y^2 = x - 1 \quad \Rightarrow \quad x = y^2 + 1$$

$$f^{-1}(x) = x^2 + 1$$

**Final answer:**

$$f^{-1}(x) = x^2 + 1, \quad \text{Domain } [0, \infty), \text{ Range } [1, \infty)$$

—

**Exercise 194**

Given

$$f(x) = \frac{1}{x+2}$$

Set $y = \frac{1}{x+2}$. Then

$$y(x+2) = 1 \quad \Rightarrow \quad x = \frac{1}{y} - 2$$

$$f^{-1}(x) = \frac{1}{x} - 2$$

**Final answer:**

$$f^{-1}(x) = \frac{1}{x} - 2, \quad \text{Domain } x \neq 0, \text{ Range } x \neq -2$$

—

**Exercise 195**

The inverse function is obtained by reflecting the graph of $f$ across the line $y = x$.

**Final answer:** The graph of $f^{-1}$ is the reflection of the given graph across $y = x$.

—

**Exercise 196**

Reflect the given decreasing curve across the line $y = x$.

**Final answer:** The inverse exists and is obtained by swapping $x$ and $y$ coordinates on the graph.

—

**Exercise 197**

The inverse of the piecewise linear graph is formed by reflecting each linear segment across $y = x$.

**Final answer:** The inverse is a piecewise linear function obtained by reflection across $y = x$.

—

**Exercise 198**

The inverse is obtained by reflecting the curve across the line $y = x$.

**Final answer:** The reflected curve represents $f^{-1}$; domain and range are swapped.

**Exercise 199**

Given $f(x) = 8x$ and $g(x) = \frac{x}{8}$.

Compute the compositions:

$$(f \circ g)(x) = f\left(\frac{x}{8}\right) = 8 \cdot \frac{x}{8} = x, \qquad (g \circ f)(x) = g(8x) = \frac{8x}{8} = x.$$

Both compositions give the identity (on $\mathbb{R}$), so $f$ and $g$ are inverses.

**Final answer:** $f^{-1}(x) = \frac{x}{8}$ and $g^{-1}(x) = 8x$.

**Exercise 200**

Given $f(x) = 8x + 3$ and $g(x) = \frac{x-3}{8}$.

$$(f \circ g)(x) = f\left(\frac{x-3}{8}\right) = 8\left(\frac{x-3}{8}\right) + 3 = x - 3 + 3 = x,$$
$$(g \circ f)(x) = g(8x+3) = \frac{(8x+3)-3}{8} = \frac{8x}{8} = x.$$

Hence $f$ and $g$ are inverses (domain $\mathbb{R}$).

**Final answer:** $f^{-1}(x) = \frac{x-3}{8}$.

**Exercise 201**

Given $f(x) = 5x - 7$ and $g(x) = \frac{x+5}{7}$.

Check:

$$(f \circ g)(x) = 5\left(\frac{x+5}{7}\right) - 7 = \frac{5x+25}{7} - \frac{49}{7} = \frac{5x-24}{7} \neq x,$$

so they are not inverses.

(Indeed, the true inverse of $f(x) = 5x - 7$ would be $f^{-1}(x) = \frac{x+7}{5}$, which is not $g$.)

**Final answer:** Not inverses.

### Exercise 202

Given $f(x) = \frac{2}{3}x + 2$ and $g(x) = \frac{3}{2}x + 3$.

$$(f \circ g)(x) = \frac{2}{3}\left(\frac{3}{2}x + 3\right) + 2 = x + 2 + 2 = x + 4 \neq x,$$

so they are not inverses.

**Final answer:** Not inverses.

### Exercise 203

Given $f(x) = \frac{1}{x-1}$ $(x \neq 1)$ and $g(x) = \frac{1}{x} + 1$ $(x \neq 0)$.

First:

$$(f \circ g)(x) = f\left(\frac{1}{x} + 1\right) = \frac{1}{\left(\frac{1}{x} + 1\right) - 1} = \frac{1}{\frac{1}{x}} = x, \quad (x \neq 0).$$

Next:

$$(g \circ f)(x) = g\left(\frac{1}{x-1}\right) = \frac{1}{\frac{1}{x-1}} + 1 = (x-1) + 1 = x, \quad (x \neq 1).$$

Thus they are inverses on their stated domains.

**Final answer:** $f^{-1}(x) = \frac{1}{x} + 1$ (with $x \neq 0$).

### Exercise 204

Given $f(x) = x^3 + 1$ and $g(x) = (x-1)^{1/3}$.

$$(f \circ g)(x) = \left((x-1)^{1/3}\right)^3 + 1 = (x-1) + 1 = x,$$

$$(g \circ f)(x) = \left((x^3+1) - 1\right)^{1/3} = (x^3)^{1/3} = x.$$

So they are inverses (domain $\mathbb{R}$).

**Final answer:** $f^{-1}(x) = (x-1)^{1/3}$.

### Exercise 205

Given $f(x) = x^2 + 2x + 1$ with $x \geq -1$ and $g(x) = -1 + \sqrt{x}$ with $x \geq 0$.

Rewrite $f(x) = (x+1)^2$ (restricted to $x \geq -1$ so it is one-to-one).

$$(f \circ g)(x) = \left((-1+\sqrt{x})+1\right)^2 = (\sqrt{x})^2 = x, \quad (x \geq 0).$$

$$(g \circ f)(x) = -1 + \sqrt{(x+1)^2} = -1 + |x+1|.$$

Since $x \geq -1$, we have $x + 1 \geq 0$, so $|x+1| = x+1$, hence

$$(g \circ f)(x) = -1 + (x+1) = x, \quad (x \geq -1).$$

Thus they are inverses with those domain restrictions.

**Final answer:** $f^{-1}(x) = -1 + \sqrt{x}$ for $x \geq 0$.

### Exercise 206

Given $f(x) = \sqrt{4-x^2}$ on $0 \leq x \leq 2$, and $g(x) = \sqrt{4-x^2}$ on $0 \leq x \leq 2$.

On $[0,2]$, $f$ is strictly decreasing, hence one-to-one, with range $[0,2]$. Solve $y = \sqrt{4-x^2}$:

$$y^2 = 4 - x^2 \;\Rightarrow\; x^2 = 4 - y^2 \;\Rightarrow\; x = \sqrt{4-y^2} \quad \text{(since } x \geq 0\text{)}.$$

So $f^{-1}(y) = \sqrt{4-y^2}$, which has the same formula as $f$ (self-inverse on $[0,2]$).

$$(f \circ f)(x) = \sqrt{4 - (\sqrt{4-x^2})^2} = \sqrt{4-(4-x^2)} = \sqrt{x^2} = x \quad (x \in [0,2]).$$

**Final answer:** $f^{-1}(x) = \sqrt{4-x^2}$ on $0 \leq x \leq 2$ (so $f$ and $g$ are inverses).

### Exercise 207

$$\tan^{-1}\left(\frac{\sqrt{3}}{3}\right) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right).$$

Since $\tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$ and $\tan^{-1}$ returns values in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$,

$$\tan^{-1}\left(\frac{\sqrt{3}}{3}\right) = \frac{\pi}{6}.$$

**Final answer:** $\dfrac{\pi}{6}$.

### Exercise 208

We need $\theta \in [0, \pi]$ such that $\cos\theta = -\frac{\sqrt{2}}{2}$. This occurs at $\theta = \frac{3\pi}{4}$.

**Final answer:** $\dfrac{3\pi}{4}$.

### Exercise 209

Using the common convention $\cot^{-1}$ has range $(0, \pi)$. Solve $\cot\theta = 1$, i.e., $\tan\theta = 1$, giving $\theta = \frac{\pi}{4}$ in $(0, \pi)$.

**Final answer:** $\dfrac{\pi}{4}$.

### Exercise 210

$\sin^{-1}$ has range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, and $\sin\left(-\frac{\pi}{2}\right) = -1$.

**Final answer:** $-\dfrac{\pi}{2}$.

### Exercise 211

Find $\theta \in [0, \pi]$ such that $\cos\theta = \frac{\sqrt{3}}{2}$. That is $\theta = \frac{\pi}{6}$.

**Final answer:** $\dfrac{\pi}{6}$.

### Exercise 212

Let $\theta = \tan^{-1}(\sqrt{3})$. Then $\tan\theta = \sqrt{3}$ with $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, so $\theta = \frac{\pi}{3}$. Thus

$$\cos\Big(\tan^{-1}(\sqrt{3})\Big) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}.$$

**Final answer:** $\dfrac{1}{2}$.

## Exercise 213

Let $\theta = \cos^{-1}\left(\frac{\sqrt{2}}{2}\right)$, so $\theta \in [0, \pi]$ and $\theta = \frac{\pi}{4}$. Then

$$\sin\left(\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)\right) = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}.$$

**Final answer:** $\dfrac{\sqrt{2}}{2}$.

## Exercise 214

Since $\frac{\pi}{3} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $\sin^{-1}$ returns the principal value in that interval,

$$\sin^{-1}(\sin(\pi/3)) = \frac{\pi}{3}.$$

**Final answer:** $\dfrac{\pi}{3}$.

## Exercise 215

Because $-\frac{\pi}{6} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$,

$$\tan^{-1}(\tan(-\pi/6)) = -\frac{\pi}{6}.$$

**Final answer:** $-\dfrac{\pi}{6}$.

## Exercise 216

Given $C = T(F) = \dfrac{5}{9}(F - 32)$.

(a) Find inverse: solve for $F$ in terms of $C$:

$$C = \frac{5}{9}(F - 32) \;\Rightarrow\; \frac{9}{5}C = F - 32 \;\Rightarrow\; F = \frac{9}{5}C + 32.$$

(b) This inverse converts degrees Celsius to degrees Fahrenheit.

**Final answer:** $T^{-1}(C) = \dfrac{9}{5}C + 32$ (Celsius $\to$ Fahrenheit).

## Exercise 217

Given $V = f(x) = 500(0.04 - x^2)$ for $0 \le x \le 0.2$.

(a) Solve for $x$:

$$\frac{V}{500} = 0.04 - x^2 \;\Rightarrow\; x^2 = 0.04 - \frac{V}{500}.$$

Because $x \ge 0$,

$$x = f^{-1}(V) = \sqrt{0.04 - \frac{V}{500}}.$$

(Here $V$ ranges from 20 down to 0, so $\text{Dom}(f^{-1}) = [0, 20]$.)

(b) Interpretation: the inverse takes a velocity value $V$ and returns the distance $x$ from the artery center that would produce that velocity.

(c) Distances:

$$x(15) = \sqrt{0.04 - \frac{15}{500}} = \sqrt{0.01} = 0.100 \text{ cm},$$
$$x(10) = \sqrt{0.04 - \frac{10}{500}} = \sqrt{0.02} \approx 0.141 \text{ cm},$$
$$x(5) = \sqrt{0.04 - \frac{5}{500}} = \sqrt{0.03} \approx 0.173 \text{ cm}.$$

**Final answer:** $f^{-1}(V) = \sqrt{0.04 - \frac{V}{500}}$, $V \in [0, 20]$ and $x(15) = 0.100$, $x(10) \approx 0.141$, $x(5) \approx 0.173$ (cm).

## Exercise 218

Given $D(x) = 2x + 24$ (U.S. size $x$ to European size $D$).

(a) Evaluate:

$$D(6) = 36, \quad D(8) = 40, \quad D(10) = 44, \quad D(12) = 48.$$

(b) Inverse (Europe $\to$ U.S.): solve $y = 2x + 24$:

$$x = \frac{y - 24}{2}.$$

(c) U.S. sizes for European sizes $46, 52, 62, 70$:

$$\frac{46 - 24}{2} = 11, \quad \frac{52 - 24}{2} = 14, \quad \frac{62 - 24}{2} = 19, \quad \frac{70 - 24}{2} = 23.$$

**Final answer:** (a) $36, 40, 44, 48$; (b) $D^{-1}(y) = \frac{y - 24}{2}$; (c) $11, 14, 19, 23$.

**Exercise 219**

Given $C(p) = \dfrac{75p}{85 - p}$, where $C$ is in *thousands* of dollars and $p < 85$.

(a)

$$C(25) = \frac{75 \cdot 25}{85 - 25} = \frac{1875}{60} = 31.25 \;\Rightarrow\; \$31{,}250,$$

$$C(40) = \frac{3000}{45} = 66.\overline{6} \;\Rightarrow\; \$66{,}667 \text{ (approx)},$$

$$C(50) = \frac{3750}{35} = 107.142857 \;\Rightarrow\; \$107{,}143 \text{ (approx)}.$$

(b) Find inverse: let $C = \dfrac{75p}{85 - p}$.

$$C(85 - p) = 75p \Rightarrow 85C - Cp = 75p \Rightarrow 85C = p(C + 75) \Rightarrow p = \frac{85C}{C + 75}.$$

(c) For \$50,000, we have $C = 50$ (thousands):

$$p = \frac{85 \cdot 50}{50 + 75} = \frac{4250}{125} = 34 \text{ ppb}.$$

**Final answer:** (a) \$31,250, \$66,667 (approx), \$107,143 (approx); (b) $C^{-1}(C) = \dfrac{85C}{C + 75}$; (c) $p = 34$ ppb.

**Exercise 220**

Given $v(t) = \dfrac{25}{t + 4} + 54$.

(a)

$$v(10) = \frac{25}{14} + 54 \approx 1.7857 + 54 = 55.7857 \text{ ft/s}.$$

(b) Inverse: solve for $t$:

$$v - 54 = \frac{25}{t + 4} \Rightarrow t + 4 = \frac{25}{v - 54} \Rightarrow t = \frac{25}{v - 54} - 4.$$

(c) Reach 150 ft/s:

$$150 = \frac{25}{t + 4} + 54 \Rightarrow 96 = \frac{25}{t + 4} \Rightarrow t + 4 = \frac{25}{96} \Rightarrow t = \frac{25}{96} - 4 = -\frac{359}{96} \approx -3.740.$$

Since this is negative, the car does *not* reach 150 ft/s for $t \geq 0$ (in fact $v(t)$ decreases toward 54).

**Final answer:** (a) $v(10) \approx 55.786$ ft/s; (b) $v^{-1}(v) = \dfrac{25}{v-54} - 4$; (c) no solution for $t \geq 0$ (formal solution $t \approx -3.740$ s).

### Exercise 221

Given $\mu = 2\sin^{-1}\left(\dfrac{1}{M}\right)$ (radians), find $\mu$ to the nearest degree.

(a) $M = 1.4$:

$$\mu = 2\sin^{-1}\left(\frac{1}{1.4}\right) \approx 91.146^\circ \approx 91^\circ.$$

(b) $M = 2.8$:

$$\mu \approx 41.798^\circ \approx 42^\circ.$$

(c) $M = 4.3$:

$$\mu \approx 26.869^\circ \approx 27^\circ.$$

**Final answer:** (a) $91^\circ$, (b) $42^\circ$, (c) $27^\circ$.

### Exercise 222

Given $\mu = 2\sin^{-1}\left(\dfrac{1}{M}\right)$, solve for $M$:

$$\frac{\mu}{2} = \sin^{-1}\left(\frac{1}{M}\right) \Rightarrow \sin\left(\frac{\mu}{2}\right) = \frac{1}{M} \Rightarrow M = \frac{1}{\sin(\mu/2)}.$$

(a) $\mu = \dfrac{\pi}{6}$:

$$M = \frac{1}{\sin(\pi/12)} = \frac{1}{\sin 15^\circ} = \sqrt{6} + \sqrt{2}.$$

(b) $\mu = \dfrac{2\pi}{7}$:

$$M = \frac{1}{\sin(\pi/7)} \approx 2.305.$$

(c) $\mu = \dfrac{3\pi}{8}$:

$$M = \frac{1}{\sin(3\pi/16)} \approx 1.555.$$

**Final answer:** (a) $\sqrt{6} + \sqrt{2}$; (b) $M \approx 2.305$; (c) $M \approx 1.555$.

**Exercise 223**

Given $T(x) = 5 + 18\sin\left(\frac{\pi}{6}(x - 4.6)\right)$, with $x = 1$ at Jan 1.

Set $T(x) = 21$:

$$21 = 5 + 18\sin\left(\frac{\pi}{6}(x - 4.6)\right) \Rightarrow \sin\left(\frac{\pi}{6}(x - 4.6)\right) = \frac{16}{18} = \frac{8}{9}.$$

Let $\theta = \frac{\pi}{6}(x - 4.6)$. Then

$$\theta = \sin^{-1}\left(\frac{8}{9}\right) \quad \text{or} \quad \theta = \pi - \sin^{-1}\left(\frac{8}{9}\right)$$

(with $2\pi k$ added for other cycles). Converting back:

$$x = 4.6 + \frac{6}{\pi}\theta.$$

Within one year, this gives

$$x \approx 6.691 \quad \text{and} \quad x \approx 8.509.$$

Interpreting $x$ as a month-count with $x = 6$ at June 1 and $x = 8$ at Aug 1, these are about late June and mid August: approximately June 21 and August 16 (using about 30 days per month).

**Final answer:** $x \approx 6.691$ and $x \approx 8.509$ (about June 21 and Aug 16).

**Exercise 224**

Given $D(t) = 5\sin\left(\frac{\pi}{6}t - \frac{7\pi}{8}\right) + 8$. Find the first $t > 0$ such that $D(t) = 11.75$.

$$11.75 = 5\sin\left(\frac{\pi}{6}t - \frac{7\pi}{8}\right) + 8 \Rightarrow \sin\left(\frac{\pi}{6}t - \frac{7\pi}{8}\right) = \frac{3.75}{5} = 0.75 = \frac{3}{4}.$$

Let $\theta = \frac{\pi}{6}t - \frac{7\pi}{8}$ and $\alpha = \sin^{-1}(3/4)$. Then solutions are $\theta = \alpha + 2\pi k$ or $\theta = \pi - \alpha + 2\pi k$. So

$$\frac{\pi}{6}t = \theta + \frac{7\pi}{8} \Rightarrow t = \frac{6}{\pi}\left(\theta + \frac{7\pi}{8}\right).$$

The smallest positive solution is

$$t \approx 6.870 \text{ hours} \approx 6 \text{ h } 52 \text{ min}.$$

**Final answer:** $t \approx 6.870$ hours after midnight (about 6:52 a.m.).

**Exercise 225**

Given $s(t) = -6\cos\left(\frac{\pi t}{2}\right)$ and the object starts at $t = 0$ with $s(0) = -6$.

"Distance moved" from the starting position after time $t$ is

$$|s(t) - s(0)| = \left|-6\cos\left(\frac{\pi t}{2}\right) + 6\right|.$$

Set this equal to 4.5:

$$\left|-6\cos\left(\frac{\pi t}{2}\right) + 6\right| = 4.5 \Rightarrow 6\left|1 - \cos\left(\frac{\pi t}{2}\right)\right| = 4.5 \Rightarrow \left|1 - \cos\left(\frac{\pi t}{2}\right)\right| = 0.75.$$

For the *first* time after $t = 0$, we have $\cos(\cdot)$ decreasing from 1, so $1 - \cos(\cdot) \geq 0$ and

$$1 - \cos\left(\frac{\pi t}{2}\right) = 0.75 \Rightarrow \cos\left(\frac{\pi t}{2}\right) = 0.25.$$

Thus

$$\frac{\pi t}{2} = \cos^{-1}(0.25) \Rightarrow t = \frac{2}{\pi}\cos^{-1}(0.25) \approx 0.839 \text{ s}.$$

**Final answer:** $t = \frac{2}{\pi}\cos^{-1}(0.25) \approx 0.839$ s.

**Exercise 226**

The viewing angle is modeled by

$$\theta(x) = \tan^{-1}\left(\frac{5.5}{x}\right) - \tan^{-1}\left(\frac{2.5}{x}\right),$$

where $x$ is the distance (in feet) from the portrait.

Substitute $x = 4$:

$$\theta(4) = \tan^{-1}\left(\frac{5.5}{4}\right) - \tan^{-1}\left(\frac{2.5}{4}\right) = \tan^{-1}(1.375) - \tan^{-1}(0.625).$$

Evaluating (in radians),

$$\tan^{-1}(1.375) \approx 0.9420, \qquad \tan^{-1}(0.625) \approx 0.5586.$$

Thus,

$$\theta(4) \approx 0.9420 - 0.5586 = 0.3834 \text{ rad}.$$

**Final answer:** $\theta \approx 0.383$ radians.

## Exercise 227

Evaluate $\tan^{-1}(\tan(2.1))$ and $\cos^{-1}(\cos(2.1))$.

$$\tan^{-1}(\tan(2.1)) \approx -1.0416.$$

This occurs because $\tan^{-1}$ has range $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, and 2.1 lies outside this interval. The angle $2.1 - \pi \approx -1.0416$ is coterminal with 2.1 and lies in the principal range.

Next,

$$\cos^{-1}(\cos(2.1)) = 2.1,$$

since $\cos^{-1}$ has range $[0, \pi]$ and $2.1 \in [0, \pi]$.

**Final answer:** $\tan^{-1}(\tan(2.1)) \approx -1.042$, while $\cos^{-1}(\cos(2.1)) = 2.1$.

## Exercise 228

Evaluate $\sin(\sin^{-1}(-2))$ and $\tan(\tan^{-1}(-2))$.

The expression $\sin^{-1}(-2)$ is *undefined* because the domain of $\sin^{-1} x$ is $[-1, 1]$. Therefore,

$$\sin\left(\sin^{-1}(-2)\right) \text{ is undefined.}$$

For the second expression,

$$\tan\left(\tan^{-1}(-2)\right) = -2,$$

since $-2$ lies in the domain of $\tan^{-1}$ and $\tan(\tan^{-1}(x)) = x$ for all real $x$.

**Final answer:** $\sin(\sin^{-1}(-2))$ is undefined, and $\tan(\tan^{-1}(-2)) = -2$.

**Exponential and Logarithmic Functions**

## Exercise 229

Given $f(x) = 5^x$.

$$\text{a) } f(3) = 5^3 = 125$$
$$\text{b) } f\left(\tfrac{1}{2}\right) = 5^{1/2} = \sqrt{5} \approx 2.24$$
$$\text{c) } f(\sqrt{2}) = 5^{\sqrt{2}} \approx 9.77$$

**Final answer:** 125, 2.24, 9.77

## Exercise 230

Given $f(x) = (0.3)^x$.

$$\text{a) } f(-1) = (0.3)^{-1} = \tfrac{1}{0.3} \approx 3.33$$

$$\text{b) } f(4) = (0.3)^4 \approx 0.01$$

$$\text{c) } f(-1.5) = (0.3)^{-1.5} \approx 6.08$$

**Final answer:** $3.33,\ 0.01,\ 6.08$

## Exercise 231

Given $f(x) = 10^x$.

$$\text{a) } f(-2) = 10^{-2} = 0.01$$

$$\text{b) } f(4) = 10^4 = 10000$$

$$\text{c) } f\left(\tfrac{5}{3}\right) = 10^{1.667} \approx 46.42$$

**Final answer:** $0.01,\ 10000,\ 46.42$

## Exercise 232

Given $f(x) = e^x$.

$$\text{a) } f(2) = e^2 \approx 7.39$$

$$\text{b) } f(-3.2) = e^{-3.2} \approx 0.04$$

$$\text{c) } f(\pi) = e^{\pi} \approx 23.14$$

**Final answer:** $7.39,\ 0.04,\ 23.14$

## Exercise 233

The graph is decreasing and approaches the horizontal asymptote $y = 2$ from above.

This matches $y = \left(\frac{1}{2}\right)^x + 2$.

**Final answer:** $\boxed{y = \left(\frac{1}{2}\right)^x + 2}$

## Exercise 234

The graph is decreasing, crosses the $y$-axis at 0, and has no vertical shift.

This matches $y = 1 - 5^x$.

**Final answer:** $y = 1 - 5^x$

## Exercise 235

The graph is increasing, passes through $(0, 1)$, and grows rapidly.

This matches $y = 2^{x+1}$.

**Final answer:** $y = 2^{x+1}$

## Exercise 236

The graph is decreasing, passes through $(0, 1)$, and approaches $y = 0$.

This matches $y = 4^{-x}$.

**Final answer:** $y = 4^{-x}$

## Exercise 237

The graph is increasing but remains below the $x$-axis and approaches 0 from below.

This matches $y = -3^{-x}$.

**Final answer:** $y = -3^{-x}$

## Exercise 238

The graph is increasing, shifted down, and has base 3 behavior.

This matches $y = 3^{x-1}$.

**Final answer:** $y = 3^{x-1}$

## Exercise 239

Given $f(x) = e^x + 2$.

- Domain: all real numbers - Since $e^x > 0$, we have $f(x) > 2$ - Horizontal asymptote: $y = 2$

**Final answer:** Domain $(-\infty, \infty)$, Range $(2, \infty)$, horizontal asymptote $y = 2$

### Exercise 240

Given $f(x) = -2^x$.

- Domain: all real numbers - Since $2^x > 0$, $-2^x < 0$ - Horizontal asymptote: $y = 0$

**Final answer:** Domain $(-\infty, \infty)$, Range $(-\infty, 0)$, horizontal asymptote $y = 0$

### Exercise 241

Given $f(x) = 3^{x+1}$.

- Domain: all real numbers - Range: positive real numbers - Horizontal asymptote: $y = 0$

**Final answer:** Domain $(-\infty, \infty)$, Range $(0, \infty)$, horizontal asymptote $y = 0$

### Exercise 242

Given $f(x) = 4^x - 1$.

- Domain: all real numbers - Since $4^x > 0$, $f(x) > -1$ - Horizontal asymptote: $y = -1$

**Final answer:** Domain $(-\infty, \infty)$, Range $(-1, \infty)$, horizontal asymptote $y = -1$

### Exercise 243

Given $f(x) = 1 - 2^{-x}$.

- Domain: all real numbers - Since $2^{-x} > 0$, $f(x) < 1$ - Horizontal asymptote: $y = 1$

**Final answer:** Domain $(-\infty, \infty)$, Range $(-\infty, 1)$, horizontal asymptote $y = 1$

### Exercise 244

Given $f(x) = 5^{x+1} + 2$.

- Domain: all real numbers - Since $5^{x+1} > 0$, $f(x) > 2$ - Horizontal asymptote: $y = 2$

**Final answer:** Domain $(-\infty, \infty)$, Range $(2, \infty)$, horizontal asymptote $y = 2$

### Exercise 245

Given $f(x) = e^{-x} - 1$.

- Domain: all real numbers - Since $e^{-x} > 0$, $f(x) > -1$ - Horizontal asymptote: $y = -1$

**Final answer:** Domain $(-\infty, \infty)$, Range $(-1, \infty)$, horizontal asymptote $y = -1$

### Exercise 246

Given $\log_3 81 = 4$.

**Final answer:** $3^4 = 81$

### Exercise 247

Given $\log_8 2 = \frac{1}{3}$.

**Final answer:** $8^{1/3} = 2$

### Exercise 248

Given $\log_5 1 = 0$.

**Final answer:** $5^0 = 1$

### Exercise 249

Given $\log_5 25 = 2$.

**Final answer:** $5^2 = 25$

### Exercise 250

Given $\log 0.1 = -1$.

**Final answer:** $10^{-1} = 0.1$

### Exercise 251

Given $\ln\left(\frac{1}{e^3}\right) = -3$.

**Final answer:** $e^{-3} = \frac{1}{e^3}$

### Exercise 252

Given $\log_9 3 = 0.5$.

**Final answer:** $9^{0.5} = 3$

### Exercise 253

Given $\ln 1 = 0$.

**Final answer:** $e^0 = 1$

### Exercise 254

Given $2^3 = 8$.

**Final answer:** $\log_2 8 = 3$

### Exercise 255

Given $4^{-2} = \frac{1}{16}$.

**Final answer:** $\log_4 \frac{1}{16} = -2$

### Exercise 256

Given $10^2 = 100$.

**Final answer:** $\log 100 = 2$

### Exercise 257

Given $9^0 = 1$.

**Final answer:** $\log_9 1 = 0$

### Exercise 258

Given $\left(\frac{1}{3}\right)^3 = \frac{1}{27}$.

**Final answer:** $\log_{1/3} \frac{1}{27} = 3$

**Exercise 259**

Given $\sqrt[3]{64} = 4$.

**Final answer:** $\log_4 64 = 3$

**Exercise 260**

Given $e^x = y$.

**Final answer:** $x = \ln y$

**Exercise 261**

Given $9^y = 150$.

**Final answer:** $y = \log_9 150$

**Exercise 262**

Given $b^3 = 45$.

**Final answer:** $b = \sqrt[3]{45}$

**Exercise 263**

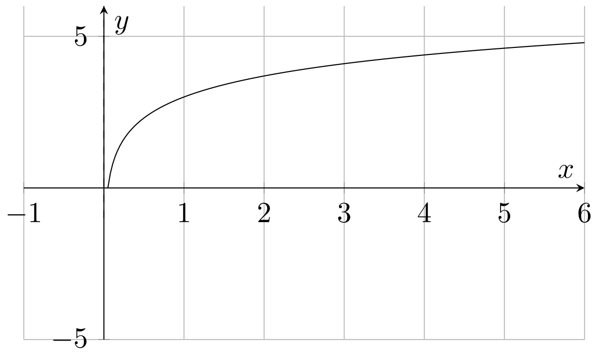
Given $4^{-3/2} = 0.125$.

**Final answer:** $\log_4 0.125 = -\frac{3}{2}$

**Exercise 264**

Given $f(x) = 3 + \ln x$.

- The logarithm requires $x > 0$, so $\mathrm{Dom}(f) = (0, \infty)$. - $\ln x$ takes all real values, and adding 3 just shifts vertically, so $\mathrm{Ran}(f) = (-\infty, \infty)$. - As $x \to 0^+$, $\ln x \to -\infty$, so the vertical asymptote is $x = 0$.

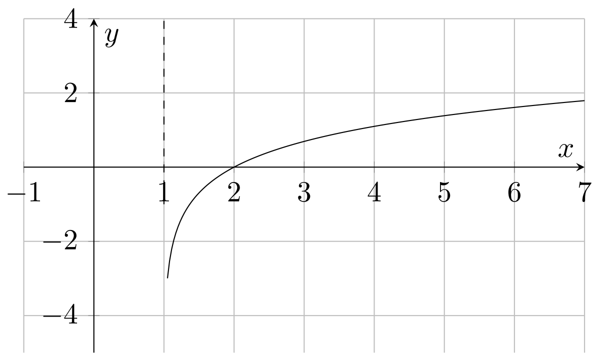



**Final answer:** Domain $(0, \infty)$, Range $(-\infty, \infty)$, vertical asymptote $x = 0$.

### Exercise 265

Given $f(x) = \ln(x - 1)$.

- Need $x - 1 > 0 \Rightarrow x > 1$, so $\mathrm{Dom}(f) = (1, \infty)$. - $\ln(\cdot)$ has range $(-\infty, \infty)$, so $\mathrm{Ran}(f) = (-\infty, \infty)$.
- As $x \to 1^+$, $\ln(x - 1) \to -\infty$, so vertical asymptote $x = 1$.

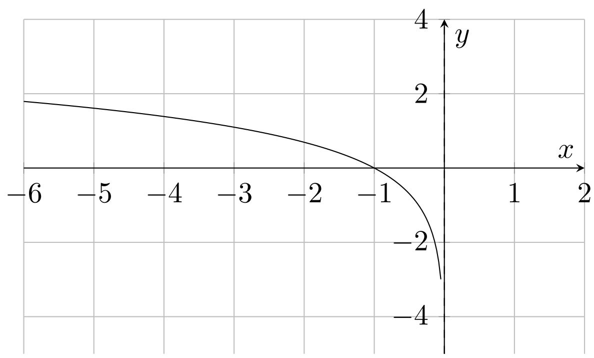



**Final answer:** Domain $(1, \infty)$, Range $(-\infty, \infty)$, vertical asymptote $x = 1$.

### Exercise 266

Given $f(x) = \ln(-x)$.

- Need $-x > 0 \Rightarrow x < 0$, so $\mathrm{Dom}(f) = (-\infty, 0)$. - $\ln(\cdot)$ ranges over all reals, so $\mathrm{Ran}(f) = (-\infty, \infty)$.
- As $x \to 0^-$, $-x \to 0^+$ and $\ln(-x) \to -\infty$, so vertical asymptote $x = 0$.

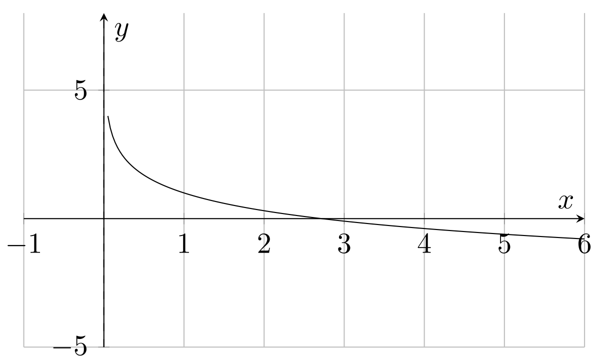



**Final answer:** Domain $(-\infty, 0)$, Range $(-\infty, \infty)$, vertical asymptote $x = 0$.

### Exercise 267

Given $f(x) = 1 - \ln x$.

- Need $x > 0$, so $\text{Dom}(f) = (0, \infty)$. - $\ln x$ has range $(-\infty, \infty)$, multiplying by $-1$ and shifting by 1 still gives all reals, so $\text{Ran}(f) = (-\infty, \infty)$. - As $x \to 0^+$, $\ln x \to -\infty$ so $1 - \ln x \to +\infty$; vertical asymptote still $x = 0$.

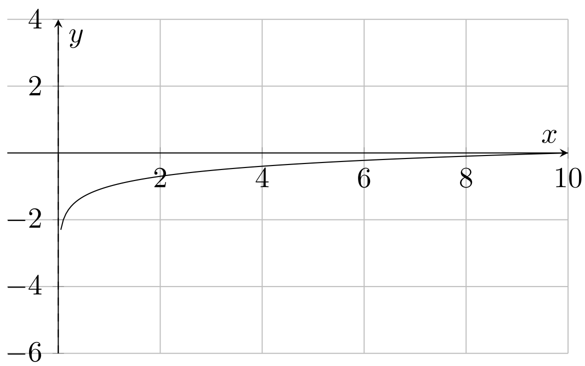



**Final answer:** Domain $(0, \infty)$, Range $(-\infty, \infty)$, vertical asymptote $x = 0$.

### Exercise 268

Given $f(x) = \log x - 1$ (here log is base 10).

- Need $x > 0$, so $\text{Dom}(f) = (0, \infty)$. - $\log_{10} x$ has range $(-\infty, \infty)$, shifting down by 1 keeps the same range, so $\text{Ran}(f) = (-\infty, \infty)$. - As $x \to 0^+$, $\log_{10} x \to -\infty$, so vertical asymptote $x = 0$.

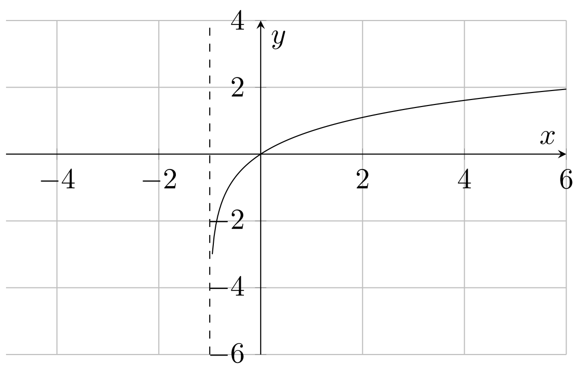



**Final answer:** Domain $(0, \infty)$, Range $(-\infty, \infty)$, vertical asymptote $x = 0$.

### Exercise 269

Given $f(x) = \ln(x+1)$.

- Need $x+1 > 0 \Rightarrow x > -1$, so $\text{Dom}(f) = (-1, \infty)$. - $\ln(\cdot)$ has range $(-\infty, \infty)$, so $\text{Ran}(f) = (-\infty, \infty)$.
- As $x \to -1^+$, $\ln(x+1) \to -\infty$, so vertical asymptote $x = -1$.




**Final answer:** Domain $(-1, \infty)$, Range $(-\infty, \infty)$, vertical asymptote $x = -1$.

### Exercise 270

Rewrite $\log(x^4 y)$ as a sum/difference of logs.

Use product rule $\log(AB) = \log A + \log B$ and power rule $\log\left(A^k\right) = k \log A$:

$$\log\left(x^4 y\right) = \log\left(x^4\right) + \log y = 4 \log x + \log y.$$

**Final answer:** $\log\left(x^4 y\right) = 4 \log x + \log y$.

**Exercise 271**

Rewrite $\log_3\left(\dfrac{9a^3}{b}\right)$.

Use quotient rule $\log\left(\frac{A}{B}\right) = \log A - \log B$, then product/power:

$$\log_3\left(\frac{9a^3}{b}\right) = \log_3(9a^3) - \log_3 b = \log_3 9 + \log_3(a^3) - \log_3 b = \log_3 9 + 3\log_3 a - \log_3 b.$$

(Also, $\log_3 9 = 2$ since $9 = 3^2$, if you want it simplified.)

**Final answer:** $\log_3\left(\dfrac{9a^3}{b}\right) = \log_3 9 + 3\log_3 a - \log_3 b$ (equivalently $2 + 3\log_3 a - \log_3 b$).

**Exercise 272**

Rewrite $\ln(a\sqrt[3]{b})$.

Use product rule and power rule; $\sqrt[3]{b} = b^{1/3}$:

$$\ln(a\sqrt[3]{b}) = \ln\left(ab^{1/3}\right) = \ln a + \ln\left(b^{1/3}\right) = \ln a + \frac{1}{3}\ln b.$$

**Final answer:** $\ln(a\sqrt[3]{b}) = \ln a + \dfrac{1}{3}\ln b$.

**Exercise 273**

Rewrite $\log_5\left(\sqrt{125xy^3}\right)$.

First, $\sqrt{125xy^3} = (125xy^3)^{1/2}$. Then use power + product:

$$\log_5\left(\sqrt{125xy^3}\right) = \log_5\left((125xy^3)^{1/2}\right) = \frac{1}{2}\log_5(125xy^3) = \frac{1}{2}\big(\log_5 125 + \log_5 x + \log_5(y^3)\big) = \frac{1}{2}\big(\log_5 125 + \log_5 x + 3\,\text{lo}$$

Since $125 = 5^3$, $\log_5 125 = 3$.

**Final answer:** $\log_5\left(\sqrt{125xy^3}\right) = \dfrac{1}{2}(3 + \log_5 x + 3\log_5 y)$.

**Exercise 274**

Rewrite $\log_4\left(\dfrac{\sqrt[3]{xy}}{64}\right)$.

Use quotient rule, then power rule:

$$\log_4\left(\frac{\sqrt[3]{xy}}{64}\right) = \log_4((xy)^{1/3}) - \log_4 64 = \frac{1}{3}\log_4(xy) - \log_4 64 = \frac{1}{3}\big(\log_4 x + \log_4 y\big) - \log_4 64.$$

Also $64 = 4^3$, so $\log_4 64 = 3$.

**Final answer:** $\log_4\left(\frac{\sqrt[3]{xy}}{64}\right) = \frac{1}{3}\log_4 x + \frac{1}{3}\log_4 y - 3.$

### Exercise 275

Rewrite $\ln\left(\frac{6}{\sqrt{e^3}}\right)$.

Note $\sqrt{e^3} = (e^3)^{1/2} = e^{3/2}$. Use quotient rule:

$$\ln\left(\frac{6}{\sqrt{e^3}}\right) = \ln 6 - \ln\left(e^{3/2}\right) = \ln 6 - \frac{3}{2}\ln e = \ln 6 - \frac{3}{2}.$$

**Final answer:** $\ln\left(\frac{6}{\sqrt{e^3}}\right) = \ln 6 - \frac{3}{2}.$

### Exercise 276

Solve $5^x = 125$.

Since $125 = 5^3$, we equate exponents:

$$5^x = 5^3 \implies x = 3.$$

**Final answer:** $x = 3$

### Exercise 277

Solve $e^{3x} - 15 = 0$.

$$e^{3x} = 15 \implies 3x = \ln 15 \implies x = \frac{1}{3}\ln 15.$$

**Final answer:** $x = \frac{\ln 15}{3}$

## Exercise 278

Solve $8^x = 4$.

Write both sides as powers of 2:

$$8 = 2^3, \quad 4 = 2^2 \implies 2^{3x} = 2^2.$$

Thus,

$$3x = 2 \implies x = \frac{2}{3}.$$

**Final answer:** $x = \frac{2}{3}$

## Exercise 279

Solve $4^{x+1} - 32 = 0$.

$$4^{x+1} = 32.$$

Write in base 2:

$$4 = 2^2, \quad 32 = 2^5 \implies 2^{2(x+1)} = 2^5.$$

So,

$$2x + 2 = 5 \implies x = \frac{3}{2}.$$

**Final answer:** $x = \frac{3}{2}$

## Exercise 280

Solve $3^{x/14} = \frac{1}{10}$.

Take natural logarithms:

$$\frac{x}{14} \ln 3 = \ln\left(\frac{1}{10}\right) = -\ln 10.$$

Thus,

$$x = -14 \frac{\ln 10}{\ln 3}.$$

**Final answer:** $x = -14 \frac{\ln 10}{\ln 3}$

**Exercise 281**

Solve $10^x = 7.21$.

Take $\log_{10}$ of both sides:

$$x = \log(7.21).$$

**Final answer:** $x = \log(7.21)$

**Exercise 282**

Solve $4 \cdot 2^{3x} - 20 = 0$.

$$4 \cdot 2^{3x} = 20 \implies 2^{3x} = 5.$$

Take logarithms:

$$3x \ln 2 = \ln 5 \implies x = \frac{\ln 5}{3 \ln 2}.$$

**Final answer:** $x = \dfrac{\ln 5}{3 \ln 2}$

**Exercise 283**

Solve $7^{3x-2} = 11$.

Take natural logarithms:

$$(3x - 2) \ln 7 = \ln 11.$$

Solve for $x$:

$$3x = \frac{\ln 11}{\ln 7} + 2 \implies x = \frac{1}{3}\left(\frac{\ln 11}{\ln 7} + 2\right).$$

**Final answer:** $x = \dfrac{1}{3}\left(\dfrac{\ln 11}{\ln 7} + 2\right)$

**Exercise 284**

Solve $\log_3 x = 0$.

Rewrite in exponential form:

$$x = 3^0 = 1.$$

**Final answer:** $x = 1$

**Exercise 285**

Solve $\log_5 x = -2$.

$$x = 5^{-2} = \frac{1}{25}.$$

**Final answer:** $x = \frac{1}{25}$

**Exercise 286**

Solve $\log_4(x+5) = 0$.

$$x + 5 = 4^0 = 1 \implies x = -4.$$

Domain check: $x + 5 > 0$ is satisfied.

**Final answer:** $x = -4$

**Exercise 287**

Solve $\log(2x - 7) = 0$.

$$2x - 7 = 10^0 = 1 \implies x = 4.$$

Domain check: $2x - 7 > 0$ holds.

**Final answer:** $x = 4$

**Exercise 288**

Solve $\ln \sqrt{x+3} = 2$.

$$\sqrt{x+3} = e^2 \implies x + 3 = e^4 \implies x = e^4 - 3.$$

**Final answer:** $x = e^4 - 3$

**Exercise 289**

Solve $\log_6(x+9) + \log_6 x = 2$.

Combine logs:

$$\log_6(x(x+9)) = 2 \implies x(x+9) = 6^2 = 36.$$

$$x^2 + 9x - 36 = 0.$$

Solve:

$$(x+12)(x-3) = 0 \implies x = 3 \text{ or } x = -12.$$

Domain requires $x > 0$, so discard $-12$.

**Final answer:** $x = 3$

### Exercise 290

Solve $\log_4(x+2) - \log_4(x-1) = 0$.

$$\log_4\left(\frac{x+2}{x-1}\right) = 0 \implies \frac{x+2}{x-1} = 1.$$

$$x + 2 = x - 1,$$

which is impossible.

**Final answer:** No solution

### Exercise 291

Solve $\ln x + \ln(x-2) = \ln 4$.

$$\ln[x(x-2)] = \ln 4 \implies x(x-2) = 4.$$

$$x^2 - 2x - 4 = 0.$$

$$x = 1 \pm \sqrt{5}.$$

Domain requires $x > 2$, so keep $1 + \sqrt{5}$.

**Final answer:** $x = 1 + \sqrt{5}$

### Exercise 292

Evaluate $\log_5 47$.

Exact form:

$$\log_5 47 = \frac{\ln 47}{\ln 5}.$$

Approximation:

$$\log_5 47 \approx 2.3879.$$

**Final answer:** $\dfrac{\ln 47}{\ln 5} \approx 2.3879$

### Exercise 293

Evaluate $\log_7 82$.

$$\log_7 82 = \frac{\ln 82}{\ln 7} \approx 2.2572.$$

**Final answer:** $\dfrac{\ln 82}{\ln 7} \approx 2.2572$

### Exercise 294

Evaluate $\log_6 103$.

$$\log_6 103 = \frac{\ln 103}{\ln 6} \approx 2.5850.$$

**Final answer:** $\dfrac{\ln 103}{\ln 6} \approx 2.5850$

### Exercise 295

Evaluate $\log_{0.5} 211$.

$$\log_{0.5} 211 = \frac{\ln 211}{\ln 0.5} \approx -7.7166.$$

**Final answer:** $\dfrac{\ln 211}{\ln 0.5} \approx -7.7166$

### Exercise 296

Evaluate $\log_2 \pi$.

$$\log_2 \pi = \frac{\ln \pi}{\ln 2} \approx 1.6515.$$

**Final answer:** $\dfrac{\ln \pi}{\ln 2} \approx 1.6515$

**Exercise 297**

Evaluate $\log_{0.2} 0.452$.

$$\log_{0.2} 0.452 = \frac{\ln 0.452}{\ln 0.2} \approx 0.4913.$$

**Final answer:** $\dfrac{\ln 0.452}{\ln 0.2} \approx 0.4913$

**Exercise 298**

Rewrite the following in terms of exponentials.

$$\text{a. } 2\cosh(\ln x) = x + \frac{1}{x}$$
$$\text{b. } \cosh 4x + \sinh 4x = e^{4x}$$
$$\text{c. } \cosh 2x - \sinh 2x = e^{-2x}$$
$$\text{d. } \ln(\cosh x + \sinh x) + \ln(\cosh x - \sinh x) = \ln 1 = 0$$

**Final answer:** Simplified exponential forms as shown above

**Exercise 299**

The number of bacteria is modeled by

$$N(t) = 1300 \cdot 2^{t/4}.$$

Evaluate at $t = 15$:

$$N(15) = 1300 \cdot 2^{15/4}.$$

**Final answer:** $N(15) = 1300 \cdot 2^{15/4} \approx 17387$ bacteria

**Exercise 300**

The demand function is

$$D(p) = 150(2.7)^{-0.25p}.$$

Evaluate at $p = 15$ and $p = 20$:

$$D(15) = 150(2.7)^{-3.75}, \qquad D(20) = 150(2.7)^{-5}.$$

**Final answer:** Demand decreases from about 5 million barrels (at \$15) to about 2 million barrels (at \$20)

### Exercise 301

The investment function is

$$A(t) = 100000e^{0.055t}.$$

At $t = 5$:

$$A(5) = 100000e^{0.275}.$$

**Final answer:** $A(5) \approx \$131{,}650$

### Exercise 302

The compound interest formula is

$$A = P\left(1 + \frac{j}{n}\right)^{nt},$$

with $P = 100000$, $j = 0.035$, $t = 5$.

$$\begin{aligned}
\text{Daily:}\ & n = 365,\ A \approx 100000\left(1 + \frac{0.035}{365}\right)^{1825}, \\
\text{Monthly:}\ & n = 12,\ A \approx 100000\left(1 + \frac{0.035}{12}\right)^{60}, \\
\text{Quarterly:}\ & n = 4,\ A \approx 100000\left(1 + \frac{0.035}{4}\right)^{20}, \\
\text{Yearly:}\ & n = 1,\ A \approx 100000(1.035)^{5}.
\end{aligned}$$

**Final answer:** Approximately \$119,100 (daily), \$118,900 (monthly), \$118,700 (quarterly), \$118,600 (yearly)

### Exercise 303

The pH formula is

$$\text{pH} = -\log[H^{+}].$$

$$\begin{aligned}
\text{Eggs: } \text{pH} &= -\log\left(1.6 \times 10^{-8}\right) \approx 7.8 \text{ (base)},\\
\text{Beer: } \text{pH} &= -\log\left(3.16 \times 10^{-3}\right) \approx 2.5 \text{ (acid)},\\
\text{Tomato juice: } \text{pH} &= -\log\left(7.94 \times 10^{-5}\right) \approx 4.1 \text{ (acid)}.
\end{aligned}$$

**Final answer:** Eggs (base), Beer (acid), Tomato juice (acid)

### Exercise 304

The decay model is

$$Q(t) = Q_0 e^{-0.08664t}.$$

For 95% decay, $Q(t) = 0.05Q_0$:

$$0.05 = e^{-0.08664t} \implies t = \frac{\ln(0.05)}{-0.08664}.$$

**Final answer:** $t \approx 35$ days

### Exercise 305

The population model is

$$P(t) = 316e^{0.0074t}.$$

$$\text{(a) } P(7) = 316e^{0.0518} \approx 333 \text{ million.}$$

For doubling:

$$2 = e^{0.0074t} \implies t = \frac{\ln 2}{0.0074}.$$

**Final answer:** (a) $\approx$ 333 million in 2020; (b) about 94 years after 2013

### Exercise 306

The accumulation function is

$$A(t) = 1000(1.04)^t.$$

$$A(5) = 1000(1.04)^5, \qquad A(10) = 1000(1.04)^{10}.$$

For tripling:

$$3 = (1.04)^t \implies t = \frac{\ln 3}{\ln 1.04}.$$

**Final answer:** $A(5) \approx \$1217$, $A(10) \approx \$1480$; tripling in about 28 years

## Exercise 307

The growth model is

$$Q = Q_0 e^{kt}.$$

Doubling in 12 hours:

$$2 = e^{12k} \implies k = \frac{\ln 2}{12}.$$

For $Q = 200{,}000$:

$$200000 = 1000e^{kt}.$$

**Final answer:** $k \approx 0.0578$; time $\approx 92$ hours

## Exercise 308

Doubling every 6 months gives

$$a = 2^{1/6}.$$

$$3500 = 120a^t.$$

**Final answer:** $a \approx 1.1225$; time $\approx 19$ months

## Exercise 309

Richter scale energy ratio:

$$\frac{E_1}{E_2} = 10^{1.5(M_1 - M_2)}.$$

$$10^{1.5(8.3-4.9)} = 10^{5.1}.$$

**Final answer:** About $1.26 \times 10^5$ times more energy

# Chapter 2

*Limits*

## Chapter contents

## A Preview of Calculus

### Exercise 1

Points $P(1,2)$ and $Q(x,y)$ lie on $f(x) = x^2 + 1$.

$$y = f(x) = x^2 + 1$$

The slope of the secant line through $P$ and $Q$ is

$$m_{\text{sec}} = \frac{f(x) - f(1)}{x - 1} = \frac{x^2 + 1 - 2}{x - 1} = \frac{x^2 - 1}{x - 1} = x + 1.$$

| $x$ | $y$ | $Q(x,y)$ | $m_{\text{sec}}$ |
|---|---|---|---|
| 1.1 | 2.21 | $(1.1, 2.21)$ | 2.1 |
| 1.01 | 2.0201 | $(1.01, 2.0201)$ | 2.01 |
| 1.001 | 2.002001 | $(1.001, 2.002001)$ | 2.001 |
| 1.0001 | 2.00020001 | $(1.0001, 2.00020001)$ | 2.0001 |

### Exercise 2

From the table, $m_{\text{sec}} \to 2$ as $x \to 1$.

**Final answer:** The slope of the tangent line at $x = 1$ is 2.

### Exercise 3

The tangent line at $P(1,2)$ has slope 2.

$$y - 2 = 2(x - 1) \quad \Rightarrow \quad y = 2x$$

**Final answer:** $y = 2x$

### Exercise 4

Points $P(1,1)$ and $Q(x,y)$ lie on $f(x) = x^3$.

$$y = x^3, \quad m_{\text{sec}} = \frac{x^3 - 1}{x - 1} = x^2 + x + 1$$

| $x$ | $y$ | $Q(x,y)$ | $m_{\text{sec}}$ |
|---|---|---|---|
| 1.1 | 1.331 | $(1.1, 1.331)$ | 3.31 |
| 1.01 | 1.030301 | $(1.01, 1.030301)$ | 3.0301 |
| 1.001 | 1.003003001 | $(1.001, 1.003003001)$ | 3.003001 |
| 1.0001 | 1.000300030001 | $(1.0001, 1.000300030001)$ | 3.00030001 |

### Exercise 5

From the table, $m_{\text{sec}} \to 3$ as $x \to 1$.

**Final answer:** The slope of the tangent line at $x = 1$ is 3.

### Exercise 6

The tangent line at $P(1, 1)$ with slope 3 is

$$y - 1 = 3(x - 1) \quad \Rightarrow \quad y = 3x - 2$$

**Final answer:** $y = 3x - 2$

### Exercise 7

Points $P(4, 2)$ and $Q(x, y)$ lie on $f(x) = \sqrt{x}$.

$$y = \sqrt{x}, \quad m_{\text{sec}} = \frac{\sqrt{x} - 2}{x - 4}$$

| $x$ | $y$ | $Q(x,y)$ | $m_{\text{sec}}$ |
|---|---|---|---|
| 4.1 | 2.0248457 | $(4.1, 2.0248457)$ | 0.2484566 |
| 4.01 | 2.0024984 | $(4.01, 2.0024984)$ | 0.2498438 |
| 4.001 | 2.00024998 | $(4.001, 2.00024998)$ | 0.2499844 |
| 4.0001 | 2.000025 | $(4.0001, 2.000025)$ | 0.2499984 |

### Exercise 8

From the table, $m_{\text{sec}} \to 0.25$ as $x \to 4$.

**Final answer:** The slope of the tangent line at $x = 4$ is $\frac{1}{4}$.

### Exercise 9

The tangent line at $P(4, 2)$ with slope $\frac{1}{4}$ is

$$y - 2 = \frac{1}{4}(x - 4) \quad \Rightarrow \quad y = \frac{1}{4}x + 1$$

**Final answer:** $y = \frac{1}{4}x + 1$

### Exercise 10

Points $P(1.5, 0)$ and $Q(\phi, y)$ lie on the graph of

$$f(\phi) = \cos(\pi\phi).$$

The slope of the secant line through $P$ and $Q$ is

$$m_{\text{sec}} = \frac{\cos(\pi\phi) - \cos(1.5\pi)}{\phi - 1.5} = \frac{\cos(\pi\phi)}{\phi - 1.5},$$

since $\cos(1.5\pi) = 0$.

| $\phi$ | $y$ | $Q(\phi, y)$ | $m_{\text{sec}}$ |
|---|---|---|---|
| 1.4 | $-0.95105652$ | $(1.4, -0.95105652)$ | 9.5105652 |
| 1.49 | $-0.03141076$ | $(1.49, -0.03141076)$ | 3.1410760 |
| 1.499 | $-0.00314159$ | $(1.499, -0.00314159)$ | 3.1415900 |
| 1.4999 | $-0.00031416$ | $(1.4999, -0.00031416)$ | 3.1416000 |

### Exercise 11

From the table, the secant slopes approach $\pi$ as $\phi \to 1.5$.

**Final answer:** The slope of the tangent line at $\phi = 1.5$ is $\pi$.

### Exercise 12

The tangent line at $P(1.5, 0)$ with slope $\pi$ is

$$y = \pi(\phi - 1.5).$$

**Final answer:** $y = \pi(\phi - 1.5)$

### Exercise 13

Points $P(-1,-1)$ and $Q(x,y)$ lie on $f(x) = \dfrac{1}{x}$.

$$m_{\text{sec}} = \frac{\frac{1}{x} - (-1)}{x - (-1)} = \frac{1 + \frac{1}{x}}{x + 1}.$$

| $x$ | $y$ | $Q(x,y)$ | $m_{\text{sec}}$ |
|---|---|---|---|
| $-1.05$ | $-0.95238095$ | $(-1.05, -0.95238095)$ | $-0.90702948$ |
| $-1.01$ | $-0.99009901$ | $(-1.01, -0.99009901)$ | $-0.98029605$ |
| $-1.005$ | $-0.99502488$ | $(-1.005, -0.99502488)$ | $-0.99007438$ |
| $-1.001$ | $-0.99900100$ | $(-1.001, -0.99900100)$ | $-0.99800300$ |

### Exercise 14

The secant slopes approach $-1$ as $x \to -1$.

**Final answer:** The slope of the tangent line at $x = -1$ is $-1$.

### Exercise 15

The tangent line at $P(-1,-1)$ with slope $-1$ is

$$y + 1 = -1(x + 1) \quad \Rightarrow \quad y = -x - 2.$$

**Final answer:** $y = -x - 2$

### Exercise 16

The position of the ball is

$$s(t) = 200 - 4.9t^2.$$

Average velocity over $[a,b]$ is

$$v_{\text{avg}} = \frac{s(b) - s(a)}{b - a}.$$

$$\text{(a) } [4.99, 5] : \; -48.951$$
$$\text{(b) } [5, 5.01] : \; -49.049$$
$$\text{(c) } [4.999, 5] : \; -48.9951$$
$$\text{(d) } [5, 5.001] : \; -49.0049$$

### Exercise 17

From the average velocities, the instantaneous velocity at $t = 5$ is approximately $-49$ m/s.

**Final answer:** $v(5) \approx -49$ m/s

### Exercise 18

For the stone, $h(t) = 15t - 4.9t^2$.

$$\begin{gathered}
\text{(a) } [1, 1.05] : 5.255 \\
\text{(b) } [1, 1.01] : 5.099 \\
\text{(c) } [1, 1.005] : 5.0495 \\
\text{(d) } [1, 1.001] : 5.0095
\end{gathered}$$

### Exercise 19

The average velocities approach 5.1 as $t \to 1$.

**Final answer:** The instantaneous velocity at $t = 1$ is 5.1 m/s.

### Exercise 20

For the rocket,

$$h(t) = 600 + 78.4t - 4.9t^2.$$

$$\begin{gathered}
\text{(a) } [9, 9.01] : -9.841 \\
\text{(b) } [8.99, 9] : -9.759 \\
\text{(c) } [9, 9.001] : -9.8049 \\
\text{(d) } [8.999, 9] : -9.7951
\end{gathered}$$

### Exercise 21

The average velocities approach $-9.8$.

**Final answer:** The instantaneous velocity at $t = 9$ is $-9.8$ m/s.

## Exercise 22

The position of the runner is given by

$$d(t) = \frac{t^3}{6} + 4t,$$

where $d$ is in meters and $t$ is in seconds.

The average velocity on an interval $[a, b]$ is

$$v_{\text{avg}} = \frac{d(b) - d(a)}{b - a}.$$

$$\begin{aligned}
\text{(a) } [1.95, 2.05]: &\quad v_{\text{avg}} = \frac{d(2.05) - d(1.95)}{0.10} = \frac{(2.05^3/6 + 8.2) - (1.95^3/6 + 7.8)}{0.10} \approx 6.6667 \\
\text{(b) } [1.995, 2.005]: &\quad v_{\text{avg}} \approx 6.6667 \\
\text{(c) } [1.9995, 2.0005]: &\quad v_{\text{avg}} \approx 6.6667 \\
\text{(d) } [2, 2.00001]: &\quad v_{\text{avg}} \approx 6.6667
\end{aligned}$$

**Final answer:** The average velocity over all intervals is approximately 6.6667 m/s.

## Exercise 23

As the time intervals shrink toward $t = 2$, the average velocity values stabilize.

**Final answer:** The instantaneous velocity at $t = 2$ is approximately

$$v(2) = 6.6667 \text{ m/s}.$$

## Exercise 24

Consider the function

$$f(x) = |x|.$$

The graph is a V-shape with vertex at $(0, 0)$, symmetric about the $y$-axis. Over $[-1, 2]$, the graph lies entirely above the $x$-axis.

**Final answer:** The graph of $f(x) = |x|$ is a V-shaped graph; the shaded region above the $x$-axis is the entire graph on $[-1, 2]$.

### Exercise 25

Using rectangles of width 1 on $[-1, 2]$, approximate the area:

Upper estimate:

$$A_{\text{upper}} = 1(1 + 2 + 2) = 5.$$

Lower estimate:

$$A_{\text{lower}} = 1(0 + 1 + 1) = 3.$$

Using geometry, the exact area consists of two right triangles:

$$A = \frac{1}{2}(1)(1) + \frac{1}{2}(2)(2) = \frac{1}{2} + 2 = \frac{5}{2}.$$

**Final answer:** The exact area is $\frac{5}{2}$.

### Exercise 26

Consider

$$f(x) = \sqrt{1 - x^2},$$

the upper half of a circle of radius 1 centered at $(0, 0)$.

Over $[-1, 1]$, the graph is a semicircle above the $x$-axis.

**Final answer:** The graph is the upper semicircle of radius 1 centered at the origin.

### Exercise 27

Using rectangles of width 0.4 on $[-1, 1]$, an approximate area can be computed numerically.

Using geometry, the exact area is the area of a semicircle:

$$A = \frac{1}{2}\pi(1)^2 = \frac{\pi}{2}.$$

**Final answer:** The exact area is $\frac{\pi}{2}$.

### Exercise 28

Consider

$$f(x) = -x^2 + 1.$$

This is a downward-opening parabola with vertex at $(0, 1)$ and $x$-intercepts at $\pm 1$.

**Final answer:** The graph is a downward parabola on $[-1, 1]$ touching the $x$-axis at $x = \pm 1$.

## Exercise 29

The area between the curve and the $x$-axis on $[-1, 1]$ is

$$A = \int_{-1}^{1} (-x^2 + 1)\, dx.$$

$$A = \left[ x - \frac{x^3}{3} \right]_{-1}^{1} = \left(1 - \frac{1}{3}\right) - \left(-1 + \frac{1}{3}\right) = \frac{4}{3}.$$

**Final answer:** The area is $\frac{4}{3}$.

### The Limit of a Function

## Exercise 30

We are given

$$f(x) = \frac{x^2 - 1}{|x - 1|}.$$

Factor the numerator:

$$x^2 - 1 = (x - 1)(x + 1).$$

Hence

$$f(x) = \frac{(x - 1)(x + 1)}{|x - 1|} = (x + 1)\frac{x - 1}{|x - 1|}.$$

For $x < 1$, $|x - 1| = -(x - 1)$, so $f(x) = -(x + 1)$. For $x > 1$, $|x - 1| = x - 1$, so $f(x) = x + 1$.

Evaluating near $x = 1$:

| $x$ | $f(x)$ |
|---|---|
| 0.9 | $-1.9$ |
| 0.99 | $-1.99$ |
| 0.999 | $-1.999$ |
| 0.9999 | $-1.9999$ |
| 1.1 | 2.1 |
| 1.01 | 2.01 |
| 1.001 | 2.001 |
| 1.0001 | 2.0001 |

**Final answer:** Left-hand values approach $-2$, right-hand values approach 2.

### Exercise 31

From Exercise 30, the left-hand limit at $x = 1$ is $-2$ and the right-hand limit is 2. Since these are not equal, the two-sided limit does not exist.

**Final answer:** $\lim_{x\to 1} f(x)$ does not exist.

### Exercise 32

Consider

$$f(x) = (1 + x)^{1/x}.$$

Evaluating numerically:

| $x$ | $f(x)$ |
|---|---|
| $-0.01$ | 2.69159 |
| $-0.001$ | 2.71692 |
| $-0.0001$ | 2.71814 |
| $-0.00001$ | 2.71827 |
| 0.01 | 2.70481 |
| 0.001 | 2.71692 |
| 0.0001 | 2.71814 |
| 0.00001 | 2.71827 |

**Final answer:** Values approach approximately 2.71828.

### Exercise 33

The table shows that $f(x)$ approaches a single finite number as $x \to 0$.

**Final answer:** The function approaches a constant near $x = 0$.

### Exercise 34

The limiting value observed in Exercise 32 is the mathematical constant $e$.

**Final answer:** $e$.

**Exercise 35**

We examine

$$\frac{\sin(2x)}{x}.$$

Using $\lim_{u\to 0}\frac{\sin u}{u}=1$ with $u=2x$,

$$\frac{\sin(2x)}{x}=2\frac{\sin(2x)}{2x}\to 2.$$

**Final answer:** 2.

**Exercise 36**

Similarly,

$$\frac{\sin(3x)}{x}=3\frac{\sin(3x)}{3x}\to 3.$$

**Final answer:** 3.

**Exercise 37**

From Exercises 35 and 36, we generalize:

$$\lim_{x\to 0}\frac{\sin(ax)}{x}=a.$$

**Final answer:** $a$.

**Exercise 38**

$$\lim_{x\to 2}\frac{x^2-4}{x^2+x-6}.$$

Factor:

$$\frac{(x-2)(x+2)}{(x-2)(x+3)}=\frac{x+2}{x+3}.$$

Evaluating at $x=2$:

$$\frac{4}{5}.$$

**Final answer:** $\frac{4}{5}$.

**Exercise 39**

$$\lim_{x\to 1}(1-2x) = 1-2(1) = -1.$$

**Final answer:** $-1$.

**Exercise 40**

$$\lim_{x\to 0} \frac{5}{1-e^{1/x}}.$$

As $x \to 0^+$, $1/x \to +\infty$, so $e^{1/x} \to \infty$, and the denominator $\to -\infty$. Hence the fraction $\to 0$.

**Final answer:** $0$.

**Exercise 41**

$$\lim_{z\to 0} \frac{z-1}{z^2(z+3)}.$$

The numerator approaches $-1$, while the denominator approaches $0^+$. Thus the quotient diverges to negative infinity.

**Final answer:** $-\infty$.

**Exercise 42**

$$\lim_{t\to 0^+} \frac{\cos t}{t}.$$

Since $\cos t \to 1$ and $t \to 0^+$, the quotient grows without bound.

**Final answer:** $+\infty$.

**Exercise 43**

$$\lim_{x\to 2} \frac{1-\frac{2}{x}}{x^2-4}.$$

Rewrite numerator:

$$1-\frac{2}{x} = \frac{x-2}{x}.$$

Then

$$\frac{(x-2)/x}{(x-2)(x+2)} = \frac{1}{x(x+2)}.$$

Evaluating at $x = 2$:

$$\frac{1}{2 \cdot 4} = \frac{1}{8}.$$

**Final answer:** $\frac{1}{8}$.

### Exercise 44

We evaluate the limit

$$\lim_{\theta \to 0} \sin\left(\frac{\pi}{\theta}\right).$$

As $\theta \to 0$, the quantity $\frac{\pi}{\theta}$ becomes unbounded and oscillates between positive and negative infinity. Since $\sin x$ oscillates between $-1$ and 1 for all real $x$, the expression $\sin(\frac{\pi}{\theta})$ oscillates infinitely often without approaching a single value.

A table of values near $\theta = 0$ confirms this oscillatory behavior: the outputs jump between values close to $-1$ and 1.

**Final answer:** The limit does not exist.

### Exercise 45

We evaluate

$$\lim_{\alpha \to 0^+} \frac{1}{\alpha} \cos\left(\frac{\pi}{\alpha}\right).$$

As $\alpha \to 0^+$, the cosine term oscillates between $-1$ and 1, while $\frac{1}{\alpha} \to +\infty$. Thus the product oscillates between large positive and negative values with unbounded magnitude.

Because the expression does not approach a finite value or $\pm\infty$ in a consistent way, the limit does not exist.

**Final answer:** The limit does not exist.

### Exercise 46

From the graph of $f(x)$, as $x \to 10$ the curve approaches the $x$-axis. Both the left-hand and right-hand limits approach 0.

**Final answer:** $\lim_{x \to 10} f(x) = 0$.

## Exercise 47

From the graph, as $x \to -2^+$ the values of $f(x)$ approach 3. The right-hand limit exists and equals 3.

**Final answer:** $\lim_{x \to -2^+} f(x) = 3$.

## Exercise 48

As $x \to -8$, the graph approaches the same value from both sides, and the point $f(-8)$ is defined at that value. Therefore the limit equals the function value.

**Final answer:** $\lim_{x \to -8} f(x) = f(-8)$.

## Exercise 49

From the graph, as $x \to 6$, the function approaches 5 from both sides.

**Final answer:** $\lim_{x \to 6} f(x) = 5$.

## Exercise 50

From the graph, as $x \to 1^-$ the function approaches 1.

**Final answer:** $\lim_{x \to 1^-} f(x) = 1$.

## Exercise 51

As $x \to 1^+$, the graph approaches 2.

**Final answer:** $\lim_{x \to 1^+} f(x) = 2$.

## Exercise 52

Because the left-hand and right-hand limits at $x = 1$ are different, the two-sided limit does not exist.

**Final answer:** The limit does not exist.

## Exercise 53

From the graph, as $x \to 2$, both sides approach the same value 0.

**Final answer:** $\lim_{x \to 2} f(x) = 0$.

## Exercise 54

The filled dot at $x = 1$ shows that $f(1) = 1$.

**Final answer:** $f(1) = 1$.

## Exercise 55

As $x \to 0^-$, the graph approaches 1.

**Final answer:** $\lim_{x \to 0^-} f(x) = 1$.

## Exercise 56

As $x \to 0^+$, the graph approaches $-4$.

**Final answer:** $\lim_{x \to 0^+} f(x) = -4$.

## Exercise 57

Since the left-hand and right-hand limits at $x = 0$ are different, the two-sided limit does not exist.

**Final answer:** The limit does not exist.

## Exercise 58

From the graph, as $x \to 2$, the function approaches 0.

**Final answer:** $\lim_{x \to 2} f(x) = 0$.

## Exercise 59

As $x \to -2^-$, the graph approaches $-1$.

**Final answer:** $\lim_{x \to -2^-} f(x) = -1$.

## Exercise 60

As $x \to -2^+$, the graph approaches 2.

**Final answer:** $\lim_{x \to -2^+} f(x) = 2$.

## Exercise 61

Because the one-sided limits at $x = -2$ are not equal, the limit does not exist.

**Final answer:** The limit does not exist.

## Exercise 62

As $x \to 2^-$, the graph approaches $-1$.

**Final answer:** $\lim_{x \to 2^-} f(x) = -1$.

## Exercise 63

As $x \to 2^+$, the graph approaches 3.

**Final answer:** $\lim_{x \to 2^+} f(x) = 3$.

## Exercise 64

Since the left-hand and right-hand limits at $x = 2$ differ, the limit does not exist.

**Final answer:** The limit does not exist.

### Exercise 65

From the graph of $g(x)$, as $x \to 0^-$ the function approaches 3.

**Final answer:** $\lim_{x \to 0^-} g(x) = 3$.

### Exercise 66

As $x \to 0^+$, the graph approaches $-1$.

**Final answer:** $\lim_{x \to 0^+} g(x) = -1$.

### Exercise 67

Because the one-sided limits at $x = 0$ are different, the two-sided limit does not exist.

**Final answer:** The limit does not exist.

### Exercise 44

We evaluate $\lim_{\theta \to 0} \sin\left(\frac{\pi}{\theta}\right)$.

As $\theta \to 0$, the quantity $\frac{\pi}{\theta}$ grows without bound and oscillates rapidly between positive and negative values. Since $\sin x$ oscillates between $-1$ and 1 and does not approach a single value as $x \to \pm\infty$, the function $\sin(\pi/\theta)$ does not settle near any number.

Tables fail here because finite numerical samples cannot capture infinite oscillation.

**Final answer:** The limit does not exist.

### Exercise 45

We evaluate $\lim_{\alpha \to 0^+} \frac{1}{\alpha} \cos\left(\frac{\pi}{\alpha}\right)$.

As $\alpha \to 0^+$, $\cos(\pi/\alpha)$ oscillates between $-1$ and 1, while $\frac{1}{\alpha} \to +\infty$. Thus the expression oscillates between arbitrarily large positive and negative values.

**Final answer:** The limit does not exist.

## Exercise 46

From the graph of $f(x)$, as $x \to 10$, the curve approaches the $x$-axis.

**Final answer:** $\lim_{x \to 10} f(x) = 0$.

## Exercise 47

From the graph, as $x \to -2^+$, the values of $f(x)$ approach 3.

**Final answer:** $\lim_{x \to -2^+} f(x) = 3$.

## Exercise 48

The graph shows that as $x \to -8$, the function approaches the value of the filled point at $x = -8$.

**Final answer:** $\lim_{x \to -8} f(x) = f(-8)$.

## Exercise 49

From the graph near $x = 6$, the curve approaches the horizontal level $y = 5$ from both sides.

**Final answer:** $\lim_{x \to 6} f(x) = 5$.

## Exercise 50

As $x \to 1^-$, the graph approaches 0 from the left branch.

**Final answer:** $\lim_{x \to 1^-} f(x) = 0$.

## Exercise 51

As $x \to 1^+$, the graph approaches 2 from the right branch.

**Final answer:** $\lim_{x \to 1^+} f(x) = 2$.

## Exercise 52

Since the one-sided limits at $x = 1$ are different, the two-sided limit does not exist.

**Final answer:** $\lim_{x \to 1} f(x)$ does not exist.

## Exercise 53

As $x \to 2$, both sides of the graph approach the same $y$-value.

**Final answer:** $\lim_{x \to 2} f(x) = 0$ (estimated).

## Exercise 54

The filled point at $x = 1$ is located at $y = 1$.

**Final answer:** $f(1) = 1$.

## Exercise 55

From the graph, as $x \to 0^-$, the function approaches 1.

**Final answer:** $\lim_{x \to 0^-} f(x) = 1$.

## Exercise 56

As $x \to 0^+$, the graph approaches $-4$.

**Final answer:** $\lim_{x \to 0^+} f(x) = -4$.

## Exercise 57

Since the one-sided limits are different, the limit at $x = 0$ does not exist.

**Final answer:** $\lim_{x \to 0} f(x)$ does not exist.

## Exercise 58

As $x \to 2^-$, the graph approaches $-1$.

**Final answer:** $\lim_{x \to 2^-} f(x) = -1$.

## Exercise 59

From the graph, as $x \to -2^-$, $f(x)$ approaches 3.

**Final answer:** $\lim_{x \to -2^-} f(x) = 3$.

## Exercise 60

As $x \to -2^+$, the graph approaches $-1$.

**Final answer:** $\lim_{x \to -2^+} f(x) = -1$.

## Exercise 61

Since left- and right-hand limits differ, the limit does not exist.

**Final answer:** $\lim_{x \to -2} f(x)$ does not exist.

## Exercise 62

As $x \to 2$, the curve approaches 2.

**Final answer:** $\lim_{x \to 2} f(x) = 2$.

## Exercise 63

As $x \to 2^+$, the graph continues upward past 2.

**Final answer:** $\lim_{x \to 2^+} f(x) = 2$.

## Exercise 64

Both one-sided limits at $x = 2$ are equal.

**Final answer:** $\lim_{x \to 2} f(x) = 2$.

## Exercise 65

From the graph of $g(x)$, as $x \to 0^-$ the curve approaches 3.

**Final answer:** $\lim_{x \to 0^-} g(x) = 3$.

## Exercise 66

As $x \to 0^+$, the graph approaches 0.

**Final answer:** $\lim_{x \to 0^+} g(x) = 0$.

### Exercise 67

Since the one-sided limits differ, the limit does not exist.

**Final answer:** $\lim_{x \to 0} g(x)$ does not exist.

### Exercise 68

From the graph of $h(x)$, as $x \to 0^-$ the curve approaches 0.

**Final answer:** $\lim_{x \to 0^-} h(x) = 0$.

### Exercise 69

As $x \to 0^+$, the graph also approaches 0.

**Final answer:** $\lim_{x \to 0^+} h(x) = 0$.

### Exercise 70

Since both one-sided limits agree,

**Final answer:** $\lim_{x \to 0} h(x) = 0$.

### Exercise 71

From the graph of $f(x)$, as $x \to 0^-$, the values approach $-2$.

**Final answer:** $\lim_{x \to 0^-} f(x) = -2$.

### Exercise 72

As $x \to 0^+$, the graph approaches 0.

**Final answer:** $\lim_{x \to 0^+} f(x) = 0$.

### Exercise 73

Because the one-sided limits are unequal, the limit does not exist.

**Final answer:** $\lim_{x \to 0} f(x)$ does not exist.

### Exercise 74

From the smooth curve, as $x \to 1$, the function approaches $-1$.

**Final answer:** $\lim_{x \to 1} f(x) = -1$.

### Exercise 75

As $x \to 2$, the curve rises sharply.

**Final answer:** $\lim_{x \to 2} f(x) = 1$ (estimated).

### Exercises 76–80

Sketches are constructed to satisfy the given one-sided limits, infinite limits, and function values. Each graph must:

- match the specified behavior near key $x$-values,
- include open or closed points as required,
- approach $\pm\infty$ where specified.

**Final answer:** Multiple valid sketches are acceptable provided all stated limit conditions are satisfied.

### Exercise 81

A shock wave creates a sudden jump (discontinuity) in the density function $\rho(x)$ at the shock front location $x_{\text{SF}}$. From the graph, the density takes a higher constant value $\rho_1$ before the shock front and a lower constant value $\rho_2$ after the shock front, with $\rho_1 > \rho_2$.

#### (a) Right-hand limit

Approaching the shock front from the right means $x > x_{\text{SF}}$. From the graph, the density on the right side of the shock is constant and equal to $\rho_2$. Therefore,

$$\lim_{x \to x_{\text{SF}}^+} \rho(x) = \rho_2.$$

**Final answer:** $\rho_2$

### (b) Left-hand limit

Approaching the shock front from the left means $x < x_{\text{SF}}$. From the graph, the density on the left side of the shock is constant and equal to $\rho_1$. Therefore,

$$\lim_{x \to x_{\text{SF}}^-} \rho(x) = \rho_1.$$

**Final answer:** $\rho_1$

### (c) Two-sided limit and physical interpretation

A two-sided limit exists only if the left-hand and right-hand limits are equal. Here,

$$\lim_{x \to x_{\text{SF}}^-} \rho(x) = \rho_1 \quad \text{and} \quad \lim_{x \to x_{\text{SF}}^+} \rho(x) = \rho_2,$$

with $\rho_1 \neq \rho_2$. Hence, the two-sided limit does not exist:

$$\lim_{x \to x_{\text{SF}}} \rho(x) \text{ does not exist.}$$

**Physical meaning:** This discontinuity represents the shock front itself. Physically, a shock wave causes an abrupt change in density over an extremely small distance, so the density does not transition smoothly. The absence of a two-sided limit reflects the real physical jump in material properties across the shock.

**Final answer:** The limit does not exist.

## Exercise 82

We are given discrete measurements of a runner's position $x(t)$ (in meters) at times $t$ (in seconds), and we are asked to evaluate

$$\lim_{t \to 2} x(t).$$

**Step 1: Examine values near $t = 2$**

From the table:

| $t$ | $x(t)$ |
|---|---|
| 1.75 | 4.5 |
| 1.95 | 6.1 |
| 1.99 | 6.42 |
| 2.01 | 6.58 |
| 2.05 | 6.9 |
| 2.25 | 8.5 |

As $t$ approaches 2 from the left $(1.95, 1.99)$, the position values approach approximately 6.4 meters. As $t$ approaches 2 from the right $(2.01, 2.05)$, the position values also cluster near 6.6 meters.

Both sides suggest the position is approaching a value close to 6.5 meters.

**Step 2: Estimate the limit**

Since the left-hand and right-hand values approach the same number, we conclude that the limit exists and is approximately

$$\lim_{t\to 2} x(t) \approx 6.5.$$

**Final answer:** $\lim_{t\to 2} x(t) \approx 6.5$ m

**Physical interpretation**

Physically, this limit represents the runner's position at time $t = 2$ seconds. Even though the table does not list $x(2)$ explicitly, the data indicate that the runner is smoothly moving and is approximately 6.5 meters from the starting point at $t = 2$ seconds.

**The Limit Laws**

### Exercise 83

$$\lim_{x\to 0}(4x^2 - 2x + 3)$$

Polynomials are continuous, so we apply direct substitution.

$$4(0)^2 - 2(0) + 3 = 3$$

Final answer: 3

**Exercise 84**

$$\lim_{x\to 1} \frac{x^3 + 3x^2 + 5}{4 - 7x}$$

Both numerator and denominator are polynomials; substitute directly.

$$\frac{1 + 3 + 5}{4 - 7} = \frac{9}{-3} = -3$$

Final answer: $-3$

**Exercise 85**

$$\lim_{x\to -2} \sqrt{x^2 - 6x + 3}$$

The square root is continuous where defined, so substitute.

$$\sqrt{(-2)^2 - 6(-2) + 3} = \sqrt{4 + 12 + 3} = \sqrt{19}$$

Final answer: $\sqrt{19}$

**Exercise 86**

$$\lim_{x\to -1} (9x + 1)^2$$

Polynomial composition is continuous.

$$(9(-1) + 1)^2 = (-8)^2 = 64$$

Final answer: 64

**Exercise 87**

$$\lim_{x\to 7} x^2$$

Direct substitution:

$$7^2 = 49$$

Final answer: 49

**Exercise 88**

$$\lim_{x \to -2} (4x^2 - 1)$$

Direct substitution:

$$4(4) - 1 = 15$$

Final answer: 15

**Exercise 89**

$$\lim_{x \to 0} \frac{1}{1 + \sin x}$$

Since $\sin x$ is continuous and $\sin 0 = 0$,

$$\frac{1}{1+0} = 1$$

Final answer: 1

**Exercise 90**

$$\lim_{x \to 2} e^{2x - x^2}$$

The exponential function is continuous.

$$e^{4-4} = e^0 = 1$$

Final answer: 1

**Exercise 91**

$$\lim_{x \to 1} \frac{2 - 7x}{x + 6}$$

Direct substitution:

$$\frac{2-7}{1+6} = -\frac{5}{7}$$

Final answer: $-\frac{5}{7}$

**Exercise 92**

$$\lim_{x \to 3} \ln\left(e^{3x}\right)$$

Use continuity of ln and exponentials.

$$\ln\left(e^9\right) = 9$$

Final answer: 9

### Exercise 93

$$\lim_{x\to 4} \frac{x^2 - 16}{x - 4}$$

Direct substitution gives 0/0. Factor:

$$\frac{(x-4)(x+4)}{x-4} = x + 4$$

Now substitute:

$$4 + 4 = 8$$

Final answer: 8

### Exercise 94

$$\lim_{x\to 2} \frac{x - 2}{x^2 - 2x}$$

Factor the denominator:

$$\frac{x-2}{x(x-2)} = \frac{1}{x}$$

Now substitute:

$$\frac{1}{2}$$

Final answer: $\frac{1}{2}$

### Exercise 95

$$\lim_{x\to 6} \frac{3x - 18}{2x - 12}$$

Factor:

$$\frac{3(x-6)}{2(x-6)} = \frac{3}{2}$$

Final answer: $\frac{3}{2}$

**Exercise 96**

$$\lim_{h \to 0} \frac{(1+h)^2 - 1}{h}$$

Expand:

$$\frac{1 + 2h + h^2 - 1}{h} = \frac{2h + h^2}{h} = 2 + h$$

Now take the limit:

$$2$$

Final answer: 2

**Exercise 97**

$$\lim_{t \to 9} \frac{t - 9}{\sqrt{t} - 3}$$

Multiply by the conjugate:

$$\frac{(t-9)(\sqrt{t}+3)}{t-9} = \sqrt{t} + 3$$

Substitute:

$$3 + 3 = 6$$

Final answer: 6

**Exercise 98**

$$\lim_{h \to 0} \frac{\frac{1}{a+h} - \frac{1}{a}}{h}$$

Combine fractions:

$$\frac{-h}{a(a+h)h} = -\frac{1}{a(a+h)}$$

Take the limit:

$$-\frac{1}{a^2}$$

Final answer: $-\dfrac{1}{a^2}$

**Exercise 99**

$$\lim_{\theta \to \pi} \frac{\sin\theta}{\tan\theta}$$

Rewrite $\tan\theta = \frac{\sin\theta}{\cos\theta}$:

$$\frac{\sin\theta}{\sin\theta/\cos\theta} = \cos\theta$$

Now substitute:

$$\cos\pi = -1$$

Final answer: $-1$

## Exercise 100

$$\lim_{x\to 1}\frac{x^3-1}{x^2-1}$$

Factor:

$$\frac{(x-1)(x^2+x+1)}{(x-1)(x+1)} = \frac{x^2+x+1}{x+1}$$

Substitute:

$$\frac{3}{2}$$

Final answer: $\dfrac{3}{2}$

## Exercise 101

$$\lim_{x\to \frac{1}{2}}\frac{2x^2+3x-2}{2x-1}$$

Factor numerator:

$$\frac{(2x-1)(x+2)}{2x-1} = x+2$$

Substitute:

$$\frac{1}{2}+2 = \frac{5}{2}$$

Final answer: $\dfrac{5}{2}$

## Exercise 102

$$\lim_{x\to -3}\frac{\sqrt{x+4}-1}{x+3}$$

Multiply by the conjugate:

$$\frac{x+4-1}{(x+3)(\sqrt{x+4}+1)} = \frac{x+3}{(x+3)(\sqrt{x+4}+1)}$$

Simplify and substitute:

$$\frac{1}{1+1} = \frac{1}{2}$$

Final answer: $\frac{1}{2}$

**Exercise 103**

$$\lim_{x \to -2^-} \frac{2x^2 + 7x - 4}{x^2 + x - 2}$$

Factor:

$$\frac{(2x-1)(x+4)}{(x+2)(x-1)}$$

As $x \to -2^-$, denominator is negative and numerator finite, so limit is $+\infty$.

Final answer: $+\infty$

**Exercise 104**

$$\lim_{x \to -2^+} \frac{2x^2 + 7x - 4}{x^2 + x - 2}$$

Same factorization; denominator is positive near $-2^+$.

Final answer: $-\infty$

**Exercise 105**

$$\lim_{x \to 1^-} \frac{2x^2 + 7x - 4}{x^2 + x - 2}$$

Denominator $\to 0^-$, numerator $\to 5$.

Final answer: $-\infty$

**Exercise 106**

$$\lim_{x \to 1^+} \frac{2x^2 + 7x - 4}{x^2 + x - 2}$$

Final answer: $+\infty$

## Exercise 107

Given $\lim_{x\to 6} f(x) = 4$, $\lim_{x\to 6} g(x) = 9$:

$$\lim_{x\to 6} 2f(x)g(x) = 2(4)(9) = 72$$

Final answer: 72

## Exercise 108

$$\lim_{x\to 6} \frac{g(x)-1}{f(x)} = \frac{9-1}{4} = 2$$

Final answer: 2

## Exercise 109

$$\lim_{x\to 6} \left( f(x) + \frac{1}{3} g(x) \right) = 4 + 3 = 7$$

Final answer: 7

## Exercise 110

$$\lim_{x\to 6} \frac{(h(x))^3}{2} = \frac{6^3}{2} = 108$$

Final answer: 108

## Exercise 111

$$\lim_{x\to 6} \sqrt{g(x) - f(x)} = \sqrt{9-4} = \sqrt{5}$$

Final answer: $\sqrt{5}$

## Exercise 112

$$\lim_{x\to 6} x \cdot h(x) = 6 \cdot 6 = 36$$

Final answer: 36

## Exercise 113

$$\lim_{x\to 6}(x+1)f(x)$$

Using the product law of limits,

$$\lim_{x\to 6}(x+1)f(x) = \left(\lim_{x\to 6}(x+1)\right)\left(\lim_{x\to 6} f(x)\right).$$

Since $\lim_{x\to 6}(x+1) = 7$ and $\lim_{x\to 6} f(x) = 4$,

$$7 \cdot 4 = 28.$$

**Final answer:** 28

## Exercise 114

$$\lim_{x\to 6}(f(x)g(x) - h(x))$$

Apply limit laws:

$$= \left(\lim_{x\to 6} f(x)\right)\left(\lim_{x\to 6} g(x)\right) - \lim_{x\to 6} h(x).$$

Substitute the given limits:

$$= (4)(9) - 6 = 36 - 6 = 30.$$

**Final answer:** 30

## Exercise 115

$$f(x) = \begin{cases} x^2, & x \le 3 \\ x+4, & x > 3 \end{cases}$$

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} x^2 = 9, \quad \lim_{x\to 3^+} f(x) = \lim_{x\to 3^+}(x+4) = 7.$$

Since the one-sided limits are not equal, the limit does not exist.

**Final answer:** DNE

## Exercise 116

$$g(x) = \begin{cases} x^3 - 1, & x \le 0 \\ 1, & x > 0 \end{cases}$$

$$\lim_{x\to 0^-} g(x) = (-1), \quad \lim_{x\to 0^+} g(x) = 1.$$

Since the one-sided limits differ, the limit does not exist.

**Final answer:** DNE

## Exercise 117

$$h(x) = \begin{cases} x^2 - 2x + 1, & x < 2 \\ 3 - x, & x \geq 2 \end{cases}$$

$$\lim_{x\to 2^-} h(x) = (2)^2 - 2(2) + 1 = 1, \quad \lim_{x\to 2^+} h(x) = 3 - 2 = 1.$$

Since both one-sided limits are equal,

$$\lim_{x\to 2} h(x) = 1.$$

**Final answer:** 1

## Exercise 118

$$\lim_{x\to -3^+} (f(x) + g(x))$$

From the graph, as $x \to -3^+$, $f(x) \to 2$ and $g(x) \to -1$. Thus,

$$2 + (-1) = 1.$$

**Final answer:** 1

## Exercise 119

$$\lim_{x\to -3^-} (f(x) - 3g(x))$$

From the graph, $f(x) \to 2$ and $g(x) \to -1$.

$$2 - 3(-1) = 2 + 3 = 5.$$

**Final answer:** 5

**Exercise 120**

$$\lim_{x\to 0} \frac{f(x)g(x)}{3}$$

From the graph, $f(x) \to 1$ and $g(x) \to -2$.

$$\frac{(1)(-2)}{3} = -\frac{2}{3}.$$

**Final answer:** $-\frac{2}{3}$

**Exercise 121**

$$\lim_{x\to -5} \frac{2+g(x)}{f(x)}$$

From the graph, $g(x) \to -1$ and $f(x) \to 2$.

$$\frac{2+(-1)}{2} = \frac{1}{2}.$$

**Final answer:** $\frac{1}{2}$

**Exercise 122**

$$\lim_{x\to 1} (f(x))^2$$

From the graph, $f(x) \to 2$.

$$(2)^2 = 4.$$

**Final answer:** 4

**Exercise 123**

$$\lim_{x\to 1} \sqrt[3]{f(x) - g(x)}$$

From the graph, $f(x) \to 2$ and $g(x) \to -1$.

$$\sqrt[3]{2-(-1)} = \sqrt[3]{3}.$$

**Final answer:** $\sqrt[3]{3}$

**Exercise 124**

$$\lim_{x\to -7} x\,g(x)$$

From the graph, $g(x) \to 2$.

$$(-7)(2) = -14.$$

**Final answer:** $-14$

**Exercise 125**

$$\lim_{x\to -9}(xf(x) + 2g(x))$$

From the graph, $f(x) \to 1$ and $g(x) \to 2$.

$$(-9)(1) + 2(2) = -9 + 4 = -5.$$

**Final answer:** $-5$

**Exercise 126**

Given

$$2x - 1 \le g(x) \le x^2 - 2x + 3,$$

evaluate

$$\lim_{x\to 2} g(x).$$

Evaluate bounds:

$$\lim_{x\to 2}(2x - 1) = 3, \quad \lim_{x\to 2}(x^2 - 2x + 3) = 3.$$

By the Squeeze Theorem,

$$\lim_{x\to 2} g(x) = 3.$$

**Final answer:** 3

**Exercise 127**

$$\lim_{\theta\to 0} \theta^2 \cos\left(\frac{1}{\theta}\right)$$

Since $-1 \le \cos(1/\theta) \le 1$,

$$-\theta^2 \le \theta^2 \cos(1/\theta) \le \theta^2.$$

Both bounds approach 0, so by the Squeeze Theorem the limit is 0.

**Final answer:** 0

### Exercise 128

$$f(x) = \begin{cases} 0, & x \text{ rational} \\ x^2, & x \text{ irrational} \end{cases}$$

As $x \to 0$, both 0 and $x^2$ approach 0. By the Squeeze Theorem,

$$\lim_{x \to 0} f(x) = 0.$$

**Final answer:** 0

### Exercise 129

$$E(r) = \frac{q}{4\pi\varepsilon_0 r^2}$$

As $r \to 0^+$, the denominator approaches 0, so $E(r) \to +\infty$. This represents the electric field becoming infinitely strong near a point charge.

**Final answer:** $+\infty$

### Exercise 130

$$V(\rho) = \frac{m}{\rho}, \quad m = 8$$

As $\rho \to 0^+$,

$$V(\rho) = \frac{8}{\rho} \to +\infty.$$

Physically, volume increases without bound as density approaches zero.

**Final answer:** $+\infty$

### Continuity

### Exercise 131

$f(x) = \dfrac{1}{\sqrt{x}}.$

The function is defined only for $x > 0$. As $x \to 0^+$, $\sqrt{x} \to 0$ and $f(x) \to \infty$. Thus there is an infinite discontinuity at $x = 0$.

**Final answer:** Discontinuous at $x = 0$ (infinite discontinuity).

### Exercise 132

$f(x) = \dfrac{2}{x^2 + 1}$.

The denominator $x^2 + 1 > 0$ for all real $x$. Hence $f$ is defined and continuous everywhere.

**Final answer:** No discontinuities; continuous for all $x \in \mathbb{R}$.

### Exercise 133

$f(x) = \dfrac{x}{x^2 - x} = \dfrac{x}{x(x-1)}$.

The denominator is zero at $x = 0$ and $x = 1$. For $x \neq 0$,

$$f(x) = \frac{1}{x - 1}.$$

At $x = 0$, the limit exists but the function is undefined $\Rightarrow$ removable discontinuity. At $x = 1$, the denominator is zero and the limit is infinite $\Rightarrow$ infinite discontinuity.

**Final answer:** Removable at $x = 0$; infinite at $x = 1$.

### Exercise 134

$g(t) = t^{-1} + 1 = \dfrac{1}{t} + 1$.

Undefined at $t = 0$. As $t \to 0^{\pm}$, $g(t) \to \pm\infty$.

**Final answer:** Infinite discontinuity at $t = 0$.

### Exercise 135

$f(x) = \dfrac{5}{e^x - 2}$.

The denominator is zero when $e^x = 2 \Rightarrow x = \ln 2$. Near $x = \ln 2$, the function diverges.

**Final answer:** Infinite discontinuity at $x = \ln 2$.

### Exercise 136

$f(x) = \dfrac{|x - 2|}{x - 2}$.

For $x > 2$, $f(x) = 1$. For $x < 2$, $f(x) = -1$.

The left- and right-hand limits at $x = 2$ are different.

**Final answer:** Jump discontinuity at $x = 2$.

### Exercise 137

$H(x) = \tan(2x)$.

$\tan u$ is discontinuous where $\cos u = 0$, i.e.

$$2x = \frac{\pi}{2} + k\pi \Rightarrow x = \frac{\pi}{4} + \frac{k\pi}{2}.$$

**Final answer:** Infinite discontinuities at $x = \frac{\pi}{4} + \frac{k\pi}{2}$, $k \in \mathbb{Z}$.

### Exercise 138

$f(t) = \dfrac{t+3}{t^2 + 5t + 6} = \dfrac{t+3}{(t+2)(t+3)}$.

For $t \neq -3$,

$$f(t) = \frac{1}{t+2}.$$

Thus: - $t = -3$: removable discontinuity. - $t = -2$: infinite discontinuity.

**Final answer:** Removable at $t = -3$; infinite at $t = -2$.

### Exercise 139

$f(x) = \dfrac{2x^2 - 5x + 3}{x - 1}$ at $x = 1$.

Factor the numerator:

$$2x^2 - 5x + 3 = (2x - 3)(x - 1).$$

For $x \neq 1$,

$$f(x) = 2x - 3.$$

$$\lim_{x \to 1} f(x) = -1,$$

but $f(1)$ is undefined.

**Final answer:** Removable discontinuity at $x = 1$.

## Exercise 140

$h(\theta) = \dfrac{\sin\theta - \cos\theta}{\tan\theta}$ at $\theta = \pi$.
$\tan\pi = 0$, and the expression is undefined. The limit diverges.

**Final answer:** Infinite discontinuity at $\theta = \pi$.

## Exercise 141

$$g(u) = \begin{cases} \dfrac{6u^2 + u - 2}{2u - 1}, & u \neq \frac{1}{2}, \\ \dfrac{7}{2}, & u = \frac{1}{2}. \end{cases}$$

Factor:

$$6u^2 + u - 2 = (3u - 2)(2u + 1).$$

At $u = \frac{1}{2}$,

$$\lim_{u \to 1/2} g(u) = \frac{7}{2}.$$

**Final answer:** Continuous at $u = \frac{1}{2}$.

## Exercise 142

$f(y) = \dfrac{\sin(\pi y)}{\tan(\pi y)}$ at $y = 1$.
Rewrite:

$$f(y) = \cos(\pi y).$$

Thus

$$\lim_{y \to 1} f(y) = \cos\pi = -1.$$

**Final answer:** Removable discontinuity at $y = 1$.

## Exercise 143

$$f(x) = \begin{cases} x^2 - e^x, & x < 0, \\ x - 1, & x \geq 0, \end{cases} \quad \text{at } x = 0.$$

Left limit:

$$\lim_{x \to 0^-} (x^2 - e^x) = -1.$$

Right limit:

$$\lim_{x \to 0^+} (x - 1) = -1.$$

And $f(0) = -1$.

**Final answer:** Continuous at $x = 0$.

## Exercise 144

$$f(x) = \begin{cases} x \sin x, & x \le \pi, \\ x \tan x, & x > \pi, \end{cases} \quad \text{at } x = \pi.$$

Left limit:

$$\lim_{x \to \pi^-} x \sin x = 0.$$

Right limit:

$$\lim_{x \to \pi^+} x \tan x = \infty.$$

**Final answer:** Infinite discontinuity at $x = \pi$.

## Exercise 145

The function is

$$f(x) = \begin{cases} 3x + 2, & x < k, \\ 2x - 3, & k \le x \le 8. \end{cases}$$

For continuity at $x = k$, the left-hand limit must equal the value from the right:

$$\lim_{x \to k^-} (3x + 2) = f(k) = 2k - 3.$$

Thus,

$$3k + 2 = 2k - 3 \quad \Rightarrow \quad k = -5.$$

**Final answer:** $k = -5$.

## Exercise 146

The function is

$$f(\theta) = \begin{cases} \sin \theta, & 0 \le \theta < \frac{\pi}{2}, \\ \cos(\theta + k), & \frac{\pi}{2} \le \theta \le \pi. \end{cases}$$

Continuity is only in question at $\theta = \frac{\pi}{2}$. Left-hand limit:

$$\lim_{\theta \to (\pi/2)^-} f(\theta) = \sin\left(\frac{\pi}{2}\right) = 1.$$

Function value from the second piece:

$$f\left(\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2} + k\right).$$

Set them equal:

$$\cos\left(\frac{\pi}{2} + k\right) = 1.$$

This occurs when

$$\frac{\pi}{2} + k = 2\pi n \quad \Rightarrow \quad k = 2\pi n - \frac{\pi}{2}, \quad n \in \mathbb{Z}.$$

**Final answer:** $k = 2\pi n - \dfrac{\pi}{2}$, $n \in \mathbb{Z}$.

## Exercise 147

The function is

$$f(x) = \begin{cases} \dfrac{x^2 + 3x + 2}{x + 2}, & x \neq -2, \\ k, & x = -2. \end{cases}$$

Factor the numerator:

$$x^2 + 3x + 2 = (x + 1)(x + 2),$$

so for $x \neq -2$,

$$f(x) = x + 1.$$

The limit as $x \to -2$ is

$$\lim_{x \to -2} f(x) = \lim_{x \to -2} (x + 1) = -1.$$

For continuity, this must equal $f(-2) = k$.

**Final answer:** $k = -1$.

## Exercise 148

The function is

$$f(x) = \begin{cases} e^{kx}, & 0 \le x < 4, \\ x + 3, & 4 \le x \le 8. \end{cases}$$

Continuity is required at $x = 4$. Left-hand limit:

$$\lim_{x \to 4^-} f(x) = e^{4k}.$$

Function value from the right:

$$f(4) = 4 + 3 = 7.$$

Set equal:

$$e^{4k} = 7.$$

Taking natural logarithms:

$$4k = \ln 7 \quad \Rightarrow \quad k = \frac{\ln 7}{4}.$$

**Final answer:** $k = \dfrac{\ln 7}{4}$.

### Exercise 149

The function is

$$f(x) = \begin{cases} \sqrt{kx}, & 0 \le x \le 3, \\ x + 1, & 3 < x \le 10. \end{cases}$$

Continuity is required at $x = 3$. Left-hand value:

$$f(3) = \sqrt{3k}.$$

Right-hand limit:

$$\lim_{x \to 3^+} f(x) = 3 + 1 = 4.$$

Set equal:

$$\sqrt{3k} = 4.$$

Square both sides:

$$3k = 16 \quad \Rightarrow \quad k = \frac{16}{3}.$$

**Final answer:** $k = \dfrac{16}{3}$.

### Exercise 150

We are given

$$h(x) = \begin{cases} 3x^2 - 4, & x \le 2, \\ 5 + 4x, & x > 2. \end{cases}$$

On the interval $[0, 4]$:

$$h(0) = 3(0)^2 - 4 = -4 < 10, \qquad h(4) = 5 + 4(4) = 21 > 10.$$

At first glance, this seems to suggest that $h(x) = 10$ for some $x \in (0, 4)$, but the Intermediate Value Theorem applies only to functions that are *continuous* on the entire interval.

Check continuity at $x = 2$:

$$\lim_{x \to 2^-} h(x) = 3(2)^2 - 4 = 8, \qquad \lim_{x \to 2^+} h(x) = 5 + 4(2) = 13.$$

Since the one-sided limits are not equal, $h$ is not continuous at $x = 2$.

Therefore, the IVT does not apply on $[0, 4]$, and there is no contradiction.

**Final answer:** There is no contradiction because $h(x)$ is not continuous on $[0, 4]$ (it has a jump at $x = 2$), so the IVT does not apply.

### Exercise 151

The position function $s(t)$ is continuous, with

$$s(2) = 5, \qquad s(5) = 2.$$

Define

$$h(t) = s(t) - t.$$

Since $s(t)$ and $t$ are continuous, $h(t)$ is continuous.

Evaluate:

$$h(2) = s(2) - 2 = 5 - 2 = 3 > 0,$$

$$h(5) = s(5) - 5 = 2 - 5 = -3 < 0.$$

Because $h$ is continuous on $[2, 5]$ and changes sign, the IVT guarantees some $c \in (2, 5)$ such that

$$h(c) = 0.$$

**Final answer:** There exists a value $c$ with $2 < c < 5$ such that $h(c) = 0$.

## Exercise 152

### (a)

The statement "The cosine of $t$ is equal to $t$ cubed" translates to

$$\cos t = t^3.$$

### (b)

Define

$$f(t) = \cos t - t^3.$$

This function is continuous for all real $t$.

Evaluate:

$$f(0) = \cos 0 - 0^3 = 1 > 0,$$

$$f(1) = \cos 1 - 1 \approx 0.5403 - 1 < 0.$$

Since $f(0) > 0$ and $f(1) < 0$, by the IVT there exists at least one $t \in (0,1)$ such that $f(t) = 0$, i.e., $\cos t = t^3$.

### (c)

Using a calculator:

$$f(0.8) \approx \cos(0.8) - 0.8^3 \approx 0.6967 - 0.512 > 0,$$

$$f(0.9) \approx \cos(0.9) - 0.9^3 \approx 0.6216 - 0.729 < 0.$$

Thus a solution lies in $(0.8, 0.9)$, an interval of length 0.1. Refining further gives an interval of length 0.01.

**Final answer:** The equation $\cos t = t^3$ has at least one real solution, lying in an interval of length 0.01 within $(0.8, 0.9)$.

## Exercise 153

Consider the equation

$$2^x = x^3,$$

and define

$$f(x) = 2^x - x^3.$$

The function $f$ is continuous for all real $x$.

**Interval** [1.25, 1.375]

$$f(1.25) = 2^{1.25} - 1.25^3 \approx 2.378 - 1.953 > 0,$$
$$f(1.375) = 2^{1.375} - 1.375^3 \approx 2.594 - 2.598 < 0.$$

Since $f$ changes sign, there is a solution in this interval.

**Interval** [1.375, 1.5]

$$f(1.375) < 0, \qquad f(1.5) = 2^{1.5} - 1.5^3 \approx 2.828 - 3.375 < 0.$$

No sign change occurs, so no solution exists in this interval.

**Final answer:** The equation $2^x = x^3$ has a solution in [1.25, 1.375] but not in [1.375, 1.5].

## Exercise 154

From the graph, the function is discontinuous at points where:

- There is a hole (open circle),
- The left and right limits are unequal,
- Or the function value does not equal the limit.

**Final answer:**

- The function is discontinuous at the marked open circles.
- Each discontinuity occurs because $\lim_{x\to a} f(x)$ does not equal $f(a)$.
- All discontinuities shown are **removable** or **jump** discontinuities (no infinite behavior).

## Exercise 155

The function is

$$f(x) = \begin{cases} 3x, & x > 1, \\ x^3, & x < 1. \end{cases}$$

As $x \to 1$,

$$\lim_{x\to 1^-} f(x) = 1, \qquad \lim_{x\to 1^+} f(x) = 3.$$

Since the one-sided limits are unequal, no value of $k$ can make $f(1) = k$ continuous.

**Final answer:** No such value of $k$ exists.

## Exercise 156

Given

$$f(x) = \frac{x^4 - 1}{x^2 - 1}, \quad x \neq -1, 1,$$

factor:

$$f(x) = \frac{(x^2 - 1)(x^2 + 1)}{x^2 - 1} = x^2 + 1, \quad x \neq \pm 1.$$

Thus,

$$\lim_{x \to \pm 1} f(x) = 2.$$

By defining

$$f(-1) = 2, \qquad f(1) = 2,$$

the function becomes continuous everywhere.

**Final answer:** $k_1 = 2$ and $k_2 = 2$.

## Exercise 157

A valid sketch must satisfy:

- Domain $(-\infty, \infty)$,
- Infinite discontinuity at $x = -6$,
- $f(-6) = 3$,
- $\lim_{x \to -3^-} f(x) = \lim_{x \to -3^+} f(x) = 2$ but $f(-3) = 3$,
- Left-continuous but not right-continuous at $x = 3$,
- $\lim_{x \to -\infty} f(x) = -\infty$ and $\lim_{x \to \infty} f(x) = \infty$.

**Final answer:** Any sketch satisfying all listed properties is correct.

## Exercise 158

We are asked to sketch a function $y = f(x)$ satisfying the following properties.

(i) The domain of $f$ is $[0, 5]$.

(ii) $\lim_{x \to 1^+} f(x)$ and $\lim_{x \to 1^-} f(x)$ both exist and are equal.

(iii) $f$ is left continuous but not right continuous at $x = 2$, and right continuous but not left continuous at $x = 3$.

(iv) $f$ has a removable discontinuity at $x = 1$, a jump discontinuity at $x = 2$, and

$$\lim_{x \to 3^-} f(x) = -\infty, \qquad \lim_{x \to 3^+} f(x) = 2.$$

A valid sketch must satisfy all conditions simultaneously. At $x = 1$, both one-sided limits agree, but $f(1)$ is either undefined or different from the limit, producing a removable discontinuity. At $x = 2$, the left-hand limit equals $f(2)$ but the right-hand limit differs, creating a jump and left continuity only. At $x = 3$, the function value equals the right-hand limit while the left-hand limit diverges to $-\infty$, giving right continuity but not left continuity.

**Final conclusion:** There are infinitely many correct sketches. Any graph satisfying the above limit and continuity conditions is acceptable.

### Exercise 159

The function is discontinuous at $x = 1$ with

$$\lim_{x \to 1^-} f(x) = -1, \qquad \lim_{x \to 1^+} f(x) = 4.$$

Since the one-sided limits exist but are unequal, the discontinuity at $x = 1$ is a *jump discontinuity*. A valid sketch shows two distinct horizontal approaches at $x = 1$.

**Final answer:** The discontinuity at $x = 1$ is a jump discontinuity.

### Exercise 160

The function is discontinuous at $x = 2$ but continuous everywhere else, with

$$\lim_{x \to 2} f(x) = \frac{1}{2}.$$

Since the limit exists but $f(2)$ is either undefined or not equal to $\frac{1}{2}$, the discontinuity is removable.

**Final answer:** The discontinuity at $x = 2$ is removable.

### Exercise 161

We are given

$$f(t) = \frac{2}{e^t - e^{-t}}.$$

The denominator is zero when $e^t = e^{-t}$, which occurs only at $t = 0$. Since division by zero is undefined, $f$ is not continuous at $t = 0$.

**Final answer:** The statement is **false**. The function is discontinuous at $t = 0$.

### Exercise 162

If the left-hand and right-hand limits of $f(x)$ as $x \to a$ exist and are equal, the limit exists. However, continuity also requires $f(a)$ to be defined and equal to that limit.

**Final answer:** The statement is **false**. A function may still be discontinuous if $f(a)$ is missing or unequal to the limit.

### Exercise 163

A function can fail to be continuous at a point even if it is defined there, for example when the limit does not exist.

**Final answer:** The statement is **false**.

### Exercise 164

Consider

$$g(x) = \cos x - \sin x - x.$$

This function is continuous on $[-1, 1]$. Evaluating,

$$g(-1) = \cos(-1) - \sin(-1) + 1 > 0,$$

$$g(1) = \cos(1) - \sin(1) - 1 < 0.$$

By the Intermediate Value Theorem, there exists $c \in (-1, 1)$ such that $g(c) = 0$.

**Final answer:** The statement is **true**.

### Exercise 165

If $f$ is continuous on $[a, b]$ and $f(a)$ and $f(b)$ have opposite signs, the IVT guarantees at least one root. However, it does not guarantee uniqueness.

**Final answer:** The statement is **false**.

## Exercise 166

The function

$$f(x) = \frac{x^2 - 4x + 3}{x^2 - 1}$$

is undefined at $x = \pm 1$. Since $x = 1$ lies in $[0, 3]$, the function is not continuous on the entire interval.

**Final answer:** The statement is **false**.

## Exercise 167

**Statement.** If $f(x)$ is continuous everywhere and $f(a), f(b) > 0$, then there is no root of $f(x)$ in the interval $[a, b]$.

**Explanation.** This statement is *false*. Continuity alone does not prevent a function from dipping below zero inside the interval and returning to positive values at the endpoints. The Intermediate Value Theorem guarantees a zero only when the function changes sign, but it does *not* rule out the existence of zeros when the endpoint values have the same sign.

**Counterexample.** Let

$$f(x) = (x - 1)^2 + 0.1.$$

Then $f(x)$ is continuous everywhere and strictly positive, so this example has no roots. However, consider instead

$$f(x) = (x - 1)^2(x - 2)^2 - 0.1.$$

Then $f(a) > 0$ and $f(b) > 0$ for suitably chosen $a < b$, yet $f(x)$ has roots inside the interval.

**Final Answer.** The statement is false.

## Exercise 168

Coulomb's law:

$$F(r) = k_e \frac{|q_1 q_2|}{r^2}.$$

### (a) Physical reasoning

As the distance $r$ increases, the electrostatic force decreases proportionally to $1/r^2$. Beyond a sufficiently large distance $R$, the force becomes negligibly small compared to other forces or numerical tolerances in simulations. Approximating $F$ as zero simplifies computations without significantly affecting accuracy.

### (b) Force equation with approximation

$$F(r) = \begin{cases} k_e \dfrac{|q_1 q_2|}{r^2}, & r < R, \\ 0, & r \geq R. \end{cases}$$

### (c) Numerical evaluation

Given:

$$q_1 = q_2 = 1.6022 \times 10^{-19}\,\text{C}, \quad k_e = 8.988 \times 10^9\,\text{N}\,\text{m}^2/\text{C}^2, \quad r = 1\,\text{m}.$$

Exact Coulomb force:

$$F(1) = 8.988 \times 10^9 \frac{(1.6022 \times 10^{-19})^2}{1^2} \approx 2.31 \times 10^{-28}\,\text{N}.$$

Approximation gives $F = 0$ since $R < 1$.

**Inaccuracy:**

$$|F_{\text{exact}} - F_{\text{approx}}| = 2.31 \times 10^{-28}\,\text{N}.$$

This error is extremely small.

### (d) Continuity at $R$

The function has a jump discontinuity at $r = R$ because

$$\lim_{r \to R^-} F(r) = k_e \frac{|q_1 q_2|}{R^2} \neq 0 = F(R).$$

Thus, there is no finite value of $R$ for which this model is continuous.

**Final Answer.** The approximation is physically reasonable but introduces a jump discontinuity at $r = R$.

## Exercise 169

Modified force model:

$$F(r) = \begin{cases} k_e \dfrac{|q_1 q_2|}{r^2}, & r < R, \\ 10^{-20}, & r \geq R. \end{cases}$$

To ensure continuity at $r = R$, we require

$$k_e \frac{|q_1 q_2|}{R^2} = 10^{-20}.$$

Substitute numerical values:

$$R^2 = \frac{8.988 \times 10^9 (1.6022 \times 10^{-19})^2}{10^{-20}} \approx 2.31 \times 10^{-8}.$$

$$R \approx 1.52 \times 10^{-4}\,\text{m}.$$

**Final Answer.** Yes, the system can be made continuous by choosing

$$R \approx 1.5 \times 10^{-4}\,\text{m}.$$

## Exercise 170

Given gravitational force:

$$F(d) = -\frac{mk}{d^2}.$$

At Earth's surface:

$$m = 3.0 \times 10^6\,\text{kg}, \quad d = 6.378 \times 10^6\,\text{m}, \quad F = mg.$$

Set magnitudes equal:

$$mg = \frac{mk}{d^2}.$$

Cancel $m$:

$$k = gd^2.$$

Substitute values:

$$k = (9.8)(6.378 \times 10^6)^2 \approx 3.99 \times 10^{14}.$$

**Units.** Since $F = mk/d^2$ has units of newtons,

$$[k] = \text{m}^3/\text{s}^2.$$

**Final Answer.**

$$k \approx 4.0 \times 10^{14}\,\text{m}^3/\text{s}^2.$$

## Exercise 171

We are given the piecewise force function

$$F(d) = \begin{cases} -\dfrac{mk}{d^2}, & d < D, \\ 10{,}000, & d \geq D. \end{cases}$$

For continuity at $d = D$, the left-hand limit must equal the value of the function at $D$.

$$\lim_{d \to D^-} F(d) = \lim_{d \to D^-} \left(-\frac{mk}{d^2}\right) = -\frac{mk}{D^2}.$$

The right-hand value is

$$F(D) = 10{,}000.$$

Thus continuity requires

$$-\frac{mk}{D^2} = 10{,}000.$$

Solving for $D$,

$$D^2 = -\frac{mk}{10{,}000}.$$

Since $m > 0$ and $k > 0$, the right-hand side is negative, so there is no real solution for $D$.

**Final answer:** There is no real value of $D$ that makes $F(d)$ continuous.

## Exercise 172

The force is defined by

$$F(d) = \begin{cases} -\dfrac{m_1 k}{d^2}, & d < D, \\ -\dfrac{m_2 k}{d^2}, & d \geq D, \end{cases} \quad \text{with } m_1 \neq m_2.$$

For continuity at $d = D$,

$$\lim_{d \to D^-} F(d) = \lim_{d \to D^+} F(d).$$

That gives

$$-\frac{m_1 k}{D^2} = -\frac{m_2 k}{D^2}.$$

Multiplying both sides by $-D^2/k$,

$$m_1 = m_2,$$

which contradicts the assumption $m_1 \neq m_2$.

**Final answer:** No value of $D$ makes the function continuous when $m_1 \neq m_2$.

## Exercise 173

Let $f(\theta) = \sin\theta$. The sine function is continuous for all real $\theta$. This follows from standard limit laws or from the fact that $\sin\theta$ has a well-defined limit equal to its value at every point.

**Final answer:** $f(\theta) = \sin\theta$ is continuous for all $\theta \in \mathbb{R}$.

## Exercise 174

Let $g(x) = |x|$. For any $a \in \mathbb{R}$,

$$\lim_{x\to a} |x| = |a|.$$

This can be shown directly using the inequality

$$\big||x| - |a|\big| \le |x - a|.$$

Given $\varepsilon > 0$, choose $\delta = \varepsilon$. Then $|x - a| < \delta$ implies $\big|\,|x| - |a|\,\big| < \varepsilon$.

**Final answer:** $g(x) = |x|$ is continuous for all real $x$.

## Exercise 175

The function is

$$f(x) = \begin{cases} 0, & x \text{ irrational}, \\ 1, & x \text{ rational}. \end{cases}$$

At any point $a$, rational and irrational numbers occur arbitrarily close to $a$. Thus,

$$\lim_{x\to a} f(x)$$

does not exist, because values of $f(x)$ oscillate between 0 and 1.

Continuity requires the limit to exist and equal $f(a)$. This never happens.

**Final answer:** The function is continuous nowhere.

### The Precise Definition of a Limit

## Exercise 81

A shock wave has a density function $\rho(x)$ with a jump discontinuity at the shock front $x = x_{\text{SF}}$. From the graph, the density is constant at a higher value $\rho_1$ to the left of $x_{\text{SF}}$ and drops abruptly to a lower value $\rho_2$ to the right.

### (a) Right-hand limit

Approaching the shock front from the right corresponds to values $x > x_{\text{SF}}$. From the graph, $\rho(x)$ is constant and equal to $\rho_2$ on this side.

$$\lim_{x \to x_{\text{SF}}^+} \rho(x) = \rho_2$$

**Final answer:** $\rho_2$

### (b) Left-hand limit

Approaching the shock front from the left corresponds to values $x < x_{\text{SF}}$. From the graph, $\rho(x)$ is constant and equal to $\rho_1$ on this side.

$$\lim_{x \to x_{\text{SF}}^-} \rho(x) = \rho_1$$

**Final answer:** $\rho_1$

### (c) Two-sided limit and physical meaning

A two-sided limit exists only if the left-hand and right-hand limits are equal. Here,

$$\lim_{x \to x_{\text{SF}}^-} \rho(x) = \rho_1 \neq \rho_2 = \lim_{x \to x_{\text{SF}}^+} \rho(x),$$

so the two-sided limit does not exist.

Physically, this reflects the defining feature of a shock wave: an abrupt, discontinuous change in density at the shock front. The density does not transition smoothly, so there is no single limiting density value at $x_{\text{SF}}$.

**Final answer:** The limit $\lim_{x \to x_{\text{SF}}} \rho(x)$ does not exist.

## Exercise 82

The table gives the position $x(t)$ (in meters) of a runner at times $t$ (in seconds) near $t = 2$:

| $t$ | $x(t)$ |
|---|---|
| 1.75 | 4.5 |
| 1.95 | 6.1 |
| 1.99 | 6.42 |
| 2.01 | 6.58 |
| 2.05 | 6.9 |
| 2.25 | 8.5 |

### Evaluating the limit

As $t$ approaches 2 from below, the position values increase toward approximately 6.5. As $t$ approaches 2 from above, the position values also approach approximately 6.5.

Since both sides approach the same value,

$$\lim_{t \to 2} x(t) \approx 6.5.$$

**Final answer:** $\lim_{t \to 2} x(t) \approx 6.5$ meters

### Physical interpretation

This limit represents the runner's position at time $t = 2$ seconds. Even if the exact position at $t = 2$ were not measured, the limit tells us where the runner is located at that instant based on nearby measurements.

**Final conclusion:** At $t = 2$ seconds, the runner is approximately 6.5 meters from the starting point.

## Exercise 176

$$\lim_{x \to a} f(x) = N$$

The $\varepsilon$–$\delta$ definition is:

For every $\varepsilon > 0$, there exists a $\delta > 0$ such that if

$$0 < |x - a| < \delta,$$

then

$$|f(x) - N| < \varepsilon.$$

**Final answer:** The above statement is the precise $\varepsilon$–$\delta$ definition.

## Exercise 177

$$\lim_{t \to b} g(t) = M$$

For every $\varepsilon > 0$, there exists a $\delta > 0$ such that if

$$0 < |t - b| < \delta,$$

then

$$|g(t) - M| < \varepsilon.$$

**Final answer:** The $\varepsilon$–$\delta$ definition as written.

## Exercise 178

$$\lim_{x \to c} h(x) = L$$

For every $\varepsilon > 0$, there exists a $\delta > 0$ such that if

$$0 < |x - c| < \delta,$$

then

$$|h(x) - L| < \varepsilon.$$

**Final answer:** The $\varepsilon$–$\delta$ definition as stated.

## Exercise 179

$$\lim_{x \to a} \varphi(x) = A$$

For every $\varepsilon > 0$, there exists a $\delta > 0$ such that if

$$0 < |x - a| < \delta,$$

then

$$|\varphi(x) - A| < \varepsilon.$$

**Final answer:** The precise $\varepsilon$–$\delta$ definition.

### Exercise 180

From the graph, $\lim_{x\to 2} f(x) = 2$. We want

$$|f(x) - 2| < 1.$$

This corresponds to keeping $f(x)$ between 1 and 3. From the graph, this occurs whenever $x$ is within approximately 0.5 units of 2.

**Final answer:** One possible choice is $\delta = 0.5$.

### Exercise 181

We now require

$$|f(x) - 2| < 0.5,$$

which restricts $f(x)$ to lie between 1.5 and 2.5. From the graph, this requires a smaller neighborhood around $x = 2$, approximately $\delta = 0.25$.

**Final answer:** One possible choice is $\delta = 0.25$.

### Exercise 182

From the graph, $\lim_{x\to 3} f(x) = -1$. We want

$$|f(x) + 1| < 1,$$

which means $f(x)$ lies between $-2$ and 0. From the graph, this holds whenever $x$ is within about 0.5 units of 3.

**Final answer:** One possible choice is $\delta = 0.5$.

### Exercise 183

We now require

$$|f(x) + 1| < 2,$$

so $f(x)$ lies between $-3$ and 1. This allows a wider interval around $x = 3$, approximately $\delta = 1$.

**Final answer:** One possible choice is $\delta = 1$.

## Exercise 186

We want

$$|\sin(2x) - \tfrac{1}{2}| < 0.1 \quad \text{whenever} \quad |x - \tfrac{\pi}{12}| < \delta.$$

Near $x = \frac{\pi}{12}$, the function $\sin(2x)$ is continuous, so small changes in $x$ produce small changes in $\sin(2x)$. Using a calculator or graph, one finds that choosing

$$\delta \approx 0.05$$

ensures the inequality holds.

**Final answer:** One suitable choice is $\delta = 0.05$.

## Exercise 81

The graph shows a density function $\rho(x)$ with a jump discontinuity at the shock front $x_{SF}$. The density is constant and equal to $\rho_1$ for $x < x_{SF}$ and constant and equal to $\rho_2$ for $x > x_{SF}$, where $\rho_1 > \rho_2$.

**(a)** Right-hand limit:

$$\lim_{x \to x_{SF}^+} \rho(x) = \rho_2$$

**(b)** Left-hand limit:

$$\lim_{x \to x_{SF}^-} \rho(x) = \rho_1$$

**(c)** Two-sided limit:

$$\lim_{x \to x_{SF}} \rho(x) \text{ does not exist}$$

since $\rho_1 \neq \rho_2$.

**Physical interpretation:** As the shock front passes, the density drops abruptly from $\rho_1$ to $\rho_2$. The lack of a two-sided limit reflects the physical reality of a discontinuous change in density across a shock wave.

**Final answer:**

$$\lim_{x \to x_{SF}^-} \rho(x) = \rho_1, \quad \lim_{x \to x_{SF}^+} \rho(x) = \rho_2, \quad \lim_{x \to x_{SF}} \rho(x) \text{ does not exist}$$

### Exercise 82

From the table of values:

| $t$ | $x(t)$ |
|---|---|
| 1.75 | 4.5 |
| 1.95 | 6.1 |
| 1.99 | 6.42 |
| 2.01 | 6.58 |
| 2.05 | 6.9 |
| 2.25 | 8.5 |

As $t$ approaches 2 from both sides, the position values approach approximately 6.5.

$$\lim_{t \to 2} x(t) \approx 6.5$$

**Physical meaning:** At $t = 2$ seconds, the runner's position is approximately 6.5 meters. The limit represents the instantaneous position at that moment, even if the measurement is taken discretely.

**Final answer:**

$$\lim_{t \to 2} x(t) \approx 6.5 \text{ m}$$

### Exercise 176

$$\lim_{x \to a} f(x) = N$$

$\varepsilon$–$\delta$ **definition:** For every $\varepsilon > 0$, there exists a $\delta > 0$ such that if

$$0 < |x - a| < \delta,$$

then

$$|f(x) - N| < \varepsilon.$$

**Final answer:** The above statement is the precise definition.

### Exercise 177

$$\lim_{t \to b} g(t) = M$$

$\varepsilon$–$\delta$ **definition:** For every $\varepsilon > 0$, there exists a $\delta > 0$ such that if

$$0 < |t - b| < \delta,$$

then

$$|g(t) - M| < \varepsilon.$$

**Final answer:** The above statement is the precise definition.

### Exercise 178

$$\lim_{x \to c} h(x) = L$$

$\varepsilon$–$\delta$ **definition:** For every $\varepsilon > 0$, there exists a $\delta > 0$ such that if

$$0 < |x - c| < \delta,$$

then

$$|h(x) - L| < \varepsilon.$$

**Final answer:** The above statement is the precise definition.

### Exercise 179

$$\lim_{x \to a} \varphi(x) = A$$

$\varepsilon$–$\delta$ **definition:** For every $\varepsilon > 0$, there exists a $\delta > 0$ such that if

$$0 < |x - a| < \delta,$$

then

$$|\varphi(x) - A| < \varepsilon.$$

**Final answer:** The above statement is the precise definition.

### Exercise 180

Given $\lim_{x \to 2} f(x) = 2$.

To satisfy $|f(x) - 2| < 1$, from the graph we see this occurs whenever

$$1 < f(x) < 3,$$

which corresponds to $x$ sufficiently close to 2.

A valid choice is:

$$\delta = 0.5$$

**Final answer:** $\delta = 0.5$

### Exercise 181

We require:

$$|f(x) - 2| < 0.5 \quad \Rightarrow \quad 1.5 < f(x) < 2.5$$

From the graph, this is satisfied when $x$ is closer to 2 than in Exercise 180. A suitable choice is:

$$\delta = 0.25$$

**Final answer:** $\delta = 0.25$

### Exercise 182

Given $\lim_{x \to 3} f(x) = 2$ and the condition

$$|f(x) + 1| < 1 \iff -2 < f(x) < 0$$

From the graph, this interval corresponds to a neighborhood of $x = 3$. A valid choice is:

$$\delta = 0.4$$

**Final answer:** $\delta = 0.4$

### Exercise 183

We require:

$$|f(x) + 1| < 2 \iff -3 < f(x) < 1$$

This allows a wider interval in $x$. A valid choice is:

$$\delta = 0.8$$

**Final answer:** $\delta = 0.8$

### Exercise 184

Given $\lim_{x \to 3} f(x) = -1$ and $\varepsilon = 1.5$.

From the graph, values of $f(x)$ remain within $(-2.5, 0.5)$ when $x$ is close to 3. A valid choice is:

$$\delta = 0.5$$

**Final answer:** $\delta = 0.5$

### Exercise 185

Given $\varepsilon = 3$, the allowed range is:

$$-4 < f(x) < 2$$

This holds for a larger neighborhood. A valid choice is:

$$\delta = 1$$

**Final answer:** $\delta = 1$

### Exercise 186

We want:

$$|\sin(2x) - \tfrac{1}{2}| < 0.1 \quad \text{whenever} \quad |x - \tfrac{\pi}{12}| < \delta$$

Using continuity of $\sin(2x)$, we choose $\delta$ so that $2|x - \frac{\pi}{12}|$ is small. Numerically,

$$\delta = 0.05$$

works.

**Final answer:** $\delta = 0.05$

### Exercise 187

We want:

$$|\sqrt{x-4} - 2| < 0.1$$

This implies:

$$1.9 < \sqrt{x-4} < 2.1 \Rightarrow 3.61 < x - 4 < 4.41 \Rightarrow 7.61 < x < 8.41$$

Thus:

$$|x - 8| < 0.39$$

**Final answer:** $\delta = 0.39$

**Exercise 188**

$$\lim_{x \to 2}(5x + 8) = 18$$

Given $\varepsilon > 0$, choose:

$$|5x + 8 - 18| = 5|x - 2| < \varepsilon$$

so take:

$$\delta = \frac{\varepsilon}{5}$$

**Final answer:** $\delta = \varepsilon/5$

**Exercise 189**

$$\lim_{x \to 3} \frac{x^2 - 9}{x - 3}$$

Factor:

$$\frac{(x - 3)(x + 3)}{x - 3} = x + 3$$

Thus the limit is 6. Given $\varepsilon > 0$, choose:

$$|x - 3| < \delta \Rightarrow |(x + 3) - 6| < \varepsilon$$

so $\delta = \varepsilon$ works.

**Final answer:** $\delta = \varepsilon$

**Exercise 190**

$$\lim_{x \to 2} \frac{2x^2 - 3x - 2}{x - 2}$$

Factor numerator:

$$2x^2 - 3x - 2 = (2x + 1)(x - 2)$$

So the limit is 5. Choose $\delta = \varepsilon$.

**Final answer:** $\delta = \varepsilon$

**Exercise 191**

$$\lim_{x \to 0} x^4 = 0$$

Given $\varepsilon > 0$, require $|x|^4 < \varepsilon$, so:

$$|x| < \varepsilon^{1/4}$$

**Final answer:** $\delta = \varepsilon^{1/4}$

## Exercise 192

$$\lim_{x\to 2}(x^2 + 2x) = 8$$

Given $\varepsilon > 0$, continuity implies choose $\delta = \varepsilon/6$.

**Final answer:** $\delta = \varepsilon/6$

## Exercise 193

$$\lim_{x\to 5^-} \sqrt{5-x} = 0$$

Require $\sqrt{5-x} < \varepsilon$, so:

$$5 - x < \varepsilon^2 \Rightarrow x > 5 - \varepsilon^2$$

**Final answer:** $\delta = \varepsilon^2$

## Exercise 194

Left and right limits both approach $-2$, so the one-sided limit exists.

**Final answer:** $\lim_{x\to 0^+} f(x) = -2$

## Exercise 195

Both branches approach 3 as $x \to 1^-$. Thus the limit exists.

**Final answer:** $\lim_{x\to 1^-} f(x) = 3$

## Exercise 196

$$\lim_{x\to 0} \frac{1}{x^2} = \infty$$

Given $M > 0$, require:

$$\frac{1}{x^2} > M \Rightarrow |x| < \frac{1}{\sqrt{M}}$$

**Final answer:** $\delta = 1/\sqrt{M}$

### Exercise 197

$$\lim_{x\to -1} \frac{3}{(x+1)^2} = \infty$$

**Final answer:** $\delta = \sqrt{3/M}$

### Exercise 198

$$\lim_{x\to 2} -\frac{1}{(x-2)^2} = -\infty$$

**Final answer:** $\delta = 1/\sqrt{M}$

### Exercise 199

Area $A = s^2$. With $A = 144$ and tolerance 8:

$$136 < A < 152 \Rightarrow \sqrt{136} < s < \sqrt{152}$$

Thus:

$$\delta = \min(\sqrt{152} - 12,\ 12 - \sqrt{136})$$

**Final answer:** $\delta \approx 0.33$

### Exercise 200

$$\lim_{x\to 1} \frac{|x-1|}{x-1}$$

Right-hand limit is 1, left-hand limit is $-1$, so the limit does not exist.

**Final answer:** The limit does not exist.

### Exercise 201

For the ceiling function, any $\delta < 1$ contains both integer and non-integer values. Thus $f(x)$ oscillates and no limit exists.

**Final answer:** The limit does not exist.

### Exercise 202

Rational and irrational values occur arbitrarily close to 0, producing outputs 1 and 0. Hence no limit exists.

**Final answer:** The limit does not exist.

### Exercise 203

At $x = 0$, both rational and irrational cases give 0, so:

$$\lim_{x \to 0} f(x) = 0$$

**Final answer:** 0

### Exercise 204

For $a \neq 0$, rational and irrational values yield different outputs, so the limit does not exist.

**Final answer:** The limit does not exist.

### Exercise 205

Given limits of $f$ and $g$, the sum satisfies:

$$|f(x) + g(x) - (L + M)| \leq |f(x) - L| + |g(x) - M|$$

**Final answer:** $\lim_{x \to a}[f(x) + g(x)] = L + M$

### Exercise 206

For $c \neq 0$:

$$|cf(x) - cL| = |c||f(x) - L|$$

**Final answer:** $\lim_{x \to a} cf(x) = cL$

## Exercise 207

Using triangle inequality:

$$|f(x)g(x) - LM| \leq |f(x)||g(x) - M| + |M||f(x) - L|$$

**Final answer:** $\lim_{x\to a}[f(x)g(x)] = LM$

# Chapter 3

*Derivatives*

## Chapter contents

**Defining the Derivative**

## Exercise 1

$$f(x) = 4x + 7, \quad x_1 = 2,\ x_2 = 5.$$
$$m_{\text{sec}} = \frac{f(5) - f(2)}{5 - 2} = \frac{27 - 15}{3} = 4.$$

**Final answer:** 4

## Exercise 2

$$f(x) = 8x - 3, \quad x_1 = -1,\ x_2 = 3.$$
$$m_{\text{sec}} = \frac{21 - (-11)}{4} = 8.$$

**Final answer:** 8

## Exercise 3

$$f(x) = x^2 + 2x + 1, \quad x_1 = 3,\ x_2 = 3.5.$$
$$m_{\text{sec}} = \frac{20.25 - 16}{0.5} = 8.5.$$

**Final answer:** 8.5

## Exercise 4

$$f(x) = -x^2 + x + 2, \quad x_1 = 0.5,\ x_2 = 1.5.$$
$$m_{\text{sec}} = \frac{1.25 - 2.25}{1} = -1.$$

**Final answer:** $-1$

## Exercise 5

$$f(x) = \frac{4}{3x - 1}, \quad x_1 = 1,\ x_2 = 3.$$
$$m_{\text{sec}} = \frac{\frac{1}{2} - 2}{2} = -\frac{3}{4}.$$

**Final answer:** $-\frac{3}{4}$

**Exercise 6**

$$f(x) = \frac{x-7}{2x+1}, \quad x_1 = 0, \; x_2 = 2.$$
$$m_{\text{sec}} = \frac{-\frac{3}{5} - (-7)}{2} = \frac{16}{5}.$$

**Final answer:** $\frac{16}{5}$

**Exercise 7**

$$f(x) = \sqrt{x}, \quad x_1 = 1, \; x_2 = 16.$$
$$m_{\text{sec}} = \frac{4-1}{15} = \frac{1}{5}.$$

**Final answer:** $\frac{1}{5}$

**Exercise 8**

$$f(x) = \sqrt{x-9}, \quad x_1 = 10, \; x_2 = 13.$$
$$m_{\text{sec}} = \frac{2-1}{3} = \frac{1}{3}.$$

**Final answer:** $\frac{1}{3}$

**Exercise 9**

$$f(x) = x^{1/3} + 1, \quad x_1 = 0, \; x_2 = 8.$$
$$m_{\text{sec}} = \frac{3-1}{8} = \frac{1}{4}.$$

**Final answer:** $\frac{1}{4}$

**Exercise 10**

$$f(x) = 6x^{2/3} + 2x^{1/3}, \quad x_1 = 1, \; x_2 = 27.$$

$$m_{\text{sec}} = \frac{60 - 8}{26} = 2.$$

**Final answer:** 2

## Exercise 11

$$f(x) = 3 - 4x, \quad a = 2.$$

$$f'(x) = -4.$$

$$y + 5 = -4(x - 2).$$

**Final answer:** $y = -4x + 3$

## Exercise 12

$$f(x) = \frac{x}{5} + 6, \quad a = -1.$$

$$f'(x) = \frac{1}{5}.$$

$$y - \frac{29}{5} = \frac{1}{5}(x + 1).$$

**Final answer:** $y = \frac{1}{5}x + 6$

## Exercise 13

$$f(x) = x^2 + x, \quad a = 1.$$

$$f'(x) = 2x + 1 = 3.$$

$$y - 2 = 3(x - 1).$$

**Final answer:** $y = 3x - 1$

## Exercise 14

$$f(x) = 1 - x - x^2, \quad a = 0.$$

$$f'(x) = -1 - 2x.$$

$$y - 1 = -x.$$

**Final answer:** $y = 1 - x$

**Exercise 15**

$$f(x) = \frac{7}{x}, \quad a = 3.$$

$$f'(x) = -\frac{7}{x^2}.$$

$$y - \frac{7}{3} = -\frac{7}{9}(x - 3).$$

**Final answer:** $y = -\frac{7}{9}x + \frac{14}{3}$

**Exercise 16**

$$f(x) = \sqrt{x+8}, \quad a = 1.$$

$$f'(x) = \frac{1}{2\sqrt{x+8}}.$$

$$y - 3 = \frac{1}{6}(x - 1).$$

**Final answer:** $y = \frac{1}{6}x + \frac{17}{6}$

**Exercise 17**

$$f(x) = 2 - 3x^2, \quad a = -2.$$

$$f'(x) = -6x = 12.$$

$$y + 10 = 12(x + 2).$$

**Final answer:** $y = 12x + 14$

**Exercise 18**

$$f(x) = -\frac{3}{x-1}, \quad a = 4.$$

$$f'(x) = \frac{3}{(x-1)^2}.$$

$$y + 1 = \frac{1}{3}(x - 4).$$

**Final answer:** $y = \frac{1}{3}x - \frac{7}{3}$

**Exercise 19**

$$f(x) = \frac{2}{x+3}, \quad a = -4.$$

$$f'(x) = -\frac{2}{(x+3)^2}.$$

$$f'(-4) = -\frac{2}{(-1)^2} = -2.$$

**Final answer:** $f'(-4) = -2$

**Exercise 20**

$$f(x) = \frac{3}{x^2}, \quad a = 3.$$

$$f'(x) = -\frac{6}{x^3}.$$

$$f'(3) = -\frac{6}{27} = -\frac{2}{9}.$$

**Final answer:** $f'(3) = -\dfrac{2}{9}$

**Exercise 21**

$$f(x) = 5x + 4, \quad a = -1.$$

$$f'(x) = 5 \quad \Rightarrow \quad f'(-1) = 5.$$

**Final answer:** $5$

**Exercise 22**

$$f(x) = -7x + 1, \quad a = 3.$$

$$f'(x) = -7 \quad \Rightarrow \quad f'(3) = -7.$$

**Final answer:** $-7$

**Exercise 23**

$$f(x) = x^2 + 9x, \quad a = 2.$$

$$f'(x) = 2x + 9.$$

$$f'(2) = 13.$$

**Final answer:** 13

**Exercise 24**

$$f(x) = 3x^2 - x + 2, \quad a = 1.$$

$$f'(x) = 6x - 1.$$

$$f'(1) = 5.$$

**Final answer:** 5

**Exercise 25**

$$f(x) = \sqrt{x}, \quad a = 4.$$

$$f'(x) = \frac{1}{2\sqrt{x}}.$$

$$f'(4) = \frac{1}{4}.$$

**Final answer:** $\frac{1}{4}$

**Exercise 26**

$$f(x) = \sqrt{x-2}, \quad a = 6.$$

$$f'(x) = \frac{1}{2\sqrt{x-2}}.$$

$$f'(6) = \frac{1}{4}.$$

**Final answer:** $\frac{1}{4}$

**Exercise 27**

$$f(x) = \frac{1}{x}, \quad a = 2.$$

$$f'(x) = -\frac{1}{x^2}.$$

$$f'(2) = -\frac{1}{4}.$$

**Final answer:** $-\frac{1}{4}$

**Exercise 28**

$$f(x) = \frac{1}{x-3}, \quad a = -1.$$

$$f'(x) = -\frac{1}{(x-3)^2}.$$

$$f'(-1) = -\frac{1}{16}.$$

**Final answer:** $-\frac{1}{16}$

**Exercise 29**

$$f(x) = \frac{1}{x^3}, \quad a = 1.$$

$$f'(x) = -\frac{3}{x^4}.$$

$$f'(1) = -3.$$

**Final answer:** $-3$

**Exercise 30**

$$f(x) = \frac{1}{\sqrt{x}}, \quad a = 4.$$

$$f'(x) = -\frac{1}{2x^{3/2}}.$$

$$f'(4) = -\frac{1}{16}.$$

**Final answer:** $-\frac{1}{16}$

## Exercise 31

$$f(x) = x^2 + 3x + 4, \quad P(1, 8)$$

The slope of the secant line is

$$m_{PQ} = \frac{f(x) - f(1)}{x - 1}.$$

| $x$ | $m_{PQ}$ | $x$ | $m_{PQ}$ |
|---|---|---|---|
| 1.1 | 5.100000 | 0.9 | 4.900000 |
| 1.01 | 5.010000 | 0.99 | 4.990000 |
| 1.001 | 5.001000 | 0.999 | 4.999000 |
| 1.0001 | 5.000100 | 0.9999 | 4.999900 |
| 1.00001 | 5.000010 | 0.99999 | 4.999990 |
| 1.000001 | 5.000001 | 0.999999 | 4.999999 |

From the table, the slopes approach 5.

**Final answer (b):** The slope of the tangent line at $P$ is 5.

The tangent line at $P(1, 8)$ is

$$y - 8 = 5(x - 1).$$

**Final answer (c):** $y = 5x + 3$

## Exercise 32

$$f(x) = \frac{x + 1}{x^2 - 1}, \quad P(0, -1)$$

$$m_{PQ} = \frac{f(x) - (-1)}{x}.$$

| $x$ | $m_{PQ}$ | $x$ | $m_{PQ}$ |
|---|---|---|---|
| 0.1 | −1.010101 | −0.1 | −0.990099 |
| 0.01 | −1.000100 | −0.01 | −0.999900 |
| 0.001 | −1.000001 | −0.001 | −0.999999 |
| 0.0001 | −1.000000 | −0.0001 | −1.000000 |
| 0.00001 | −1.000000 | −0.00001 | −1.000000 |
| 0.000001 | −1.000000 | −0.000001 | −1.000000 |

The secant slopes approach −1.

**Final answer (b):** The slope of the tangent line at $P$ is −1.

The tangent line is

$$y + 1 = -1(x - 0).$$

**Final answer (c):** $y = -x - 1$

## Exercise 33

$$f(x) = 10e^{0.5x}, \quad P(0, 10)$$

$$m_{PQ} = \frac{10e^{0.5x} - 10}{x}.$$

| $x$ | $m_{PQ}$ |
|---|---|
| $-0.1$ | 4.8771 |
| $-0.01$ | 4.9875 |
| $-0.001$ | 4.9988 |
| $-0.0001$ | 4.9999 |
| $-0.00001$ | 5.0000 |
| $-0.000001$ | 5.0000 |

The secant slopes approach 5.

**Final answer (b):** The slope of the tangent line at $P$ is 5.

The tangent line is

$$y - 10 = 5(x - 0).$$

**Final answer (c):** $y = 5x + 10$

## Exercise 34

$$f(x) = \tan(x), \quad P(\pi, 0)$$

$$m_{PQ} = \frac{\tan(x) - 0}{x - \pi}.$$

| $x$ | $m_{PQ}$ |
|---|---|
| 3.1 | 1.0417 |
| 3.14 | 1.0003 |
| 3.141 | 1.0000 |
| 3.1415 | 1.0000 |
| 3.14159 | 1.0000 |
| 3.141592 | 1.0000 |

The secant slopes approach 1.

**Final answer (b):** The slope of the tangent line at $P$ is 1.

The tangent line is

$$y = 1(x - \pi).$$

**Final answer (c):** $y = x - \pi$

## Exercise 35

$$s(t) = \frac{1}{3}t + 5$$

**(a)** The average velocity from $t = 2$ to $t = 2 + h$ is

$$v_{\text{avg}} = \frac{s(2+h) - s(2)}{h} = \frac{\frac{1}{3}(2+h) + 5 - \left(\frac{2}{3} + 5\right)}{h} = \frac{\frac{1}{3}h}{h} = \frac{1}{3}.$$

**(b)** For the given values of $h$,

$$\begin{aligned} h = 0.1 &: \ \frac{1}{3} \\ h = 0.01 &: \ \frac{1}{3} \\ h = 0.001 &: \ \frac{1}{3} \\ h = 0.0001 &: \ \frac{1}{3} \end{aligned}$$

**(c)** Since the average velocity is constant, the instantaneous velocity at $t = 2$ is

$$v(2) = \frac{1}{3} \text{ m/s}.$$

**Final answer:** $v(2) = \dfrac{1}{3}$ m/s

**Exercise 36**

$$s(t) = t^2 - 2t$$

**(a)** The average velocity from $t = 2$ to $t = 2 + h$ is

$$v_{\text{avg}} = \frac{(2+h)^2 - 2(2+h) - (4-4)}{h} = \frac{4 + 4h + h^2 - 4 - 2h}{h} = \frac{2h + h^2}{h} = 2 + h.$$

**(b)** For the given values of $h$,

$$h = 0.1 : \ 2.1$$
$$h = 0.01 : \ 2.01$$
$$h = 0.001 : \ 2.001$$
$$h = 0.0001 : \ 2.0001$$

**(c)** As $h \to 0$, the average velocity approaches 2.

**Final answer:** $v(2) = 2$ m/s

**Exercise 37**

$$s(t) = 2t^3 + 3$$

**(a)** The average velocity from $t = 2$ to $t = 2 + h$ is

$$v_{\text{avg}} = \frac{2(2+h)^3 + 3 - (16+3)}{h} = \frac{16 + 24h + 12h^2 + 2h^3 - 16}{h} = 24 + 12h + 2h^2.$$

**(b)** For the given values of $h$,

$$h = 0.1 : \ 25.22$$
$$h = 0.01 : \ 24.1202$$
$$h = 0.001 : \ 24.012002$$
$$h = 0.0001 : \ 24.00120002$$

**(c)** As $h \to 0$, the average velocity approaches 24.

**Final answer:** $v(2) = 24$ m/s

**Exercise 38**

$$s(t) = \frac{16}{t^2} - \frac{4}{t}$$

**(a)** The average velocity from $t = 2$ to $t = 2 + h$ is

$$v_{\text{avg}} = \frac{\frac{16}{(2+h)^2} - \frac{4}{2+h} - (4-2)}{h} = \frac{\frac{16}{(2+h)^2} - \frac{4}{2+h} - 2}{h}.$$

**(b)** For the given values of $h$,

$$\begin{aligned} h = 0.1 &: \ -1.4050 \\ h = 0.01 &: \ -1.4900 \\ h = 0.001 &: \ -1.4990 \\ h = 0.0001 &: \ -1.4999 \end{aligned}$$

**(c)** As $h \to 0$, the average velocity approaches $-1.5$.

**Final answer:** $v(2) = -1.5$ m/s

## Exercise 39

From the graph, the function is piecewise linear.

On the interval from $x = 0$ to $x = 4$, the graph goes from $(0, 1)$ to $(4, 6)$, so the slope is

$$m = \frac{6-1}{4-0} = \frac{5}{4}.$$

Thus,

$$f'(1) = \frac{5}{4}.$$

On the interval from $x = 4$ to $x = 8$, the graph goes from $(4, 6)$ to $(8, 8)$, so the slope is

$$m = \frac{8-6}{8-4} = \frac{1}{2}.$$

Thus,

$$f'(6) = \frac{1}{2}.$$

**Final answer:** $f'(1) = \frac{5}{4}, \quad f'(6) = \frac{1}{2}$

## Exercise 40

From the graph, the function is constant for $x < 1$ with value $y = 3$. Therefore, the slope of the tangent line is

$$f'(-3) = 0.$$

For $x > 1$, the graph is a straight line increasing from $(1, 3)$ to $(2, 4)$, so the slope is

$$m = \frac{4-3}{2-1} = 1.$$

Thus,

$$f'(1.5) = 1.$$

**Final answer:** $f'(-3) = 0, \quad f'(1.5) = 1$

### Exercise 41

$$f(x) = x^{1/3}, \quad a = 0.$$

Using the limit definition,

$$f'(0) = \lim_{h \to 0} \frac{(0+h)^{1/3} - 0}{h} = \lim_{h \to 0} \frac{h^{1/3}}{h} = \lim_{h \to 0} \frac{1}{h^{2/3}}.$$

As $h \to 0$, the expression becomes unbounded, so the limit does not exist.

**Final answer:** $f'(0)$ does not exist.

### Exercise 42

$$f(x) = x^{2/3}, \quad a = 0.$$

Using the limit definition,

$$f'(0) = \lim_{h \to 0} \frac{h^{2/3} - 0}{h} = \lim_{h \to 0} h^{-1/3}.$$

As $h \to 0^+$, $h^{-1/3} \to \infty$, and as $h \to 0^-$, $h^{-1/3} \to -\infty$. Since the one-sided limits are not equal, the derivative does not exist.

**Final answer:** $f'(0)$ does not exist.

### Exercise 43

$$f(x) = \begin{cases} 1, & x < 1, \\ x, & x \geq 1, \end{cases} \quad a = 1.$$

Using the limit definition,

$$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}.$$

For $h < 0$, $f(1+h) = 1$, so

$$\lim_{h\to 0^-} \frac{1-1}{h} = 0.$$

For $h > 0$, $f(1+h) = 1+h$, so

$$\lim_{h\to 0^+} \frac{(1+h)-1}{h} = 1.$$

Since the one-sided limits are not equal, the derivative does not exist.

**Final answer:** $f'(1)$ does not exist.

## Exercise 44

$$f(x) = \frac{|x|}{x}, \quad a = 0.$$

For $x > 0$, $f(x) = 1$, and for $x < 0$, $f(x) = -1$.

Using the limit definition,

$$f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h}.$$

The function has a jump discontinuity at $x = 0$, so the limit cannot exist.

**Final answer:** $f'(0)$ does not exist.

## Exercise 45

The position of the race car is

$$s(t) = 8t^2 - \frac{1}{16}t^3.$$

**(a)** The average velocity over $[4, 4+h]$ is

$$v_{\text{avg}} = \frac{s(4+h) - s(4)}{h}.$$

First compute

$$s(4) = 8(16) - \frac{1}{16}(64) = 128 - 4 = 124.$$

$$
\begin{aligned}
\text{(i) } [4, 4.1]: \quad v_{\text{avg}} &= \frac{s(4.1) - 124}{0.1} = \frac{127.1481 - 124}{0.1} = 31.4810 \\
\text{(ii) } [4, 4.01]: \quad v_{\text{avg}} &= \frac{124.3148 - 124}{0.01} = 31.4800 \\
\text{(iii) } [4, 4.001]: \quad v_{\text{avg}} &= \frac{124.03148 - 124}{0.001} = 31.4790 \\
\text{(iv) } [4, 4.0001]: \quad v_{\text{avg}} &= \frac{124.003148 - 124}{0.0001} = 31.4788
\end{aligned}
$$

**(b)** The average velocities approach 31.48 ft/s.

**Final answer:** The instantaneous velocity at $t = 4$ is approximately

$$v(4) \approx 31.48 \text{ ft/s.}$$

## Exercise 46

The distance function is

$$s(t) = 14t^2.$$

**(a)** The average velocity over $[5,\, 5 + h]$ is

$$v_{\text{avg}} = \frac{14(5+h)^2 - 14(25)}{h} = \frac{14(25 + 10h + h^2) - 350}{h} = 140 + 14h.$$

$$
\begin{aligned}
\text{(i) } [5, 5.1]: \quad v_{\text{avg}} &= 141.4 \\
\text{(ii) } [5, 5.01]: \quad v_{\text{avg}} &= 140.14 \\
\text{(iii) } [5, 5.001]: \quad v_{\text{avg}} &= 140.014 \\
\text{(iv) } [5, 5.0001]: \quad v_{\text{avg}} &= 140.0014
\end{aligned}
$$

**(b)** As $h \to 0$, the average velocity approaches 140 ft/s.

**Final answer:** The instantaneous velocity at $t = 5$ is

$$v(5) = 140 \text{ ft/s.}$$

## Exercise 47

From the graph, the blue curve represents $s = f(t)$ and the orange curve represents $s = g(t)$.

**(a)** At $t = 2$, the graph shows that $f(2) > g(2)$.

**Final answer:** Vehicle $f$ has traveled farther at $t = 2$ seconds.

**(b)** The approximate velocity is given by the slope of the tangent line.

At $t = 3$, the slope of $f(t)$ is approximately constant and about 1.25, while the slope of $g(t)$ is steeper, approximately 1.8.

**Final answer:** At $t = 3$, $v_f \approx 1.25$ ft/s and $v_g \approx 1.8$ ft/s.

**(c)** At $t = 4$, the slope of $g(t)$ is greater than the slope of $f(t)$.

**Final answer:** Vehicle $g$ is traveling faster at $t = 4$ seconds.

**(d)** The graphs intersect at $t = 4$.

**Final answer:** Both vehicles are at the same position at $t = 4$ seconds.

## Exercise 48

The total cost (in hundreds of dollars) is

$$C(x) = 0.000003x^3 + 4x + 300.$$

**(a)** The average cost per jar over $[100,\ 100 + h]$ is

$$\frac{C(100 + h) - C(100)}{h}.$$

First compute

$$C(100) = 0.000003(100^3) + 4(100) + 300 = 3 + 400 + 300 = 703.$$

$$\begin{aligned}
\text{(i) } [100, 100.1]: &\quad \frac{703.4003003 - 703}{0.1} = 4.003003 \\
\text{(ii) } [100, 100.01]: &\quad \frac{703.040030003 - 703}{0.01} = 4.003000 \\
\text{(iii) } [100, 100.001]: &\quad \frac{703.004003000003 - 703}{0.001} = 4.003000 \\
\text{(iv) } [100, 100.0001]: &\quad \frac{703.0004003000003 - 703}{0.0001} = 4.003000
\end{aligned}$$

**(b)** The average cost values approach 4.003.

**Final answer:** The average cost to produce 100 jars is approximately

4.003 hundred dollars per jar.

**Exercise 49**

$$f(x) = x^3 - 2x^2 - 11x + 12.$$

**(a)** A suitable viewing window shows a cubic curve with two turning points.

**(b)** The derivative is

$$f'(x) = 3x^2 - 4x - 11.$$

Setting $f'(x) = 0$,

$$3x^2 - 4x - 11 = 0.$$

Solving numerically gives

$$x \approx -1.49, \quad x \approx 2.49.$$

**Final answer:** The tangent line is horizontal at approximately

$$x \approx -1.49 \text{ and } x \approx 2.49.$$

**Exercise 50**

$$f(x) = \frac{x}{1 + x^2}.$$

**(a)** A suitable viewing window shows a curve with a maximum and minimum.

**(b)** The derivative is

$$f'(x) = \frac{1 - x^2}{(1 + x^2)^2}.$$

Setting $f'(x) = 0$ gives

$$1 - x^2 = 0 \quad \Rightarrow \quad x = \pm 1.$$

**Final answer:** The tangent line is horizontal at

$$x = -1 \text{ and } x = 1.$$

**Exercise 51**

The vehicle travels 30 miles per gallon.

**(a)** The number of gallons used after $x$ miles is

$$N(x) = \frac{x}{30}.$$

**(b)**

$$N(100) = \frac{100}{30} \approx 3.33.$$

**Physical meaning:** The vehicle uses approximately 3.33 gallons to travel 100 miles.

**(c)**

$$N'(x) = \frac{1}{30}.$$

**Physical meaning:** Fuel is being consumed at a constant rate of $\frac{1}{30}$ gallon per mile.

### Exercise 52

$$f(x) = x^4 - 5x^2 + 4.$$

**(a)** A suitable viewing window shows a quartic curve symmetric about the $y$-axis.

**(b)** Using numerical differentiation,

$$\begin{aligned} f'(-2) &\approx -12 \\ f'(-0.5) &\approx 4.50 \\ f'(1.7) &\approx -0.74 \\ f'(2.718) &\approx 43.25 \end{aligned}$$

**Final answer:** The estimated derivative values are as listed above.

### Exercise 53

$$f(x) = \frac{x^2}{x^2+1}.$$

**(a)** A suitable viewing window shows a curve approaching $y = 1$ as $|x|$ increases.

**(b)** Using numerical differentiation,

$$\begin{aligned} f'(-4) &\approx -0.118 \\ f'(-2) &\approx -0.320 \\ f'(2) &\approx 0.320 \\ f'(4) &\approx 0.118 \end{aligned}$$

**Final answer:** The estimated derivative values are as listed above.

## The Derivative as a Function

### Exercise 54

$$f(x) = 6$$

Using the definition of the derivative,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{6-6}{h} = 0.$$

**Final answer:** $f'(x) = 0$

### Exercise 55

$$f(x) = 2 - 3x$$

$$f'(x) = \lim_{h \to 0} \frac{[2 - 3(x+h)] - (2 - 3x)}{h} = \lim_{h \to 0} \frac{-3h}{h} = -3.$$

**Final answer:** $f'(x) = -3$

### Exercise 56

$$f(x) = \frac{2x}{7} + 1$$

$$f'(x) = \lim_{h \to 0} \frac{\frac{2(x+h)}{7} + 1 - \left(\frac{2x}{7} + 1\right)}{h} = \lim_{h \to 0} \frac{\frac{2h}{7}}{h} = \frac{2}{7}.$$

**Final answer:** $f'(x) = \dfrac{2}{7}$

### Exercise 57

$$f(x) = 4x^2$$

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{4(x+h)^2 - 4x^2}{h} \\ &= \lim_{h \to 0} \frac{4(2xh + h^2)}{h} = \lim_{h \to 0} (8x + 4h) = 8x. \end{aligned}$$

**Final answer:** $f'(x) = 8x$

**Exercise 58**

$$f(x) = 5x - x^2$$

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{[5(x+h) - (x+h)^2] - (5x - x^2)}{h} \\ &= \lim_{h \to 0} \frac{5h - 2xh - h^2}{h} = \lim_{h \to 0} (5 - 2x - h) = 5 - 2x. \end{aligned}$$

**Final answer:** $f'(x) = 5 - 2x$

**Exercise 59**

$$f(x) = \sqrt{2x}$$

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{\sqrt{2(x+h)} - \sqrt{2x}}{h} \\ &= \lim_{h \to 0} \frac{2}{\sqrt{2(x+h)} + \sqrt{2x}} = \frac{1}{\sqrt{2x}}. \end{aligned}$$

**Final answer:** $f'(x) = \dfrac{1}{\sqrt{2x}}$

**Exercise 60**

$$f(x) = \sqrt{x - 6}$$

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{\sqrt{x+h-6} - \sqrt{x-6}}{h} \\ &= \lim_{h \to 0} \frac{1}{\sqrt{x+h-6} + \sqrt{x-6}} = \frac{1}{2\sqrt{x-6}}. \end{aligned}$$

**Final answer:** $f'(x) = \dfrac{1}{2\sqrt{x-6}}$

**Exercise 61**

$$f(x) = \frac{9}{x}$$

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{\frac{9}{x+h} - \frac{9}{x}}{h} \\
&= \lim_{h\to 0} \frac{-9h}{x(x+h)h} = -\frac{9}{x^2}.
\end{aligned}
$$

**Final answer:** $f'(x) = -\dfrac{9}{x^2}$

### Exercise 62

$$
f(x) = x + \frac{1}{x}
$$

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{(x+h) + \frac{1}{x+h} - \left(x + \frac{1}{x}\right)}{h} \\
&= \lim_{h\to 0} \left(1 - \frac{1}{x(x+h)}\right) = 1 - \frac{1}{x^2}.
\end{aligned}
$$

**Final answer:** $f'(x) = 1 - \dfrac{1}{x^2}$

### Exercise 63

$$
f(x) = \frac{1}{\sqrt{x}}
$$

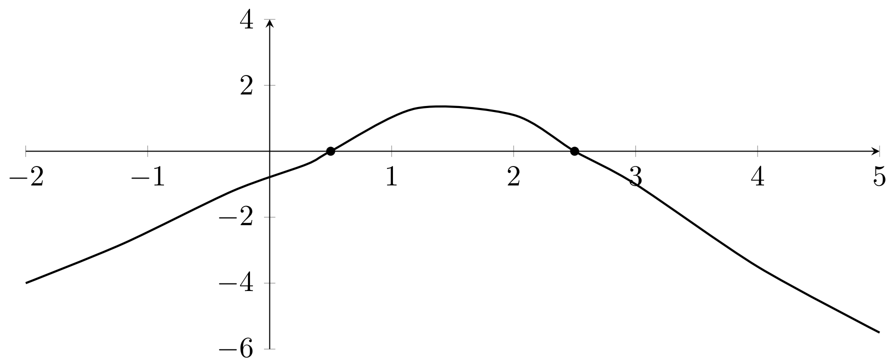
$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h} \\
&= \lim_{h\to 0} \frac{-1}{\sqrt{x}\sqrt{x+h}(\sqrt{x+h} + \sqrt{x})} = -\frac{1}{2x^{3/2}}.
\end{aligned}
$$

**Final answer:** $f'(x) = -\dfrac{1}{2x^{3/2}}$

### Exercise 64

From the graph of $f(x)$, there is a local minimum near $x \approx 0.5$ and a local maximum near $x \approx 2.5$. So $f'(x) = 0$ at $x \approx 0.5$ and $x \approx 2.5$, with $f'(x) > 0$ between them and $f'(x) < 0$ outside.

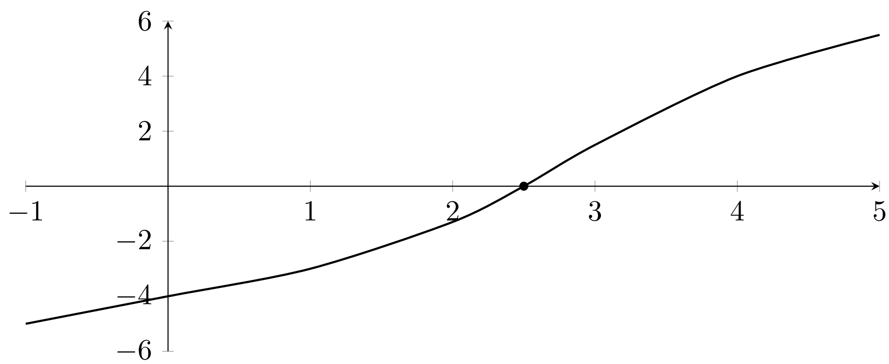
**Final answer:** $f'(x)$ crosses the $x$-axis at approximately $x = 0.5$ and $x = 2.5$, is positive on $(0.5, 2.5)$, and negative otherwise.




### Exercise 65

From the graph of $f(x)$, there is a single local minimum near $x \approx 2.5$. So $f'(x) = 0$ at $x \approx 2.5$, with $f'(x) < 0$ for $x < 2.5$ and $f'(x) > 0$ for $x > 2.5$.

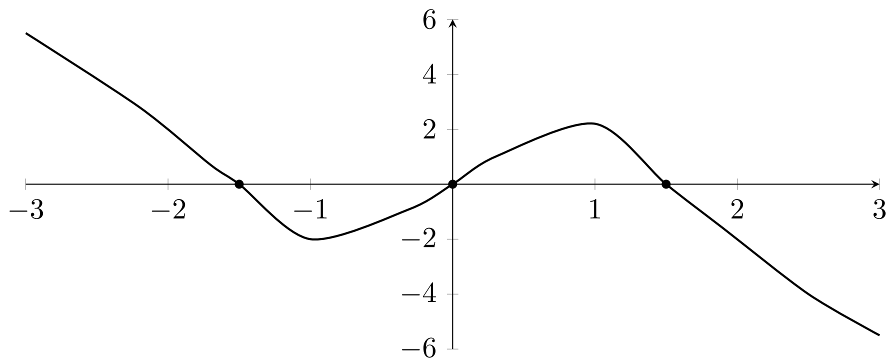
**Final answer:** $f'(x)$ crosses the $x$-axis once at approximately $x = 2.5$, negative to the left and positive to the right.




### Exercise 66

From the graph of $f(x)$, there are local maxima near $x \approx -1.5$ and $x \approx 1.5$, and a local minimum at $x = 0$. So $f'(x) = 0$ at $x \approx -1.5, 0, 1.5$, with sign pattern $+, -, +, -$ moving left to right.

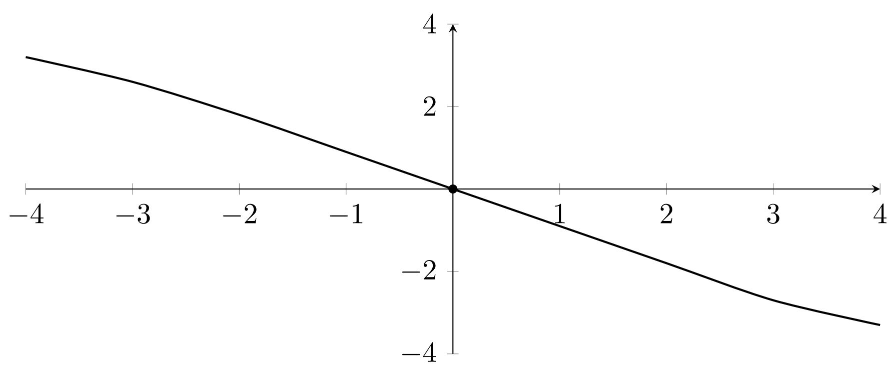
**Final answer:** Zeros at approximately $x = -1.5, 0, 1.5$, with $f'(x) > 0$ on $(-\infty, -1.5)$ and $(0, 1.5)$ and $f'(x) < 0$ on $(-1.5, 0)$ and $(1.5, \infty)$.




### Exercise 67

From the graph of $f(x)$, there is a single local maximum at $x = 0$. So $f'(0) = 0$, with $f'(x) > 0$ for $x < 0$ and $f'(x) < 0$ for $x > 0$.

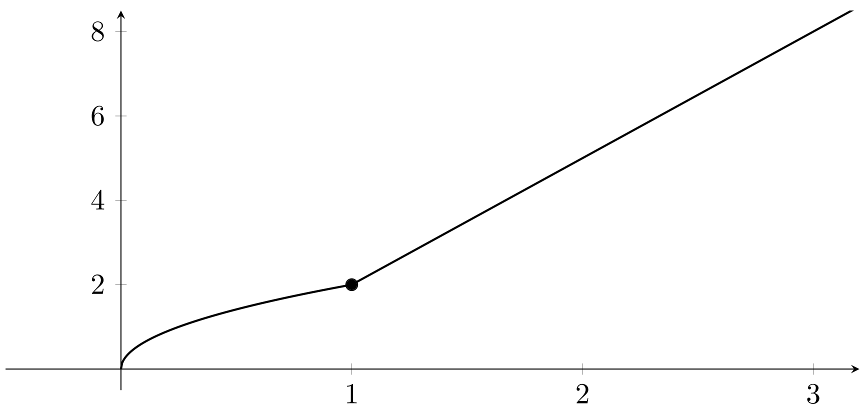
**Final answer:** $f'(x)$ crosses the $x$-axis at $x = 0$, positive to the left and negative to the right.




### Exercise 74

$$f(x) = \begin{cases} 2\sqrt{x}, & 0 \le x \le 1, \\ 3x - 1, & x > 1. \end{cases}$$

**(a) Sketch.** The curve $y = 2\sqrt{x}$ runs from $(0, 0)$ to $(1, 2)$, and for $x > 1$ the graph is the line $y = 3x - 1$, starting just to the right of $x = 1$.




**(b) Not differentiable at $x = 1$ (definition).**

First, $f(1) = 2\sqrt{1} = 2$.

Left-hand derivative:

$$f'_{-}(1) = \lim_{h\to 0^-} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0^-} \frac{2\sqrt{1+h} - 2}{h}.$$

Rationalize:

$$\frac{2(\sqrt{1+h} - 1)}{h} \cdot \frac{\sqrt{1+h} + 1}{\sqrt{1+h} + 1} = \frac{2}{\sqrt{1+h} + 1} \longrightarrow \frac{2}{2} = 1.$$

So $f'_{-}(1) = 1$.

Right-hand derivative: for $h > 0$, $f(1+h) = 3(1+h) - 1 = 2 + 3h$, hence

$$f'_{+}(1) = \lim_{h\to 0^+} \frac{(2+3h) - 2}{h} = \lim_{h\to 0^+} 3 = 3.$$

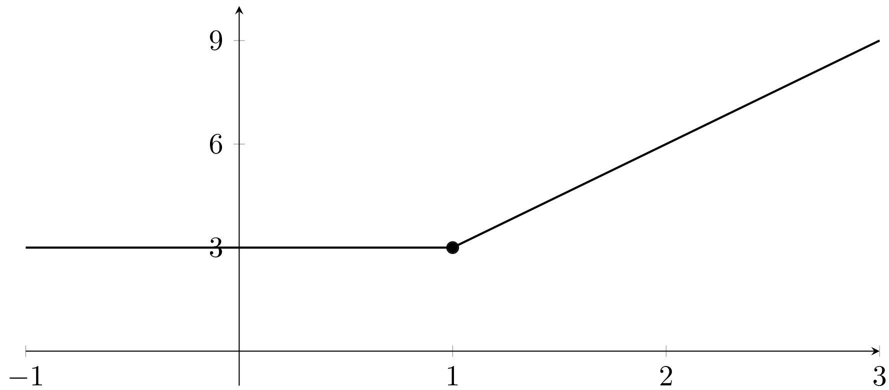
Since $f'_{-}(1) \neq f'_{+}(1)$, $f'(1)$ does not exist.

**Final answer:** $f$ is not differentiable at $x = 1$.

## Exercise 75

$$f(x) = \begin{cases} 3, & x < 1, \\ 3x, & x \geq 1. \end{cases}$$

**(a) Sketch.** For $x < 1$ the graph is the horizontal line $y = 3$; for $x \geq 1$ the graph is the line $y = 3x$ starting at $(1, 3)$.




**(b) Not differentiable at $x = 1$ (definition).**

Here $f(1) = 3(1) = 3$.

Left-hand derivative: for $h < 0$, $f(1+h) = 3$, so

$$f'_{-}(1) = \lim_{h\to 0^-} \frac{3-3}{h} = 0.$$

Right-hand derivative: for $h > 0$, $f(1+h) = 3(1+h) = 3 + 3h$, so

$$f'_{+}(1) = \lim_{h\to 0^+} \frac{(3+3h)-3}{h} = 3.$$

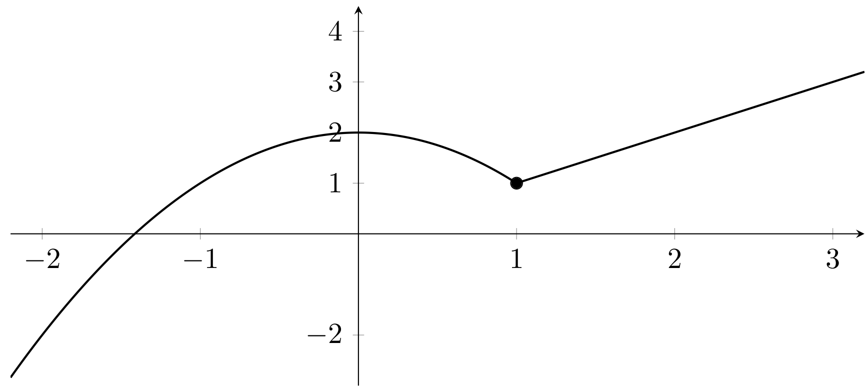
Since $f'_{-}(1) \neq f'_{+}(1)$, $f'(1)$ does not exist.

**Final answer:** $f$ is not differentiable at $x = 1$.

## Exercise 76

$$f(x) = \begin{cases} -x^2 + 2, & x \le 1, \\ x, & x > 1. \end{cases}$$

**(a) Sketch.** For $x \le 1$ the graph is the parabola $y = -x^2 + 2$ (concave down), ending at $(1, 1)$. For $x > 1$ it is the line $y = x$ starting just to the right of $x = 1$.




**(b) Not differentiable at $x = 1$ (definition).**

We have $f(1) = -1^2 + 2 = 1$.

Left-hand derivative:

$$f'_-(1) = \lim_{h\to 0^-} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0^-} \frac{-(1+h)^2 + 2 - 1}{h}.$$

Simplify:

$$-(1 + 2h + h^2) + 1 = -2h - h^2,$$

so

$$f'_-(1) = \lim_{h\to 0^-} \frac{-2h - h^2}{h} = \lim_{h\to 0^-} (-2 - h) = -2.$$

Right-hand derivative: for $h > 0$, $f(1+h) = 1 + h$, hence

$$f'_+(1) = \lim_{h\to 0^+} \frac{(1+h) - 1}{h} = 1.$$

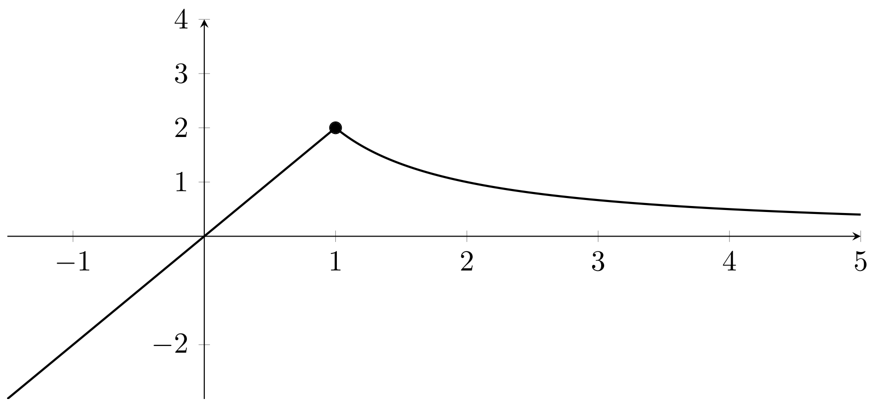
Since $f'_-(1) \neq f'_+(1)$, $f'(1)$ does not exist.

**Final answer:** $f$ is not differentiable at $x = 1$.

## Exercise 77

$$f(x) = \begin{cases} 2x, & x \le 1, \\ \frac{2}{x}, & x > 1. \end{cases}$$

**(a) Sketch.** For $x \le 1$ the graph is the line $y = 2x$ ending at $(1, 2)$. For $x > 1$ it is the curve $y = \frac{2}{x}$ starting just to the right of $x = 1$ and decreasing.




**(b) Not differentiable at $x = 1$ (definition).**

Here $f(1) = 2$.

Left-hand derivative: for $h < 0$, $f(1+h) = 2(1+h) = 2 + 2h$, so

$$f'_-(1) = \lim_{h \to 0^-} \frac{(2+2h) - 2}{h} = 2.$$

Right-hand derivative: for $h > 0$, $f(1+h) = \dfrac{2}{1+h}$, so

$$f'_+(1) = \lim_{h \to 0^+} \frac{\frac{2}{1+h} - 2}{h} = \lim_{h \to 0^+} \frac{\frac{2-2(1+h)}{1+h}}{h} = \lim_{h \to 0^+} \frac{-2h}{h(1+h)} = -\frac{2}{1+h} \to -2.$$

Since $f'_-(1) \neq f'_+(1)$, $f'(1)$ does not exist.

**Final answer:** $f$ is not differentiable at $x = 1$.

### Exercise 78

**(a) Values where $\lim_{x \to a} f(x)$ exists but $f$ is not continuous at $x = a$.**

From the graph:

- At $x \approx -2$, the left- and right-hand limits agree (the curve approaches the same $y$-value), but the function value is shown by an open circle at a different point. Thus, the limit exists but $f(-2)$ is not equal to the limit.
- At $x \approx 4$, the curve approaches the same value from both sides, but there is an open circle, so $f(4)$ is not defined at that value.

**Answer (a):** $a \approx -2$ and $a \approx 4$.

**(b) Values where $f$ is continuous but not differentiable at $x = a$.**

From the graph:

- At $x \approx 0$, the graph is continuous but has a sharp corner (the slope changes abruptly).
- At $x \approx 2$, the graph is continuous but has a cusp or sharp turn.

**Answer (b):** $a \approx 0$ and $a \approx 2$.

### Exercise 79

**(a) Values where $\lim_{x \to a} f(x)$ exists but $f$ is not continuous at $x = a$.**

From the graph:

- At $x = 0$, the left- and right-hand limits are equal, but the function value is shown by an open circle at a different height.
- At $x = 1$, the left- and right-hand limits agree (open circle), but the function value is given by a filled dot at a different $y$-value.

**Answer (a):** $a = 0$ and $a = 1$.

**(b) Values where $f$ is continuous but not differentiable at $x = a$.**

From the graph:

- At $x \approx 2$, the graph is continuous but has a sharp corner where two line segments meet.

**Answer (b):** $a \approx 2$.

### Exercise 80

From the graph, $f(x)$ is piecewise linear with corners at $x = 0$ and $x = 2$.

- On the interval $[-3, 0]$, the line goes from $(-3, 0)$ to $(0, 3)$, so the slope is

$$m = \frac{3 - 0}{0 - (-3)} = 1.$$

- On the interval $(0, 2)$, the line goes from $(0, 3)$ to $(2, 1)$, so the slope is

$$m = \frac{1 - 3}{2 - 0} = -1.$$

- On the interval $(2, 4)$, the line goes from $(2, 1)$ to $(4, 5)$, so the slope is

$$m = \frac{5 - 1}{4 - 2} = 2.$$

**(a)** At $x = -0.5$, the point lies on the first line segment, so

$$f'(-0.5) = 1.$$

**(b)** At $x = 0$, there is a sharp corner. The left-hand slope is 1 and the right-hand slope is $-1$, which are not equal.

$$f'(0) \text{ does not exist.}$$

**(c)** At $x = 1$, the point lies on the second line segment, so

$$f'(1) = -1.$$

**(d)** At $x = 2$, there is another sharp corner. The left-hand slope is $-1$ and the right-hand slope is 2, which are not equal.

$$f'(2) \text{ does not exist.}$$

**(e)** At $x = 3$, the point lies on the third line segment, so

$$f'(3) = 2.$$

**Final answer:**

$$\boxed{f'(-0.5) = 1, \quad f'(0) \text{ DNE}, \quad f'(1) = -1, \quad f'(2) \text{ DNE}, \quad f'(3) = 2}$$

## Exercise 81

$$f(x) = 2 - 3x$$

$$f'(x) = -3$$

Using the definition of the second derivative,

$$f''(x) = \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h} = \lim_{h \to 0} \frac{-3 - (-3)}{h} = 0.$$

**Final answer:** $f''(x) = 0$

## Exercise 82

$$f(x) = 4x^2$$

$$f'(x) = 8x$$

$$f''(x) = \lim_{h \to 0} \frac{8(x+h) - 8x}{h} = \lim_{h \to 0} 8 = 8.$$

**Final answer:** $f''(x) = 8$

**Exercise 83**

$$f(x) = x + \frac{1}{x}$$

$$f'(x) = 1 - \frac{1}{x^2}$$

$$f''(x) = \lim_{h \to 0} \frac{\left(1 - \frac{1}{(x+h)^2}\right) - \left(1 - \frac{1}{x^2}\right)}{h} = \frac{2}{x^3}.$$

**Final answer:** $f''(x) = \dfrac{2}{x^3}$

**Exercise 84**

$$f(x) = -\frac{5}{x}$$

$$f'(x) = \frac{5}{x^2}.$$

**Final answer:** $f'(x) = \dfrac{5}{x^2}$

**Exercise 85**

$$f(x) = 3x^2 + 2x + 4$$

$$f'(x) = 6x + 2.$$

**Final answer:** $f'(x) = 6x + 2$

**Exercise 86**

$$f(x) = \sqrt{x} + 3x$$

$$f'(x) = \frac{1}{2\sqrt{x}} + 3.$$

**Final answer:** $f'(x) = \dfrac{1}{2\sqrt{x}} + 3$

**Exercise 87**

$$f(x) = \frac{1}{\sqrt{2x}}$$
$$f'(x) = -\frac{1}{(2x)^{3/2}}.$$

**Final answer:** $f'(x) = -\dfrac{1}{(2x)^{3/2}}$

**Exercise 88**

$$f(x) = 1 + x + \frac{1}{x}$$
$$f'(x) = 1 - \frac{1}{x^2}.$$

**Final answer:** $f'(x) = 1 - \dfrac{1}{x^2}$

**Exercise 89**

$$f(x) = x^3 + 1$$
$$f'(x) = 3x^2.$$

**Final answer:** $f'(x) = 3x^2$

**Exercise 90**

$$\frac{f(x+h) - f(x)}{h}$$

represents the *average rate of change* of the population over $h$ years, measured in people per year.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

represents the *instantaneous rate of population change* at time $x$, in people per year.

**Exercise 91**

$$\frac{C(x+h)-C(x)}{h}$$

is the average money spent per customer, in thousands of dollars per customer.

$$C'(x)$$

is the instantaneous spending rate per additional customer, in thousands of dollars per customer.

**Exercise 92**

$$\frac{R(x+h)-R(x)}{h}$$

is the average manufacturing cost per radio, in thousands of dollars per radio.

$$R'(x)$$

is the marginal cost of producing one more radio, in thousands of dollars per radio.

**Exercise 93**

$$\frac{g(x+h)-g(x)}{h}$$

is the average increase in test score per hour of study, in percentage points per hour.

$$g'(x)$$

is the instantaneous rate of score improvement per hour of study, in percentage points per hour.

**Exercise 94**

$$\frac{B(x+h)-B(x)}{h}$$

is the average rate of textbook price change per year, in dollars per year.

$$B'(x)$$

is the instantaneous rate of change of textbook price, in dollars per year.

### Exercise 95

$$\frac{p(x+h)-p(x)}{h}$$

is the average change in atmospheric pressure per foot of altitude, in pressure units per foot.

$$p'(x)$$

is the instantaneous rate of pressure change with altitude, in pressure units per foot.

### Exercise 96

One possible sketch is a function that increases on $[-2, 1)$, has a sharp corner at $x = 1$, reaches a minimum at $x = 2$, then increases afterward.

**Key features:**

- Increasing on $[-2, 1)$
- Not differentiable at $x = 1$
- Local minimum at $(2, 2)$
- $f(0) = 1$
- $\lim_{x\to-\infty} f(x) = 0$ and $\lim_{x\to\infty} f(x) = \infty$

### Exercise 97

**(a)** $T'(x)$ represents the rate of change of temperature with respect to height, measured in degrees Fahrenheit per foot.

**(b)** $T'(1000) = -0.1$ means that at 1000 feet above ground, temperature is decreasing at a rate of 0.1°F per foot.

### Exercise 98

**(a)**

$$\frac{P(b)-P(a)}{b-a}$$

measures the average profit per item sold between $a$ and $b$, in thousand dollars per item.

**(b)**

$$P'(x)$$

measures the marginal profit per additional item sold, in thousand dollars per item.

**(c)** If $P'(30) = 5$, then increasing sales from 30 to 31 units increases profit by approximately

$$5 \text{ thousand dollars.}$$

### Exercise 99

Let $N(t)$ denote the number of people who have the flu $t$ weeks after the outbreak.

**(a)** $N'(t)$ represents the *rate at which new people are becoming infected* at time $t$, measured in people per week. From the graph, $N'(t)$ is initially increasing (the outbreak accelerates), then reaches a maximum, and eventually decreases toward 0 as the number of new cases slows.

**(b)** The derivative shows that the outbreak spreads rapidly at first, then slows down as fewer susceptible individuals remain. Eventually, the town approaches a steady state where almost no new infections occur.

### Exercise 100

The function $h(t)$ represents the height of the rocket in meters after $t$ seconds.

**Physical meaning of $h'(t)$:** $h'(t)$ is the *vertical velocity* of the rocket at time $t$.

**Units:** meters per second (m/s).

### Exercise 101

Using the table values, $h'(t)$ is estimated by

$$h'(t) \approx \frac{h(t + \Delta t) - h(t)}{\Delta t}.$$

At interior points of the interval, both left-hand and right-hand differences are computed and averaged to estimate $h'(t)$.

**Graphical interpretation:** On the same axes, $h(t)$ is increasing and concave up, while $h'(t)$ is positive and increasing, representing increasing upward velocity.

### Exercise 102

Given

$$H(t) = 7.229t - 4.905,$$

$$H'(t) = 7.229.$$

**Interpretation:** The rocket has constant upward velocity.

**Units:** meters per second.

**Graph description:** $H(t)$ is a straight line; $H'(t)$ is a horizontal line at $y = 7.229$.

### Exercise 103

Given

$$G(t) = 1.429t^2 + 0.0857t - 0.1429,$$

$$G'(t) = 2.858t + 0.0857.$$

**Interpretation:** The rocket's velocity increases linearly with time.

**Graph description:** $G(t)$ is a parabola opening upward; $G'(t)$ is a straight line.

### Exercise 104

Given

$$F(t) = 0.2037t^3 + 2.956t^2 - 2.705t + 0.4683,$$

$$F'(t) = 0.6111t^2 + 5.912t - 2.705.$$

**Interpretation:** The velocity changes nonlinearly, indicating varying acceleration.

**Conclusion:** The cubic model fits the data best because it allows for changing acceleration.

### Exercise 105

**Physical meanings:**

- $H'(t)$: constant velocity (m/s)
- $G'(t)$: velocity changing at a constant rate (m/s)
- $F'(t)$: velocity changing at a variable rate (m/s)

**Units:** meters per second for all derivatives.

**Conclusion:** $F'(t)$ best represents the rocket's true velocity because real motion involves changing acceleration.

**Differentiation Rules**

## Exercise 106

$$f(x) = x^7 + 10$$
$$f'(x) = 7x^6.$$

**Final answer:** $f'(x) = 7x^6$

## Exercise 107

$$f(x) = 5x^3 - x + 1$$
$$f'(x) = 15x^2 - 1.$$

**Final answer:** $f'(x) = 15x^2 - 1$

## Exercise 108

$$f(x) = 4x^2 - 7x$$
$$f'(x) = 8x - 7.$$

**Final answer:** $f'(x) = 8x - 7$

## Exercise 109

$$f(x) = 8x^4 + 9x^2 - 1$$
$$f'(x) = 32x^3 + 18x.$$

**Final answer:** $f'(x) = 32x^3 + 18x$

## Exercise 110

$$f(x) = x^4 + \frac{2}{x}$$
$$f'(x) = 4x^3 - \frac{2}{x^2}.$$

**Final answer:** $f'(x) = 4x^3 - \dfrac{2}{x^2}$

**Exercise 111**

$$f(x) = 3x\left(18x^4 + \frac{13}{x+1}\right)$$

First expand:

$$f(x) = 54x^5 + \frac{39x}{x+1}.$$

Differentiate:

$$f'(x) = 270x^4 + \frac{39(x+1) - 39x}{(x+1)^2} = 270x^4 + \frac{39}{(x+1)^2}.$$

**Final answer:** $f'(x) = 270x^4 + \dfrac{39}{(x+1)^2}$

**Exercise 112**

$$f(x) = (x+2)(2x^2 - 3)$$

Expand:

$$f(x) = 2x^3 + 4x^2 - 3x - 6.$$

$$f'(x) = 6x^2 + 8x - 3.$$

**Final answer:** $f'(x) = 6x^2 + 8x - 3$

**Exercise 113**

$$f(x) = x^2\left(\frac{2}{x^2} + \frac{5}{x^3}\right)$$

Simplify:

$$f(x) = 2 + \frac{5}{x}.$$

$$f'(x) = -\frac{5}{x^2}.$$

**Final answer:** $f'(x) = -\dfrac{5}{x^2}$

**Exercise 114**

$$f(x) = \frac{x^3 + 2x^2 - 4}{3}$$

$$f'(x) = \frac{1}{3}(3x^2 + 4x) = x^2 + \frac{4}{3}x.$$

**Final answer:** $f'(x) = x^2 + \frac{4}{3}x$

**Exercise 115**

$$f(x) = \frac{4x^3 - 2x + 1}{x^2}$$

Rewrite:

$$f(x) = 4x - \frac{2}{x} + \frac{1}{x^2}.$$

$$f'(x) = 4 + \frac{2}{x^2} - \frac{2}{x^3}.$$

**Final answer:** $f'(x) = 4 + \frac{2}{x^2} - \frac{2}{x^3}$

**Exercise 116**

$$f(x) = \frac{x^2 + 4}{x^2 - 4}$$

Using the quotient rule:

$$f'(x) = \frac{(2x)(x^2 - 4) - (x^2 + 4)(2x)}{(x^2 - 4)^2} = \frac{-16x}{(x^2 - 4)^2}.$$

**Final answer:** $f'(x) = -\frac{16x}{(x^2 - 4)^2}$

**Exercise 117**

$$f(x) = \frac{x + 9}{x^2 - 7x + 1}$$

$$f'(x) = \frac{(x^2 - 7x + 1) - (x + 9)(2x - 7)}{(x^2 - 7x + 1)^2}.$$

**Final answer:**

$$f'(x) = \frac{x^2 - 7x + 1 - (x + 9)(2x - 7)}{(x^2 - 7x + 1)^2}$$

**Exercise 118**

$$y = 3x^2 + 4x + 1, \quad (0, 1)$$

$$y' = 6x + 4, \quad y'(0) = 4.$$

Tangent line:

$$y - 1 = 4(x - 0) \Rightarrow y = 4x + 1.$$

**Final answer:** $y = 4x + 1$

## Exercise 119

$$y = 2\sqrt{x} + 1, \quad (4, 5)$$
$$y' = \frac{1}{\sqrt{x}}, \quad y'(4) = \frac{1}{2}.$$
$$y - 5 = \frac{1}{2}(x - 4).$$

**Final answer:** $y = \frac{1}{2}x + 3$

## Exercise 120

$$y = \frac{2x}{x - 1}, \quad (-1, 1)$$
$$y' = \frac{2(x - 1) - 2x}{(x - 1)^2} = -\frac{2}{(x - 1)^2}.$$
$$y'(-1) = -\frac{1}{2}.$$
$$y - 1 = -\frac{1}{2}(x + 1).$$

**Final answer:** $y = -\frac{1}{2}x + \frac{1}{2}$

## Exercise 121

$$y = \frac{2}{x} - \frac{3}{x^2}, \quad (1, -1)$$
$$y' = -\frac{2}{x^2} + \frac{6}{x^3}, \quad y'(1) = 4.$$
$$y + 1 = 4(x - 1).$$

**Final answer:** $y = 4x - 5$

**Exercise 122**

$$h(x) = 4f(x) + \frac{g(x)}{7}$$
$$h'(x) = 4f'(x) + \frac{g'(x)}{7}.$$

**Final answer:** $h'(x) = 4f'(x) + \dfrac{g'(x)}{7}$

**Exercise 123**

$$h(x) = x^3 f(x)$$

Using the product rule:
$$h'(x) = 3x^2 f(x) + x^3 f'(x).$$

**Final answer:** $h'(x) = 3x^2 f(x) + x^3 f'(x)$

**Exercise 124**

$$h(x) = \frac{f(x)g(x)}{2}$$
$$h'(x) = \frac{1}{2}\big(f'(x)g(x) + f(x)g'(x)\big).$$

**Final answer:** $h'(x) = \dfrac{f'(x)g(x) + f(x)g'(x)}{2}$

**Exercise 125**

$$h(x) = \frac{3f(x)}{g(x)+2}$$
$$h'(x) = \frac{3f'(x)(g(x)+2) - 3f(x)g'(x)}{(g(x)+2)^2}.$$

**Final answer:**

$$h'(x) = \frac{3\big(f'(x)(g(x)+2) - f(x)g'(x)\big)}{(g(x)+2)^2}$$

**Exercise 126**

$$h(x) = xf(x) + 4g(x)$$

$$h'(x) = f(x) + xf'(x) + 4g'(x).$$

At $x = 1$:

$$h'(1) = 3 + 1(-1) + 4(4) = 18.$$

**Final answer:** $h'(1) = 18$

## Exercise 127

$$h(x) = \frac{f(x)}{g(x)}$$

$$h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

At $x = 2$:

$$h'(2) = \frac{7(3) - 5(1)}{3^2} = \frac{16}{9}.$$

**Final answer:** $h'(2) = \frac{16}{9}$

## Exercise 128

$$h(x) = 2x + f(x)g(x)$$

$$h'(x) = 2 + f'(x)g(x) + f(x)g'(x).$$

At $x = 3$:

$$h'(3) = 2 + 8(-4) + (-2)(2) = -34.$$

**Final answer:** $h'(3) = -34$

## Exercise 129

$$h(x) = \frac{1}{x} + \frac{g(x)}{f(x)}$$

$$h'(x) = -\frac{1}{x^2} + \frac{g'(x)f(x) - g(x)f'(x)}{f(x)^2}.$$

At $x = 4$:

$$h'(4) = -\frac{1}{16} + \frac{9(0) - 6(-3)}{0^2}.$$

Since $f(4) = 0$, the derivative is undefined.

**Final answer:** $h'(4)$ does not exist.

### Exercise 130

Let $h(x) = f(x) + g(x)$. Since both functions are differentiable at the given points (except where noted),

$$h'(x) = f'(x) + g'(x).$$

From the graph:

- For $0 < x < 3$, $f(x)$ is linear from $(0, 5)$ to $(3, 1)$ with slope

$$f'(x) = \frac{1-5}{3-0} = -\frac{4}{3}.$$

- For $3 < x < 5$, $f(x)$ is linear from $(3, 1)$ to $(5, 3)$ with slope

$$f'(x) = \frac{3-1}{5-3} = 1.$$

- At $x = 3$, $f$ has a sharp corner, so $f'(3)$ does not exist.
- For $0 < x < 2$, $g(x)$ is linear from $(0, 0)$ to $(2, 2)$ with slope 1.
- For $x > 2$, $g(x)$ is constant, so $g'(x) = 0$.

**(a)** At $x = 1$:

$$h'(1) = f'(1) + g'(1) = -\frac{4}{3} + 1 = -\frac{1}{3}.$$

**(b)** At $x = 3$: Since $f'(3)$ does not exist,

$$h'(3) \text{ does not exist.}$$

**(c)** At $x = 4$:

$$h'(4) = f'(4) + g'(4) = 1 + 0 = 1.$$

**Final answer:**

$$h'(1) = -\frac{1}{3}, \qquad h'(3) \text{ DNE}, \qquad h'(4) = 1$$

### Exercise 131

Let $h(x) = f(x)g(x)$. Then

$$h'(x) = f'(x)g(x) + f(x)g'(x).$$

From the graph, the function values are:

$$f(1) = \frac{11}{3}, \quad g(1) = 1, \qquad f(3) = 1, \quad g(3) = 2.5, \qquad f(4) = 2, \quad g(4) = 2.5.$$

**(a)** At $x = 1$:

$$h'(1) = \left(-\frac{4}{3}\right)(1) + \left(\frac{11}{3}\right)(1) = \frac{7}{3}.$$

**(b)** At $x = 3$: Since $f'(3)$ does not exist,

$$h'(3) \text{ does not exist.}$$

**(c)** At $x = 4$:

$$h'(4) = 1(2.5) + 2(0) = 2.5.$$

**Final answer:**

$$h'(1) = \frac{7}{3}, \qquad h'(3) \text{ DNE}, \qquad h'(4) = 2.5$$

## Exercise 132

Let

$$h(x) = \frac{f(x)}{g(x)}.$$

Then

$$h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}.$$

**(a)** At $x = 1$:

$$h'(1) = \frac{\left(-\frac{4}{3}\right)(1) - \left(\frac{11}{3}\right)(1)}{1^2} = -5.$$

**(b)** At $x = 3$: Since $f'(3)$ does not exist,

$$h'(3) \text{ does not exist.}$$

**(c)** At $x = 4$:

$$h'(4) = \frac{(1)(2.5) - 2(0)}{(2.5)^2} = \frac{2.5}{6.25} = 0.4.$$

**Final answer:**

$$h'(1) = -5, \qquad h'(3) \text{ DNE}, \qquad h'(4) = 0.4$$

**Exercise 133**

$$f(x) = 2x^3 + 3x - x^2, \quad a = 2$$

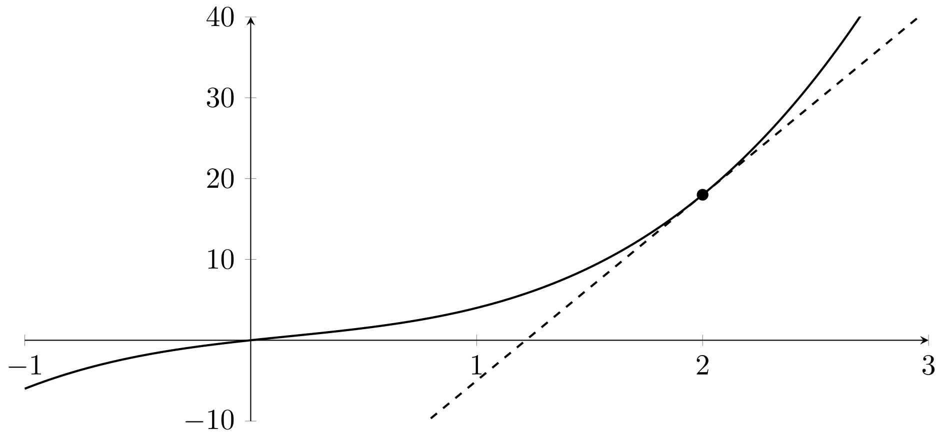
$$f'(x) = 6x^2 + 3 - 2x$$

$$f'(2) = 6(2)^2 + 3 - 2(2) = 24 + 3 - 4 = 23, \qquad f(2) = 2(2)^3 + 3(2) - (2)^2 = 16 + 6 - 4 = 18$$

Tangent line at $x = 2$:

$$y - 18 = 23(x - 2) \quad \Rightarrow \quad y = 23x - 28$$

**Final answer:** $f'(2) = 23$, tangent line $y = 23x - 28$




**Exercise 134**

$$f(x) = \frac{1}{x} - x^2, \quad a = 1$$

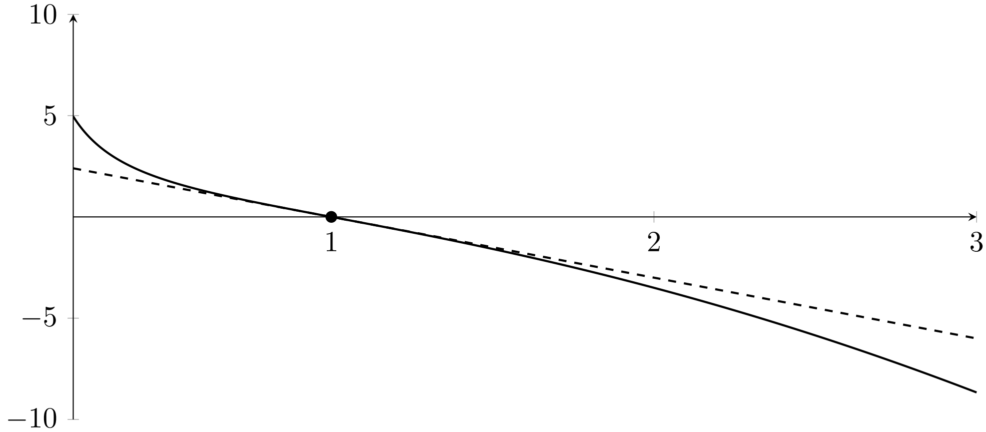
$$f'(x) = -\frac{1}{x^2} - 2x$$

$$f'(1) = -1 - 2 = -3, \qquad f(1) = 1 - 1 = 0$$

Tangent line at $x = 1$:

$$y - 0 = -3(x - 1) \quad \Rightarrow \quad y = -3x + 3$$

**Final answer:** $f'(1) = -3$, tangent line $y = -3x + 3$




**Exercise 135**

$$f(x) = x^2 - x^{12} + 3x + 2, \quad a = 0$$

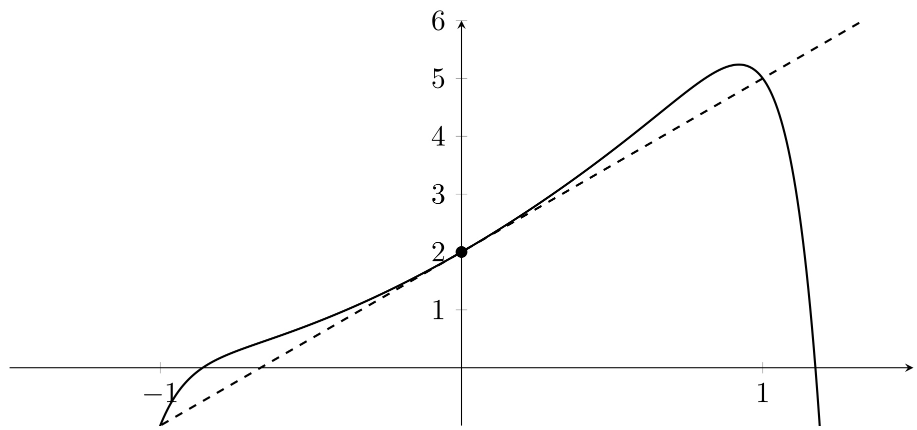
$$f'(x) = 2x - 12x^{11} + 3$$

$$f'(0) = 3, \qquad f(0) = 2$$

Tangent line at $x = 0$:

$$y - 2 = 3(x - 0) \quad \Rightarrow \quad y = 3x + 2$$

**Final answer:** $f'(0) = 3$, tangent line $y = 3x + 2$




**Exercise 136**

$$f(x) = \frac{1}{x} - x^{2/3}, \quad a = -1$$

$$f'(x) = -\frac{1}{x^2} - \frac{2}{3}x^{-1/3}$$

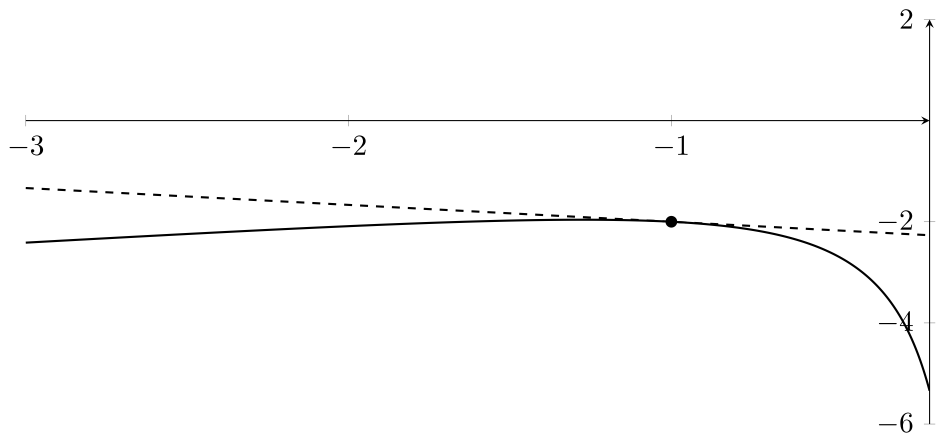
Since $(-1)^{-1/3} = -1$,

$$f'(-1) = -1 - \frac{2}{3}(-1) = -1 + \frac{2}{3} = -\frac{1}{3}, \qquad f(-1) = \frac{1}{-1} - (-1)^{2/3} = -1 - 1 = -2$$

Tangent line at $x = -1$:

$$y + 2 = -\frac{1}{3}(x+1) \quad \Rightarrow \quad y = -\frac{1}{3}x - \frac{7}{3}$$

**Final answer:** $f'(-1) = -\frac{1}{3}$, tangent line $y = -\frac{1}{3}x - \frac{7}{3}$




### Exercise 137

$$f(x) = 2x^3 + 4x^2 - 5x - 3$$

$$f'(x) = 6x^2 + 8x - 5$$

At $x = -1$:

$$f'(-1) = 6(1) - 8 - 5 = -7, \qquad f(-1) = -2 + 4 + 5 - 3 = 4$$

Tangent line:

$$y - 4 = -7(x+1) \quad \Rightarrow \quad y = -7x - 3$$

**Final answer:** $\boxed{y = -7x - 3}$

### Exercise 138

$$f(x) = x^2 + \frac{4}{x} - 10$$

$$f'(x) = 2x - \frac{4}{x^2}$$

At $x = 8$:

$$f'(8) = 16 - \frac{4}{64} = \frac{255}{16}, \qquad f(8) = 64 + \frac{1}{2} - 10 = \frac{109}{2}$$

Tangent line:

$$y - \frac{109}{2} = \frac{255}{16}(x - 8)$$

**Final answer:** $\boxed{y = \frac{255}{16}x - \frac{401}{2}}$

## Exercise 139

$$f(x) = (3x - x^2)(3 - x - x^2)$$

Differentiate using the product rule:

$$f'(x) = (3 - 2x)(3 - x - x^2) + (3x - x^2)(-1 - 2x)$$

At $x = 1$:

$$f'(1) = (1)(1) + (2)(-3) = 1 - 6 = -5, \qquad f(1) = 2(1) = 2$$

Tangent line:

$$y - 2 = -5(x - 1) \quad \Rightarrow \quad y = -5x + 7$$

**Final answer:** $\boxed{y = -5x + 7}$

## Exercise 140

$$f(x) = x^3, \quad f'(x) = 3x^2$$

Let the point be $(a, a^3)$. Tangent line:

$$y - a^3 = 3a^2(x - a)$$

The $x$-intercept occurs when $y = 0$:

$$-a^3 = 3a^2(x - a) \Rightarrow x = \frac{2a}{3}$$

Given the $x$-intercept is 6:

$$\frac{2a}{3} = 6 \Rightarrow a = 9$$

**Final answer:** $\boxed{(9, 729)}$

**Exercise 141**

$$f(x) = \frac{6}{x-1}$$

$$f'(x) = -\frac{6}{(x-1)^2}$$

At $x = 3$:

$$f'(3) = -\frac{6}{4} = -\frac{3}{2}$$

Line through $(3, 3)$:

$$y - 3 = -\frac{3}{2}(x - 3)$$

**Final answer:** $\boxed{y = -\frac{3}{2}x + \frac{15}{2}}$

**Exercise 142**

$$f(x) = x^3 + x^2 - x - 1$$

$$f'(x) = 3x^2 + 2x - 1$$

**(a) Horizontal tangents:**

$$3x^2 + 2x - 1 = 0 \Rightarrow x = \frac{1}{3}, -1$$

Points:

$$\left(\frac{1}{3}, -\frac{32}{27}\right), \quad (-1, 0)$$

**(b) Slope $-1$:**

$$3x^2 + 2x - 1 = -1 \Rightarrow 3x^2 + 2x = 0 \Rightarrow x = 0, -\frac{2}{3}$$

Points:

$$(0, -1), \quad \left(-\frac{2}{3}, \frac{1}{27}\right)$$

**Final answer:** Horizontal at $x = -1,\ \frac{1}{3}$; slope $-1$ at $x = 0,\ -\frac{2}{3}$.

**Exercise 143**

Let

$$f(x) = ax^2 + bx + c$$

Given:

$$f(1) = 5, \quad f'(1) = 3, \quad f''(1) = -6$$

$$f'(x) = 2ax + b, \quad f''(x) = 2a$$

From $f''(1) = -6$:

$$2a = -6 \Rightarrow a = -3$$

From $f'(1) = 3$:

$$-6 + b = 3 \Rightarrow b = 9$$

From $f(1) = 5$:

$$-3 + 9 + c = 5 \Rightarrow c = -1$$

**Final answer:** $\boxed{f(x) = -3x^2 + 9x - 1}$

### Exercise 144

$$s(t) = t^3 - 6t^2 + 9t$$

Velocity:

$$v(t) = s'(t) = 3t^2 - 12t + 9$$

**(a)** Set $v(t) = 0$:

$$3(t^2 - 4t + 3) = 0 \Rightarrow t = 1,\ 3$$

**(b)** Acceleration:

$$a(t) = v'(t) = 6t - 12$$

At $t = 1$: $a(1) = -6$ At $t = 3$: $a(3) = 6$

**Final answer:** Velocity is zero at $t = 1, 3$; acceleration is $-6$ m/s$^2$ at $t = 1$ and 6 m/s$^2$ at $t = 3$.

### Exercise 145

The position of the herring is

$$s(t) = \frac{t^2}{t^2 + 2}.$$

Differentiate using the quotient rule:

$$s'(t) = \frac{(2t)(t^2 + 2) - t^2(2t)}{(t^2 + 2)^2} = \frac{4t}{(t^2 + 2)^2}.$$

At $t = 3$:

$$s'(3) = \frac{4(3)}{(9 + 2)^2} = \frac{12}{121}.$$

**Final answer:** The velocity at $t = 3$ seconds is $\dfrac{12}{121}$ ft/s.

**Exercise 146**

The population (in millions) is
$$P(t) = \frac{8t+3}{0.2t^2+1}.$$

**(a)** Initial population:
$$P(0) = \frac{3}{1} = 3.$$

So the initial population is 3 million flounder.

**(b)** Differentiate using the quotient rule:

$$P'(t) = \frac{8(0.2t^2+1) - (8t+3)(0.4t)}{(0.2t^2+1)^2}.$$

At $t = 10$:
$$P'(10) = \frac{8(21) - (83)(4)}{21^2} = \frac{168 - 332}{441} = -\frac{164}{441} \approx -0.372.$$

**Final answer:** $P(0) = 3$ million. $P'(10) \approx -0.372$ million flounder per year, meaning the population is decreasing at year 10.

**Exercise 147**

The concentration function is
$$C(t) = \frac{2t^2+t}{t^3+50}.$$

**(a)** Differentiate:
$$C'(t) = \frac{(4t+1)(t^3+50) - (2t^2+t)(3t^2)}{(t^3+50)^2}.$$

**(b)** Evaluate:

$$C'(8) \approx 0.0097, \quad C'(12) \approx 0.0018, \quad C'(24) \approx 0.0002, \quad C'(36) \approx 0.00005.$$

**(c)** Interpretation: As time increases, the rate of change of concentration approaches 0. The antibiotic concentration stabilizes and changes very slowly.

**Final answer:** The concentration increases at first, then levels off over time.

## Exercise 148

The cost per book is

$$C(x) = \frac{x^3 + 2x + 3}{x^2}.$$

Rewrite:

$$C(x) = x + \frac{2}{x} + \frac{3}{x^2}.$$

Differentiate:

$$C'(x) = 1 - \frac{2}{x^2} - \frac{6}{x^3}.$$

At $x = 2$:

$$C'(2) = 1 - \frac{2}{4} - \frac{6}{8} = 1 - \frac{1}{2} - \frac{3}{4} = -\frac{1}{4}.$$

**Final answer:** $C'(2) = -0.25$. This means that when 2000 books are produced, the cost per book is decreasing by \$0.25 for each additional thousand books produced.

## Exercise 149

The gravitational force is

$$F = \frac{Gm_1m_2}{d^2}.$$

**(a)** Differentiate with respect to $d$:

$$\frac{dF}{dd} = -\frac{2Gm_1m_2}{d^3}.$$

**(b)** Substitute values:

$$G = 6.67 \times 10^{-11}, \quad m_1 = m_2 = 1000, \quad d = 10.$$

$$\frac{dF}{dd} = -\frac{2(6.67 \times 10^{-11})(10^6)}{10^3} = -1.334 \times 10^{-7}.$$

**Final answer:**

$$\frac{dF}{dd} = -1.334 \times 10^{-7} \text{ N/m}.$$

The force decreases as the distance between the bodies increases.

**Derivatives as Rates of Change**

## Exercise 150

$$s(t) = 2t^3 - 3t^2 - 12t + 8$$

Velocity:

$$v(t) = s'(t) = 6t^2 - 6t - 12$$

Acceleration:

$$a(t) = v'(t) = 12t - 6$$

Speeding up occurs when $v(t)$ and $a(t)$ have the same sign.

Critical points:

$$v(t) = 6(t-2)(t+1) = 0 \Rightarrow t = -1, 2$$

$$a(t) = 0 \Rightarrow t = \tfrac{1}{2}$$

Sign analysis gives: - Speeding up on $(-1, \frac{1}{2}) \cup (2, \infty)$ - Slowing down on $(-\infty, -1) \cup (\frac{1}{2}, 2)$

## Exercise 151

$$s(t) = 2t^3 - 15t^2 + 36t - 10$$

Velocity:

$$v(t) = 6t^2 - 30t + 36 = 6(t-2)(t-3)$$

Acceleration:

$$a(t) = 12t - 30$$

Critical points:

$$v(t) = 0 \Rightarrow t = 2, 3, \qquad a(t) = 0 \Rightarrow t = \tfrac{5}{2}$$

Speeding up on $(2, \frac{5}{2}) \cup (3, \infty)$ Slowing down on $(-\infty, 2) \cup (\frac{5}{2}, 3)$

## Exercise 152

$$s(t) = \frac{t}{1+t^2}$$

Velocity:

$$v(t) = \frac{1-t^2}{(1+t^2)^2}$$

Acceleration:

$$a(t) = \frac{-2t(3-t^2)}{(1+t^2)^3}$$

Speeding up where $v(t)$ and $a(t)$ have the same sign:

$$(-\sqrt{3}, -1) \cup (0, 1) \cup (\sqrt{3}, \infty)$$

Slowing down elsewhere.

### Exercise 153

$$s(t) = -16t^2 + 560t$$

Velocity:

$$v(t) = -32t + 560$$

Acceleration:

$$a(t) = -32$$

At $t = 3$:

$$v(3) = 464 \text{ ft/s}, \qquad a(3) = -32 \text{ ft/s}^2$$

### Exercise 154

$$s(t) = -16t^2 - 8t + 64$$

**(a)** Ground hit when $s(t) = 0$:

$$-16t^2 - 8t + 64 = 0 \Rightarrow t = 2$$

**(b)** Velocity:

$$v(t) = -32t - 8 \Rightarrow v(2) = -72 \text{ ft/s}$$

### Exercise 155

$$s(t) = t^2 - 3t - 4$$

Velocity:

$$v(t) = 2t - 3$$

**(a)** When $s(t) = 0$:

$$t^2 - 3t - 4 = 0 \Rightarrow t = 4$$

$$v(4) = 5 \text{ ft/s}$$

**(b)** When $s(t) = 14$:

$$t^2 - 3t - 18 = 0 \Rightarrow t = 6$$

$$v(6) = 9 \text{ ft/s}$$

## Exercise 156

$$s(t) = 3t^3 - 7t$$

Velocity:

$$v(t) = 9t^2 - 7$$

Acceleration:

$$a(t) = 18t$$

**(a)** Velocity at $t = 1$:

$$v(1) = 2 \text{ m/s}$$

**(b)** Acceleration at $t = 1$:

$$a(1) = 18 \text{ m/s}^2$$

**(c)** Velocity $= 0$:

$$9t^2 - 7 = 0 \Rightarrow t = \sqrt{\frac{7}{9}}$$

$$a\left(\sqrt{\frac{7}{9}}\right) = 18\sqrt{\frac{7}{9}}$$

**Final Answer:** Acceleration is positive when velocity is zero.

## Exercise 157

The position function is

$$s(t) = -16t^2 + 100t + 85.$$

Velocity:

$$v(t) = s'(t) = -32t + 100.$$

Acceleration:

$$a(t) = v'(t) = -32.$$

**(a) Velocity**

$$v(0.5) = -32(0.5) + 100 = 84 \text{ ft/s},$$

$$v(5.75) = -32(5.75) + 100 = -84 \text{ ft/s}.$$

**(b) Speed** (magnitude of velocity)

$$\text{Speed at } 0.5 = 84 \text{ ft/s}, \qquad \text{Speed at } 5.75 = 84 \text{ ft/s}.$$

**(c) Maximum height**

Occurs when $v(t) = 0$:

$$-32t + 100 = 0 \Rightarrow t = \frac{25}{8} = 3.125 \text{ s}.$$

**(d) Acceleration**

$$a(0.5) = -32 \text{ ft/s}^2, \qquad a(1.5) = -32 \text{ ft/s}^2.$$

**(e) Time in the air**

Ground is reached when $s(t) = 0$:

$$-16t^2 + 100t + 85 = 0 \Rightarrow t = \frac{100 + \sqrt{15440}}{32} \approx 7.17 \text{ s}.$$

**(f) Velocity upon hitting the ground**

$$v(7.17) \approx -129.4 \text{ ft/s}.$$

**Final answers:**

$$\begin{aligned} &v(0.5) = 84, \quad v(5.75) = -84, \\ &\text{Max height at } t = 3.125 \text{ s}, \\ &\text{Time in air } \approx 7.17 \text{ s}, \\ &v_{\text{impact}} \approx -129.4 \text{ ft/s}. \end{aligned}$$

## Exercise 158

$$s(t) = t^3 - 8t.$$

Velocity:

$$v(t) = 3t^2 - 8.$$

Acceleration:

$$a(t) = 6t.$$

**(a) Direction when $s(t) = 0$**

$$t(t^2 - 8) = 0 \Rightarrow t = 0,\ \sqrt{8}.$$

At $t = \sqrt{8}$:

$$v(\sqrt{8}) = 16 > 0 \Rightarrow \text{east}.$$

**(b) Direction when $a(t) = 0$**

$$a(t) = 0 \Rightarrow t = 0, \quad v(0) = -8 < 0 \Rightarrow \text{west}.$$

**(c) Speeding up vs slowing down**

Critical points:

$$v(t) = 0 \Rightarrow t = \pm\sqrt{\tfrac{8}{3}}, \qquad a(t) = 0 \Rightarrow t = 0.$$

Speeding up when $v$ and $a$ have the same sign:

$$(-\sqrt{\tfrac{8}{3}}, 0) \cup (\sqrt{\tfrac{8}{3}}, \infty).$$

Slowing down elsewhere.

### Exercise 159

**(a) Velocity sign from the position graph**

From the graph:

$$\begin{aligned} &v(t) > 0 \text{ on } (0,1) \cup (6,7),\\ &v(t) = 0 \text{ on } [2,5],\\ &v(t) < 0 \text{ on } (1,2) \cup (5,6). \end{aligned}$$

**(b) Velocity graph**

The velocity graph is positive where $s(t)$ increases, negative where $s(t)$ decreases, and zero on flat segments.

**(c) Acceleration**

Acceleration is determined from the slope of $v(t)$:

$$\begin{aligned} &a(t) > 0 \text{ where } v(t) \text{ increases},\\ &a(t) < 0 \text{ where } v(t) \text{ decreases},\\ &a(t) = 0 \text{ where } v(t) \text{ is constant}. \end{aligned}$$

**(d) Speeding up vs slowing down**

Speeding up when $v$ and $a$ have the same sign:

$$(0,1) \cup (5,6) \cup (6,7).$$

Slowing down when $v$ and $a$ have opposite signs:

$$(1,2) \cup (2,5).$$

**Final summary:** The object speeds up when velocity and acceleration agree in sign, slows down otherwise, and remains at rest where velocity is zero.

### Exercise 160

$$C(x) = 200 + \frac{7}{x} + \frac{x^2}{7}$$

**(a) Marginal cost**

$$C'(x) = -\frac{7}{x^2} + \frac{2x}{7}$$

**(b) Estimated cost of the 13th processor**

$$C'(12) = -\frac{7}{144} + \frac{24}{7} = \frac{24}{7} - \frac{7}{144} \approx 3.38$$

**(c) Actual cost of the 13th processor**

$$C(13) - C(12) \approx 3.33$$

### Exercise 161

$$p = 10 - 0.001x$$

**(a) Revenue**

$$R(x) = xp = x(10 - 0.001x) = 10x - 0.001x^2$$

**(b) Marginal revenue**

$$R'(x) = 10 - 0.002x$$

**(c)**

$$R'(2000) = 6, \qquad R'(5000) = 0$$

## Exercise 162

$$P(x) = 30x - 0.3x^2 - 250$$

**(a) Exact profit from the 30th skateboard**

$$P(30) - P(29) = 12.7$$

**(b) Marginal profit**

$$P'(x) = 30 - 0.6x$$

$$P'(29) = 12.6$$

## Exercise 163

$$p = 143 - 0.03x, \qquad C(x) = 75000 + 65x$$

**(a) Marginal cost**

$$C'(x) = 65$$

**(b) Revenue**

$$R(x) = xp = x(143 - 0.03x) = 143x - 0.03x^2$$

$$R'(x) = 143 - 0.06x$$

**(c)**

$$R'(1000) = 83, \qquad R'(4000) = -97$$

**(d) Profit**

$$P(x) = R(x) - C(x)$$

$$P'(x) = R'(x) - C'(x) = 78 - 0.06x$$

**(e)**

$$P'(1000) = 18, \qquad P'(4000) = -162$$

## Exercise 164

$$P(t) = -\frac{1}{3}t^3 + 64t + 3000$$

**(a)**

$$P'(t) = -t^2 + 64$$

**(b)**

$$P'(1) = 63, \quad P'(2) = 60, \quad P'(3) = 55, \quad P'(4) = 48$$

Population is increasing but at a decreasing rate.

**(c)**

$$P''(t) = -2t$$

$$P''(1) = -2,\ P''(2) = -4,\ P''(3) = -6,\ P''(4) = -8$$

Negative acceleration indicates slowing population growth.

### Exercise 165

$$N(t) = 3000\left(1 + \frac{4t}{t^2 + 100}\right)$$

**(a)**

$$N'(t) = \frac{12000(100 - t^2)}{(t^2 + 100)^2}$$

**(b)**

$$N'(0) = 120, \quad N'(10) = 60, \quad N'(20) = -48, \quad N'(30) = -96$$

**(c)** Growth increases initially, peaks, then declines.

**(d)**

$$N''(t) = \frac{-24000t(t^2 - 300)}{(t^2 + 100)^3}$$

### Exercise 166

$$F(r) = \frac{mv^2}{r}$$

**(a)**

$$\frac{dF}{dr} = -\frac{mv^2}{r^2}$$

**(b)**

$$\frac{dF}{dr} = -\frac{(1000)(13.89)^2}{200^2} \approx -4.82 \text{ N/m}$$

**Final interpretation:** Centripetal force decreases as distance from the center increases.

### Exercise 167

Population of London data (years since 1800, population in millions).

**(a) Best–fit linear model**

Using linear regression on the data,

$$P(t) \approx 0.0399t + 0.661$$

where $t$ is years since 1800 and $P(t)$ is population (millions).

**(b) First derivative and meaning**

$$P'(t) = 0.0399$$

This represents a constant population growth rate of approximately

$$0.0399 \text{ million people per year } \approx 39{,}900 \text{ people/year.}$$

**(c) Second derivative and meaning**

$$P''(t) = 0$$

This indicates no acceleration in population growth; the linear model assumes a constant growth rate.

### Exercise 168

**(a) Best–fit quadratic model**

Using quadratic regression,

$$P(t) \approx 0.00033t^2 + 0.012t + 0.83$$

**(b) First derivative and meaning**

$$P'(t) = 0.00066t + 0.012$$

Population growth rate increases with time, indicating accelerating growth.

**(c) Second derivative and meaning**

$$P''(t) = 0.00066 > 0$$

The positive second derivative confirms accelerating population growth.

### Exercise 169

Falling sensor data (time in seconds, position in meters).

**(a) Best–fit quadratic position function**

Regression gives

$$s(t) \approx -0.56t^2 - 0.08t$$

**(b) First derivative and meaning**

$$v(t) = s'(t) = -1.12t - 0.08$$

This represents velocity (m/s). The velocity becomes more negative over time, meaning the object is speeding up downward.

**(c) Second derivative and meaning**

$$a(t) = s''(t) = -1.12$$

This constant negative acceleration indicates uniform downward acceleration (similar to gravity).

### Exercise 170

**(a) Best–fit cubic model**

A cubic regression gives

$$s(t) \approx -0.06t^3 - 0.02t^2 - 0.9t$$

**(b) First derivative and meaning**

$$v(t) = s'(t) = -0.18t^2 - 0.04t - 0.9$$

Velocity decreases nonlinearly with time.

**(c) Second derivative and meaning**

$$a(t) = s''(t) = -0.36t - 0.04$$

Acceleration changes with time, which is not physically realistic for free fall.

#### (d) Why a cubic model is inappropriate

Free-fall motion should have constant acceleration. Since the cubic model produces a time-varying acceleration, it does not accurately represent the physical situation.

### Exercise 171

Holling Type I equation:

$$f(x) = ax, \qquad a = 0.5$$

#### (a) Graph

The graph is a straight line through the origin with slope 0.5.

#### (b) First derivative and meaning

$$f'(x) = 0.5$$

This means the predator consumes prey at a constant rate per unit increase in prey availability.

#### (c) Second derivative and meaning

$$f''(x) = 0$$

No curvature; consumption increases linearly without saturation.

#### (d) Why Holling Type I may not be realistic

In reality, predators have handling time and saturation limits. A linear model ignores these biological constraints, making it unrealistic for high prey densities.

### Exercise 172

The Holling type II functional response is

$$f(x) = \frac{ax}{n+x}, \qquad a > 0.$$

#### (a) Graph for $a = 0.5,\ n = 5$

$$f(x) = \frac{0.5x}{5+x}$$

This curve increases rapidly for small $x$ and then levels off, approaching the horizontal asymptote

$$y = a = 0.5.$$

*Difference between Holling type I and II:* Type I ($f(x) = ax$) is linear and assumes unlimited consumption. Type II incorporates saturation: as prey increases, the predator's consumption rate approaches a maximum $a$.

**(b) First derivative and interpretation**

$$f'(x) = \frac{an}{(n+x)^2}.$$

This represents the marginal increase in the predator's consumption rate per additional unit of prey. As $x$ increases, $f'(x)$ decreases, reflecting predator saturation and handling time.

**(c) Show that** $f(n) = \frac{1}{2}a$

$$f(n) = \frac{an}{n+n} = \frac{a}{2}.$$

Thus, $n$ is the prey level at which the predator consumes at half of its maximum rate. *Interpretation:* $n$ is the half-saturation constant.

**(d) Second derivative and interpretation**

$$f''(x) = -\frac{2an}{(n+x)^3} < 0.$$

The function is concave down for all $x > 0$, meaning consumption increases at a decreasing rate. This makes the Holling type II function more realistic than type I.

## Exercise 173

The Holling type III functional response is

$$f(x) = \frac{ax^2}{n^2 + x^2}.$$

**(a) Graph for** $a = 0.5,\ n = 5$

$$f(x) = \frac{0.5x^2}{25 + x^2}.$$

This graph is sigmoidal: slow growth at low prey density, rapid increase at moderate levels, then saturation.

*Difference between Holling type II and III:* Type II rises quickly at low prey levels, while type III rises slowly, reflecting prey refuge or predator learning.

**(b) First derivative and interpretation**

$$f'(x) = \frac{2axn^2}{(n^2 + x^2)^2}.$$

At low prey levels, the derivative is small, meaning predators consume prey inefficiently. As prey availability increases, consumption accelerates.

**(c) Second derivative and interpretation**

The second derivative changes sign, indicating an inflection point:

$$f''(x) > 0 \text{ for small } x, \qquad f''(x) < 0 \text{ for large } x.$$

This confirms the S-shaped behavior of the curve.

**(d) Ecological interpretation**

The Holling type III model accounts for:

- prey refuge at low densities,
- predator learning or prey switching,
- delayed predator response.

These features are not captured by the Holling type II model.

## Exercise 174

Snowshoe hare (thousands) vs. lynx (hundreds):

$$(20, 10),\ (55, 15),\ (65, 55),\ (95, 60)$$

**(a) Best-fitting Holling-type model**

The data show:

- low predator population at low prey levels,
- rapid increase at moderate prey levels,
- saturation at high prey levels.

This behavior is most consistent with a Holling type III response.

**(b) Estimating parameters**

The maximum lynx population is approximately 60 (hundreds), so

$$a \approx 60.$$

Half-maximum occurs near 30, corresponding to a prey population around 60 (thousands), so

$$n \approx 60.$$

**(c) Model comparison**

- Type I does not model saturation.
- Type II saturates but increases too rapidly at low prey levels.
- Type III best matches the observed data trend.

**Conclusion:** The Holling type III function provides the most realistic fit for the snowshoe hare–lynx data.

**Derivatives of Trigonometric Functions**

## Exercise 175

Given $y = x^2 - \sec x + 1$.

$$\frac{dy}{dx} = 2x - \sec x \tan x.$$

**Final answer:** $y' = 2x - \sec x \tan x$

## Exercise 176

Given $y = 3\csc x + \frac{5}{x}$.

$$\frac{dy}{dx} = 3\,(-\csc x \cot x) + 5\left(-x^{-2}\right) = -3\csc x \cot x - \frac{5}{x^2}.$$

**Final answer:** $y' = -3\csc x \cot x - \frac{5}{x^2}$

## Exercise 177

Given $y = x^2 \cot x$.

Using the product rule:

$$\frac{dy}{dx} = (2x)\cot x + x^2\left(-\csc^2 x\right) = 2x\cot x - x^2\csc^2 x.$$

**Final answer:** $y' = 2x\cot x - x^2\csc^2 x$

## Exercise 178

Given $y = x - x^3 \sin x$.

Differentiate:

$$\frac{dy}{dx} = 1 - \frac{d}{dx}\left(x^3\sin x\right).$$

Product rule on $x^3 \sin x$:

$$\frac{d}{dx}\left(x^3\sin x\right) = 3x^2\sin x + x^3\cos x.$$

So

$$\frac{dy}{dx} = 1 - \left(3x^2\sin x + x^3\cos x\right).$$

**Final answer:** $y' = 1 - 3x^2\sin x - x^3\cos x$

## Exercise 179

Given $y = \dfrac{\sec x}{x}$.

Using the quotient rule:

$$\frac{dy}{dx} = \frac{x(\sec x\tan x) - (\sec x)}{x^2} = \frac{x\sec x\tan x - \sec x}{x^2}.$$

**Final answer:** $y' = \dfrac{x\sec x\tan x - \sec x}{x^2}$

## Exercise 180

Given $y = \sin x \tan x$.

Product rule:

$$\frac{dy}{dx} = (\cos x)\tan x + \sin x(\sec^2 x).$$

**Final answer:** $y' = \cos x \tan x + \sin x \sec^2 x$

## Exercise 181

Given $y = (x + \cos x)(1 - \sin x)$.

Let $u = x + \cos x$ and $v = 1 - \sin x$. Then

$$u' = 1 - \sin x, \qquad v' = -\cos x.$$

Product rule:

$$y' = u'v + uv' = (1 - \sin x)(1 - \sin x) - (x + \cos x)\cos x.$$

**Final answer:** $y' = (1 - \sin x)^2 - (x + \cos x)\cos x$

## Exercise 182

Given $y = \dfrac{\tan x}{1 - \sec x}$.

Let $u = \tan x$ and $v = 1 - \sec x$. Then

$$u' = \sec^2 x, \qquad v' = -\sec x \tan x.$$

Quotient rule:

$$y' = \frac{u'v - uv'}{v^2} = \frac{\sec^2 x(1 - \sec x) - \tan x(-\sec x \tan x)}{(1 - \sec x)^2}.$$

So

$$y' = \frac{\sec^2 x(1 - \sec x) + \sec x \tan^2 x}{(1 - \sec x)^2}.$$

**Final answer:** $y' = \dfrac{\sec^2 x(1 - \sec x) + \sec x \tan^2 x}{(1 - \sec x)^2}$

## Exercise 183

Given $y = \dfrac{1 - \cot x}{1 + \cot x}$.

Let $u = 1 - \cot x$, $v = 1 + \cot x$. Then

$$u' = \csc^2 x, \qquad v' = -\csc^2 x.$$

Quotient rule:

$$y' = \frac{u'v - uv'}{v^2} = \frac{\csc^2 x(1 + \cot x) - (1 - \cot x)(-\csc^2 x)}{(1 + \cot x)^2}.$$

Factor $\csc^2 x$:

$$y' = \frac{\csc^2 x\big[(1+\cot x)+(1-\cot x)\big]}{(1+\cot x)^2} = \frac{2\csc^2 x}{(1+\cot x)^2}.$$

**Final answer:** $y' = \dfrac{2\csc^2 x}{(1+\cot x)^2}$

### Exercise 184

Given $y = \cos x(1+\csc x)$.

Expand:

$$y = \cos x + \cos x\ \csc x = \cos x + \cot x.$$

Differentiate:

$$y' = -\sin x - \csc^2 x.$$

**Final answer:** $y' = -\sin x - \csc^2 x$

### Exercise 185

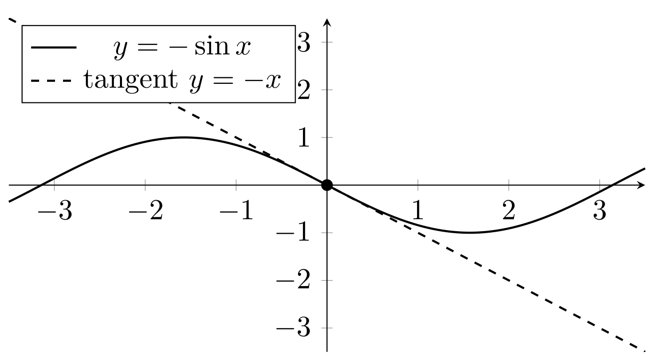
Find the tangent line to $f(x) = -\sin x$ at $x = 0$.

$$f'(x) = -\cos x \quad \Rightarrow \quad f'(0) = -1.$$

Also $f(0) = 0$. Point-slope form:

$$y - 0 = -1(x-0) \quad \Rightarrow \quad y = -x.$$

**Final answer:** $y = -x$



## Exercise 186

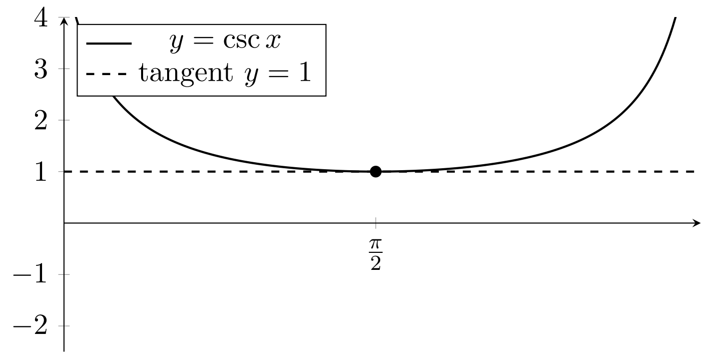
Find the tangent line to $f(x) = \csc x$ at $x = \dfrac{\pi}{2}$.

$$f'(x) = -\csc x \cot x \quad \Rightarrow \quad f'\left(\tfrac{\pi}{2}\right) = -\csc\left(\tfrac{\pi}{2}\right)\cot\left(\tfrac{\pi}{2}\right) = -(1)(0) = 0.$$

Also $f\left(\tfrac{\pi}{2}\right) = 1$. Horizontal tangent:

$$y = 1.$$

**Final answer:** $y = 1$




## Exercise 187

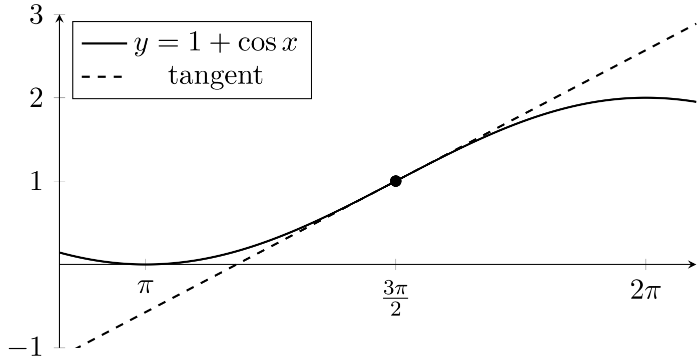
Find the tangent line to $f(x) = 1 + \cos x$ at $x = \dfrac{3\pi}{2}$.

$$f'(x) = -\sin x \quad \Rightarrow \quad f'\left(\tfrac{3\pi}{2}\right) = -\sin\left(\tfrac{3\pi}{2}\right) = 1.$$

Also $f\left(\tfrac{3\pi}{2}\right) = 1 + \cos\left(\tfrac{3\pi}{2}\right) = 1$.

$$y - 1 = 1\left(x - \tfrac{3\pi}{2}\right).$$

**Final answer:** $y = x - \dfrac{3\pi}{2} + 1$




**Exercise 188**

Find the tangent line to $f(x) = \sec x$ at $x = \dfrac{\pi}{4}$.

$$f'(x) = \sec x \tan x.$$

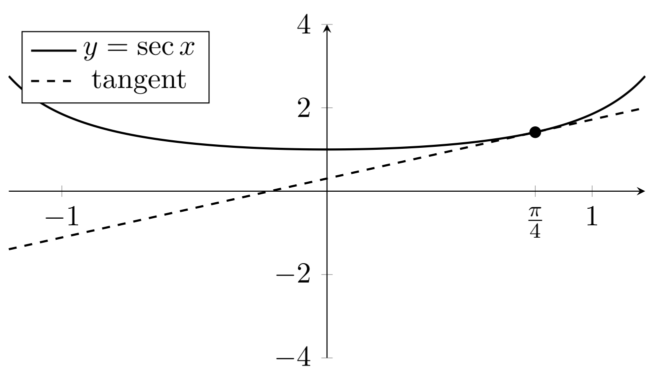
At $x = \frac{\pi}{4}$:

$$\sec(\tfrac{\pi}{4}) = \sqrt{2}, \qquad \tan(\tfrac{\pi}{4}) = 1 \Rightarrow f'(\tfrac{\pi}{4}) = \sqrt{2}.$$

Also $f(\frac{\pi}{4}) = \sqrt{2}$. Thus

$$y - \sqrt{2} = \sqrt{2}\left(x - \tfrac{\pi}{4}\right).$$

**Final answer:** $y = \sqrt{2}\,x + \sqrt{2}\left(1 - \dfrac{\pi}{4}\right)$




**Exercise 189**

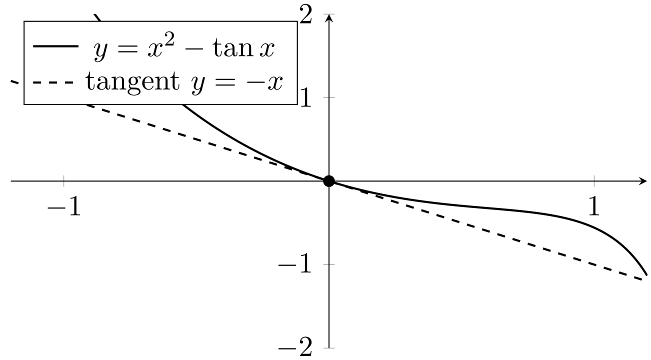
Find the tangent line to $f(x) = x^2 - \tan x$ at $x = 0$.

$$f'(x) = 2x - \sec^2 x \quad \Rightarrow \quad f'(0) = 0 - 1 = -1.$$

Also $f(0) = 0$. Hence

$$y = -x.$$

**Final answer:** $y = -x$




## Exercise 190

Find the tangent line to $f(x) = 5\cot x$ at $x = \dfrac{\pi}{4}$.

$$f'(x) = 5(-\csc^2 x) = -5\csc^2 x.$$

At $x = \frac{\pi}{4}$, $\csc(\frac{\pi}{4}) = \sqrt{2}$, so

$$f'\left(\tfrac{\pi}{4}\right) = -5(\sqrt{2})^2 = -10.$$

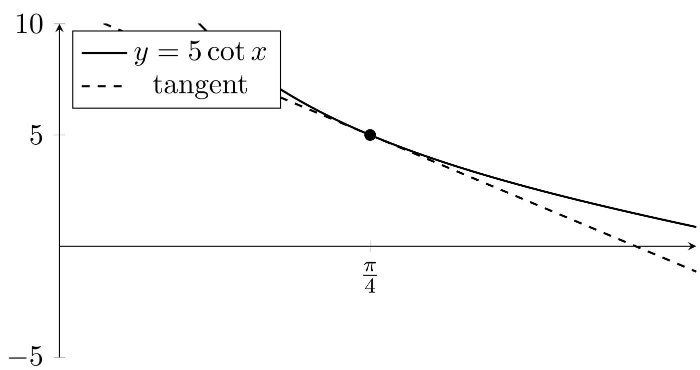
Also

$$f\left(\tfrac{\pi}{4}\right) = 5\cot\left(\tfrac{\pi}{4}\right) = 5.$$

Thus

$$y - 5 = -10\left(x - \tfrac{\pi}{4}\right).$$

**Final answer:** $y = -10x + \dfrac{5\pi}{2} + 5$



**Exercise 191**

Given $y = x \sin x - \cos x$. Find $y''$.

First derivative:

$$y' = \frac{d}{dx}(x \sin x) - \frac{d}{dx}(\cos x) = (\sin x + x \cos x) + \sin x = 2 \sin x + x \cos x.$$

Second derivative:

$$y'' = 2 \cos x + (\cos x - x \sin x) = 3 \cos x - x \sin x.$$

**Final answer:** $y'' = 3 \cos x - x \sin x$

**Exercise 192**

Given $y = \sin x \cos x$. Find $y''$.

$$y' = (\cos x) \cos x + \sin x(-\sin x) = \cos^2 x - \sin^2 x.$$

$$y'' = 2 \cos x(-\sin x) - 2 \sin x(\cos x) = -4 \sin x \cos x.$$

**Final answer:** $y'' = -4 \sin x \cos x$

**Exercise 193**

Given $y = x - \frac{1}{2} \sin x$. Find $y''$.

$$y' = 1 - \tfrac{1}{2} \cos x, \qquad y'' = \tfrac{1}{2} \sin x.$$

**Final answer:** $y'' = \dfrac{1}{2} \sin x$

**Exercise 194**

Given $y = \dfrac{1}{x} + \tan x$. Find $y''$.

$$y' = -\frac{1}{x^2} + \sec^2 x.$$

$$y'' = \frac{2}{x^3} + 2 \sec^2 x \tan x.$$

**Final answer:** $y'' = \dfrac{2}{x^3} + 2\sec^2 x \tan x$

### Exercise 195

Given $y = 2\csc x$. Find $y''$.

$$y' = -2\csc x \cot x.$$

Differentiate using product rule on $\csc x \cot x$:

$$y'' = -2\left[(-\csc x \cot x)\cot x + \csc x(-\csc^2 x)\right] = 2\csc x \cot^2 x + 2\csc^3 x.$$

So

$$y'' = 2\csc x\left(\cot^2 x + \csc^2 x\right).$$

**Final answer:** $y'' = 2\csc x(\cot^2 x + \csc^2 x)$

### Exercise 196

Given $y = \sec^2 x$. Find $y''$.

$$y' = \frac{d}{dx}(\sec^2 x) = 2\sec^2 x \tan x.$$

Differentiate:

$$y'' = 2\left[(2\sec^2 x \tan x)\tan x + \sec^2 x(\sec^2 x)\right] = 4\sec^2 x \tan^2 x + 2\sec^4 x.$$

**Final answer:** $y'' = 4\sec^2 x \tan^2 x + 2\sec^4 x$

### Exercise 197

Find all $x$ where the tangent line to $f(x) = -3\sin x \cos x$ is horizontal.

Horizontal tangents occur where $f'(x) = 0$.

$$f(x) = -3\sin x \cos x.$$

Differentiate:

$$f'(x) = -3(\cos^2 x - \sin^2 x).$$

Set equal to 0:

$$\cos^2 x - \sin^2 x = 0 \quad \Rightarrow \quad \cos^2 x = \sin^2 x \quad \Rightarrow \quad \tan^2 x = 1.$$

Thus

$$x = \frac{\pi}{4} + \frac{k\pi}{2}, \quad k \in \mathbb{Z}.$$

**Final answer:** $x = \dfrac{\pi}{4} + \dfrac{k\pi}{2}$

### Exercise 198

Find all $x$ with $0 < x < 2\pi$ where the tangent line to $f(x) = x - 2\cos x$ has slope 2.

$$f'(x) = 1 + 2\sin x.$$

Slope 2 means:

$$1 + 2\sin x = 2 \quad \Rightarrow \quad \sin x = \frac{1}{2}.$$

On $0 < x < 2\pi$,

$$x = \frac{\pi}{6}, \ \frac{5\pi}{6}.$$

**Final answer:** $x = \dfrac{\pi}{6}, \ \dfrac{5\pi}{6}$

### Exercise 199

Let $f(x) = \cot x$. Find points on $0 < x < 2\pi$ where the tangent line is parallel to $y = -2x$.

Parallel means slope $-2$, so solve $f'(x) = -2$.

$$f'(x) = -\csc^2 x.$$

$$-\csc^2 x = -2 \quad \Rightarrow \quad \csc^2 x = 2 \quad \Rightarrow \quad \sin^2 x = \frac{1}{2}.$$

Thus in $0 < x < 2\pi$,

$$x = \frac{\pi}{4}, \ \frac{3\pi}{4}, \ \frac{5\pi}{4}, \ \frac{7\pi}{4}.$$

The corresponding points are $(x, \cot x)$:

$$\cot(\tfrac{\pi}{4}) = 1, \quad \cot\left(\tfrac{3\pi}{4}\right) = -1, \quad \cot\left(\tfrac{5\pi}{4}\right) = 1, \quad \cot\left(\tfrac{7\pi}{4}\right) = -1.$$

**Final answer:**

$$\left(\tfrac{\pi}{4}, 1\right), \ \left(\tfrac{3\pi}{4}, -1\right), \ \left(\tfrac{5\pi}{4}, 1\right), \ \left(\tfrac{7\pi}{4}, -1\right).$$

## Exercise 200

Given

$$s(t) = -6\cos t$$

Velocity:

$$v(t) = s'(t) = 6\sin t$$

Rate of oscillation at $t = 5$:

$$v(5) = 6\sin 5$$

## Exercise 201

$$s(t) = a\cos t + b\sin t$$

Position at $t = 0$:

$$s(0) = a = 0$$

Velocity:

$$v(t) = -a\sin t + b\cos t$$

Velocity at $t = 0$:

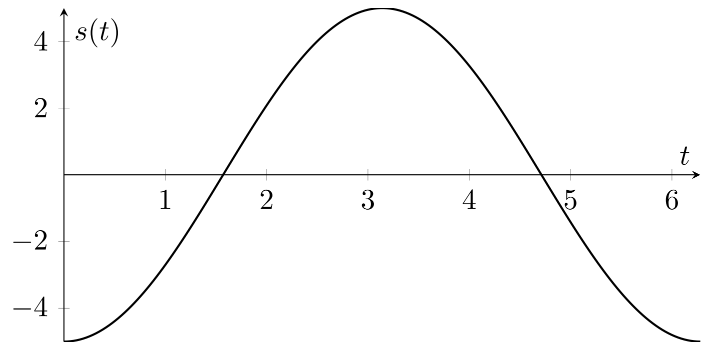
$$v(0) = b = 3$$

$$\boxed{a = 0, \quad b = 3}$$

## Exercise 202

$$s(t) = -5\cos t$$

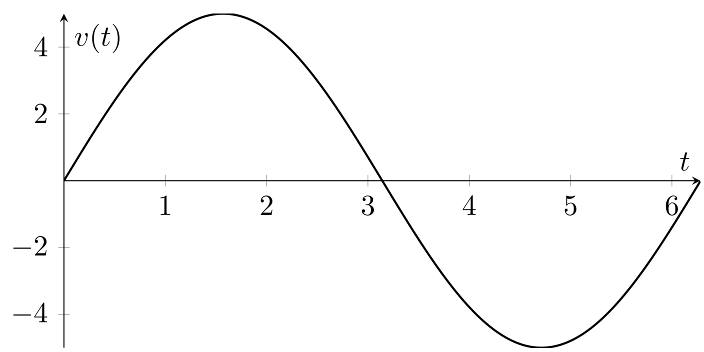
**(a) Position graph**




**(b) Velocity**

$$v(t) = 5 \sin t$$

**(c) Velocity graph**



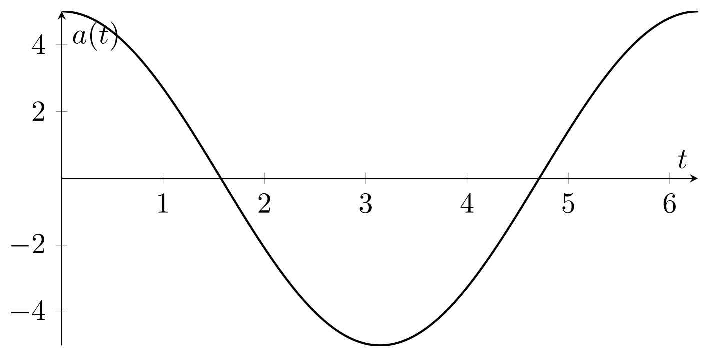

**(d) Velocity zero**

$$5 \sin t = 0 \Rightarrow t = 0, \pi, 2\pi$$

**(e) Acceleration**

$$a(t) = v'(t) = 5 \cos t$$

**(f) Acceleration graph**




## Exercise 203

$$y = 10 + 5\sin x$$

$$y' = 5\cos x$$

Increasing when:

$$\cos x > 0$$

$$\boxed{(-\tfrac{\pi}{2}, \tfrac{\pi}{2})}$$

## Exercise 204

$$y(t) = 0.5 + 0.3\cos t$$

$$y'(t) = -0.3\sin t$$

Decreasing when:

$$\sin t > 0 \Rightarrow (0, \pi)$$

## Exercise 205

$$\frac{d}{dx}(\cot x) = \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right) = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = -\csc^2 x$$

**Exercise 206**

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

**Exercise 207**

$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$

**Exercise 208**

$$\frac{d}{dx}(\cos x) = \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h}$$

Using

$$\cos(x+h) = \cos x \cos h - \sin x \sin h$$

$$\Rightarrow -\sin x$$

**Exercise 209**

$$y = 3\cos x$$

$$y' = -3\sin x, \quad y'' = -3\cos x, \quad y''' = 3\sin x$$

**Exercise 210**

$$y = 3\sin x + x^2 \cos x$$

$$y' = 3\cos x + 2x\cos x - x^2 \sin x$$

$$y'' = -3\sin x + 2\cos x - 4x\sin x - x^2\cos x$$

**Exercise 211**

$$y = 5\cos x$$

$$\frac{d^4y}{dx^4} = 5\cos x$$

**Exercise 212**

$$y = \sec x + \cot x$$

$$y' = \sec x \tan x - \csc^2 x$$

$$y'' = \sec x \tan^2 x + \sec^3 x + 2\csc^2 x \cot x$$

**Exercise 213**

$$y = x^{10} - \sec x$$

$$y' = 10x^9 - \sec x \tan x$$

$$y'' = 90x^8 - (\sec x \tan^2 x + \sec^3 x)$$

$$y''' = 720x^7 - (\sec x \tan^3 x + 5\sec^3 x \tan x)$$

**The Chain Rule**

**Exercise 214**

Given

$$y = 3u - 6, \quad u = 2x^2$$
$$\frac{dy}{du} = 3, \quad \frac{du}{dx} = 4x$$
$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 12x$$

**Final Answer:** $\frac{dy}{dx} = 12x$.

**Exercise 215**

$$y = 6u^3, \quad u = 7x - 4$$
$$\frac{dy}{du} = 18u^2, \quad \frac{du}{dx} = 7$$
$$\frac{dy}{dx} = 18u^2 \cdot 7 = 126(7x-4)^2$$

**Final Answer:** $\dfrac{dy}{dx} = 126(7x-4)^2$.

**Exercise 216**

$$y = \sin u, \quad u = 5x - 1$$
$$\frac{dy}{du} = \cos u, \quad \frac{du}{dx} = 5$$
$$\frac{dy}{dx} = 5\cos(5x-1)$$

**Final Answer:** $\dfrac{dy}{dx} = 5\cos(5x-1)$.

**Exercise 217**

$$y = \cos u, \quad u = -\frac{x}{8}$$
$$\frac{dy}{du} = -\sin u, \quad \frac{du}{dx} = -\frac{1}{8}$$
$$\frac{dy}{dx} = (-\sin u)\left(-\frac{1}{8}\right) = \frac{1}{8}\sin\left(-\frac{x}{8}\right) = -\frac{1}{8}\sin\left(\frac{x}{8}\right)$$

**Final Answer:** $\dfrac{dy}{dx} = -\dfrac{1}{8}\sin\left(\dfrac{x}{8}\right)$.

**Exercise 218**

$$y = \tan u, \quad u = 9x + 2$$
$$\frac{dy}{du} = \sec^2 u, \quad \frac{du}{dx} = 9$$
$$\frac{dy}{dx} = 9\sec^2(9x+2)$$

**Final Answer:** $\dfrac{dy}{dx} = 9\sec^2(9x+2)$.

**Exercise 219**

$$y = \sqrt{4u+3}, \quad u = x^2 - 6x$$

$$\frac{dy}{du} = \frac{1}{2\sqrt{4u+3}} \cdot 4 = \frac{2}{\sqrt{4u+3}}, \quad \frac{du}{dx} = 2x - 6$$

$$\frac{dy}{dx} = \frac{2}{\sqrt{4u+3}}(2x-6) = \frac{4x-12}{\sqrt{4(x^2-6x)+3}} = \frac{4x-12}{\sqrt{4x^2-24x+3}}$$

**Final Answer:** $\dfrac{dy}{dx} = \dfrac{4x-12}{\sqrt{4x^2-24x+3}}$.

**Exercise 220**

$$y = (3x-2)^6$$

Let $u = 3x - 2$. Then $y = u^6$.

$$\frac{dy}{du} = 6u^5, \quad \frac{du}{dx} = 3$$

$$\frac{dy}{dx} = 6u^5 \cdot 3 = 18(3x-2)^5$$

**Final Answer:** $\dfrac{dy}{dx} = 18(3x-2)^5$.

**Exercise 221**

$$y = (3x^2+1)^3$$

Let $u = 3x^2 + 1$. Then $y = u^3$.

$$\frac{dy}{du} = 3u^2, \quad \frac{du}{dx} = 6x$$

$$\frac{dy}{dx} = 3(3x^2+1)^2(6x) = 18x(3x^2+1)^2$$

**Final Answer:** $\dfrac{dy}{dx} = 18x(3x^2+1)^2$.

**Exercise 222**

$$y = \sin^5 x$$

Let $u = \sin x$. Then $y = u^5$.

$$\frac{dy}{du} = 5u^4, \quad \frac{du}{dx} = \cos x$$

$$\frac{dy}{dx} = 5\sin^4 x \cos x$$

**Final Answer:** $\dfrac{dy}{dx} = 5\sin^4 x \cos x$.

**Exercise 223**

$$y = \left(\frac{x}{7} + \frac{7}{x}\right)^7$$

Let $u = \frac{x}{7} + \frac{7}{x}$. Then $y = u^7$.

$$\frac{dy}{du} = 7u^6, \quad \frac{du}{dx} = \frac{1}{7} - \frac{7}{x^2}$$

$$\frac{dy}{dx} = 7\left(\frac{x}{7} + \frac{7}{x}\right)^6 \left(\frac{1}{7} - \frac{7}{x^2}\right)$$

**Final Answer:** $\dfrac{dy}{dx} = 7\left(\dfrac{x}{7} + \dfrac{7}{x}\right)^6 \left(\dfrac{1}{7} - \dfrac{7}{x^2}\right)$.

**Exercise 224**

$$y = \tan(\sec x)$$

Let $u = \sec x$. Then $y = \tan u$.

$$\frac{dy}{du} = \sec^2 u, \quad \frac{du}{dx} = \sec x \tan x$$

$$\frac{dy}{dx} = \sec^2(\sec x)\ \sec x \tan x$$

**Final Answer:** $\dfrac{dy}{dx} = \sec^2(\sec x) \sec x \tan x$.

**Exercise 225**

$$y = \csc(\pi x + 1)$$

Let $u = \pi x + 1$. Then $y = \csc u$.

$$\frac{dy}{du} = -\csc u \cot u, \quad \frac{du}{dx} = \pi$$

$$\frac{dy}{dx} = -\pi\ \csc(\pi x + 1) \cot(\pi x + 1)$$

**Final Answer:** $\dfrac{dy}{dx} = -\pi\ \csc(\pi x + 1) \cot(\pi x + 1)$.

**Exercise 226**

$$y = \cot^2 x$$

Let $u = \cot x$. Then $y = u^2$.

$$\frac{dy}{du} = 2u, \quad \frac{du}{dx} = -\csc^2 x$$

$$\frac{dy}{dx} = 2\cot x(-\csc^2 x) = -2\cot x\ \csc^2 x$$

**Final Answer:** $\dfrac{dy}{dx} = -2\cot x\ \csc^2 x$.

**Exercise 227**

$$y = -6\sin^{-3} x = -6(\sin x)^{-3}$$

Let $u = \sin x$. Then $y = -6u^{-3}$.

$$\frac{dy}{du} = -6(-3)u^{-4} = 18u^{-4}, \quad \frac{du}{dx} = \cos x$$

$$\frac{dy}{dx} = 18(\sin x)^{-4}\cos x = \frac{18\cos x}{\sin^4 x}$$

**Final Answer:** $\dfrac{dy}{dx} = \dfrac{18\cos x}{\sin^4 x}$.

**Exercise 228**

$$y = (3x^2 + 3x - 1)^4$$

Let $u = 3x^2 + 3x - 1$. Then $y = u^4$.

$$\frac{dy}{du} = 4u^3, \quad \frac{du}{dx} = 6x + 3$$

$$\frac{dy}{dx} = 4(3x^2 + 3x - 1)^3(6x + 3)$$

**Final Answer:** $\dfrac{dy}{dx} = 4(3x^2 + 3x - 1)^3(6x + 3)$.

**Exercise 229**

$$y = (5 - 2x)^{-2}$$

Let $u = 5 - 2x$. Then $y = u^{-2}$.

$$\frac{dy}{du} = -2u^{-3}, \quad \frac{du}{dx} = -2$$

$$\frac{dy}{dx} = (-2)u^{-3}(-2) = 4(5-2x)^{-3}$$

**Final Answer:** $\dfrac{dy}{dx} = 4(5-2x)^{-3}$.

## Exercise 230

$$y = \cos^3(\pi x)$$

Let $u = \cos(\pi x)$. Then $y = u^3$.

$$\frac{dy}{du} = 3u^2$$

Also,

$$\frac{du}{dx} = \frac{d}{dx}\big(\cos(\pi x)\big) = -\sin(\pi x) \cdot \pi$$

Thus

$$\frac{dy}{dx} = 3\cos^2(\pi x)\big(-\pi\sin(\pi x)\big) = -3\pi\cos^2(\pi x)\sin(\pi x)$$

**Final Answer:** $\dfrac{dy}{dx} = -3\pi\cos^2(\pi x)\sin(\pi x)$.

## Exercise 231

$$y = (2x^3 - x^2 + 6x + 1)^3$$

Let $u = 2x^3 - x^2 + 6x + 1$. Then $y = u^3$.

$$\frac{dy}{du} = 3u^2, \quad \frac{du}{dx} = 6x^2 - 2x + 6$$

$$\frac{dy}{dx} = 3(2x^3 - x^2 + 6x + 1)^2(6x^2 - 2x + 6)$$

**Final Answer:** $\dfrac{dy}{dx} = 3(2x^3 - x^2 + 6x + 1)^2(6x^2 - 2x + 6)$.

## Exercise 232

$$y = \frac{1}{\sin^2 x} = \csc^2 x$$

$$\frac{dy}{dx} = \frac{d}{dx}(\csc^2 x) = 2\csc x \cdot (-\csc x \cot x) = -2\csc^2 x \cot x$$

**Final Answer:** $\dfrac{dy}{dx} = -2\csc^2 x \cot x$.

**Exercise 233**

$$y = (\tan x + \sin x)^{-3}$$

Let $u = \tan x + \sin x$. Then $y = u^{-3}$.

$$\frac{dy}{du} = -3u^{-4}, \quad \frac{du}{dx} = \sec^2 x + \cos x$$

$$\frac{dy}{dx} = -3(\tan x + \sin x)^{-4}(\sec^2 x + \cos x)$$

**Final Answer:** $\dfrac{dy}{dx} = -3(\tan x + \sin x)^{-4}(\sec^2 x + \cos x)$.

**Exercise 234**

$$y = x^2 \cos^4 x$$

Use the product rule with $f = x^2$ and $g = \cos^4 x$:

$$\frac{dy}{dx} = f'g + fg'$$

$$f' = 2x$$

For $g = \big(\cos x\big)^4$, let $u = \cos x$, $g = u^4$:

$$g' = 4u^3u' = 4\cos^3 x(-\sin x) = -4\cos^3 x \sin x$$

Thus

$$\frac{dy}{dx} = 2x\cos^4 x + x^2(-4\cos^3 x \sin x) = 2x\cos^4 x - 4x^2\cos^3 x \sin x$$

**Final Answer:** $\dfrac{dy}{dx} = 2x\cos^4 x - 4x^2\cos^3 x \sin x$.

**Exercise 235**

$$y = \sin(\cos 7x)$$

Let $u = \cos 7x$. Then $y = \sin u$.

$$\frac{dy}{du} = \cos u$$

Also,

$$\frac{du}{dx} = \frac{d}{dx}(\cos 7x) = -\sin 7x \cdot 7 = -7\sin 7x$$

Therefore

$$\frac{dy}{dx} = \cos(\cos 7x)\,(-7\sin 7x)$$

**Final Answer:** $\frac{dy}{dx} = -7\sin 7x\ \cos(\cos 7x)$.

### Exercise 236

$$y = \sqrt{6 + \sec(\pi x^2)}$$

Let $u = 6 + \sec(\pi x^2)$. Then $y = \sqrt{u} = u^{1/2}$.

$$\frac{dy}{du} = \frac{1}{2\sqrt{u}}$$

Now differentiate $u$:

$$\frac{du}{dx} = \frac{d}{dx}\sec\Big(\pi x^2\Big) = \sec\Big(\pi x^2\Big)\tan\Big(\pi x^2\Big) \cdot \frac{d}{dx}(\pi x^2)$$

$$\frac{d}{dx}(\pi x^2) = 2\pi x$$

So

$$\frac{du}{dx} = 2\pi x\ \sec\Big(\pi x^2\Big)\tan\Big(\pi x^2\Big)$$

Finally,

$$\frac{dy}{dx} = \frac{1}{2\sqrt{6+\sec(\pi x^2)}} \cdot 2\pi x\ \sec\Big(\pi x^2\Big)\tan\Big(\pi x^2\Big)$$

$$\frac{dy}{dx} = \frac{\pi x\ \sec(\pi x^2)\tan(\pi x^2)}{\sqrt{6+\sec(\pi x^2)}}$$

**Final Answer:** $\frac{dy}{dx} = \frac{\pi x\ \sec(\pi x^2)\tan(\pi x^2)}{\sqrt{6+\sec(\pi x^2)}}$.

### Exercise 237

$$y = \cot^3(4x+1)$$

Let $u = \cot(4x+1)$. Then $y = u^3$.

$$\frac{dy}{du} = 3u^2$$

Also,

$$\frac{du}{dx} = \frac{d}{dx}\big(\cot(4x+1)\big) = -\csc^2(4x+1)\cdot(4) = -4\csc^2(4x+1)$$

So

$$\frac{dy}{dx} = 3\cot^2(4x+1)\big(-4\csc^2(4x+1)\big) = -12\cot^2(4x+1)\csc^2(4x+1)$$

**Final Answer:** $\dfrac{dy}{dx} = -12\cot^2(4x+1)\csc^2(4x+1)$.

### Exercise 238

Let $y = [f(x)]^3$. By the chain rule:

$$\frac{dy}{dx} = 3[f(x)]^2 f'(x)$$

At $x = 1$, we are told $f'(1) = 4$ and $\frac{dy}{dx} = 10$. Substitute:

$$10 = 3[f(1)]^2(4) = 12[f(1)]^2$$

$$[f(1)]^2 = \frac{10}{12} = \frac{5}{6}$$

$$f(1) = \sqrt{\frac{5}{6}}$$

**Final Answer:** $f(1) = \sqrt{\dfrac{5}{6}}$.

### Exercise 239

$$y = (f(x) + 5x^2)^4$$

Let $u = f(x) + 5x^2$. Then $y = u^4$.

$$\frac{dy}{dx} = 4u^3 \cdot u' = 4(f(x) + 5x^2)^3(f'(x) + 10x)$$

At $x = -1$: $f(-1) = -4$ and $\frac{dy}{dx} = 3$. First compute $u$ at $-1$:

$$u = f(-1) + 5(-1)^2 = -4 + 5 = 1$$

So

$$3 = 4(1)^3\big(f'(-1) + 10(-1)\big) = 4\big(f'(-1) - 10\big)$$

$$f'(-1) - 10 = \frac{3}{4} \quad \Rightarrow \quad f'(-1) = 10 + \frac{3}{4} = \frac{43}{4}$$

**Final Answer:** $f'(-1) = \dfrac{43}{4}$.

## Exercise 240

$$y = (f(u) + 3x)^2, \qquad u = x^3 - 2x$$

Let $v = f(u) + 3x$. Then $y = v^2$ and

$$\frac{dy}{dx} = 2v \cdot v'$$

Compute $v'$:

$$v' = \frac{d}{dx}(f(u)) + \frac{d}{dx}(3x) = f'(u)\, u' + 3$$

Also,

$$u' = 3x^2 - 2$$

So

$$\frac{dy}{dx} = 2(f(u) + 3x)\big(f'(u)(3x^2 - 2) + 3\big)$$

At $x = 2$: $u = 2^3 - 2(2) = 8 - 4 = 4$, and we are told $f(4) = 6$, $\frac{dy}{dx} = 18$. Then

$$f(u) + 3x = f(4) + 6 = 6 + 6 = 12, \qquad 3x^2 - 2 = 3(4) - 2 = 10$$

Substitute into the derivative:

$$18 = 2(12)\big(f'(4) \cdot 10 + 3\big) = 24(10f'(4) + 3)$$

$$10f'(4) + 3 = \frac{18}{24} = \frac{3}{4}$$

$$10f'(4) = \frac{3}{4} - 3 = \frac{3}{4} - \frac{12}{4} = -\frac{9}{4}$$

$$f'(4) = -\frac{9}{40}$$

**Final Answer:** $f'(4) = -\dfrac{9}{40}$.

## Exercise 241

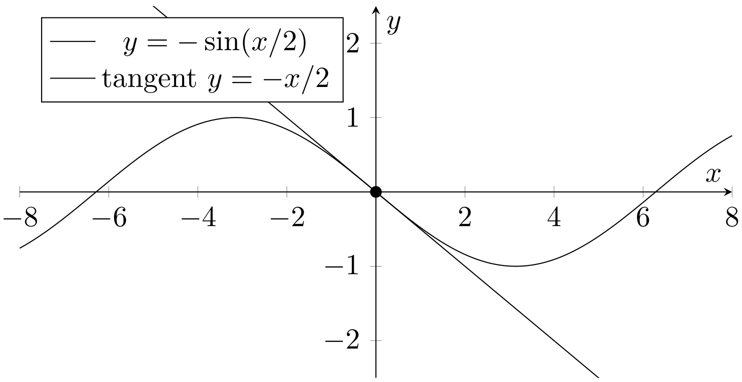
$$y = -\sin\left(\frac{x}{2}\right) \qquad y' = -\frac{1}{2}\cos\left(\frac{x}{2}\right)$$

At $x = 0$,

$$m = y'(0) = -\frac{1}{2}, \qquad y(0) = 0$$

So the tangent line at the origin is

$$y = -\frac{1}{2}x$$




**Final Answer:** tangent line is $y = -\frac{1}{2}x$.

## Exercise 242

$$y = \left(3x + \frac{1}{x}\right)^2$$

Differentiate using the chain rule:

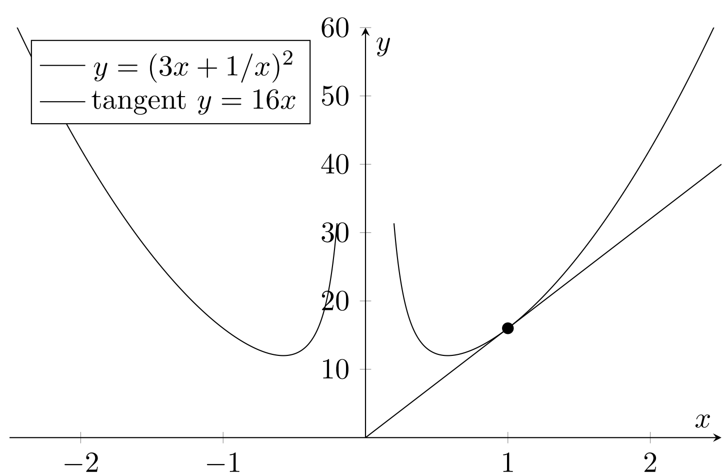
$$y' = 2\left(3x + \frac{1}{x}\right)\left(3 - \frac{1}{x^2}\right)$$

At $x = 1$,

$$y(1) = (3+1)^2 = 16, \qquad y'(1) = 2(4)(2) = 16$$

So the tangent line at $(1, 16)$ is

$$y - 16 = 16(x - 1) \quad \Longrightarrow \quad y = 16x$$




**Final Answer:** $f'(1) = 16$ and the tangent line is $y = 16x$.

## Exercise 243

$$y = \left(x - \frac{6}{x}\right)^8$$

Horizontal tangents occur when $y' = 0$. Differentiate:

$$y' = 8\left(x - \frac{6}{x}\right)^7 \left(1 + \frac{6}{x^2}\right)$$

Set equal to 0:

$$8\left(x - \frac{6}{x}\right)^7 \left(1 + \frac{6}{x^2}\right) = 0$$

So either

$$x - \frac{6}{x} = 0 \quad \Rightarrow \quad x^2 - 6 = 0 \quad \Rightarrow \quad x = \pm\sqrt{6},$$

or

$$1 + \frac{6}{x^2} = 0 \quad \Rightarrow \quad x^2 = -6,$$

which has no real solutions. Therefore the horizontal tangents happen at $x = \pm\sqrt{6}$. **Final Answer:** $x = \pm\sqrt{6}$.

## Exercise 244

$$g(\theta) = \sin^2(\pi\theta)$$

Differentiate:

$$g'(\theta) = 2\sin(\pi\theta) \cdot \cos(\pi\theta) \cdot \pi$$

At $\theta = \frac{1}{4}$,

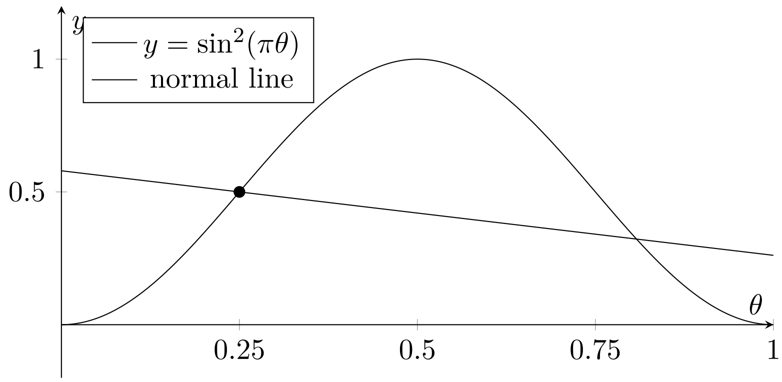
$$g\left(\frac{1}{4}\right) = \sin^2\left(\frac{\pi}{4}\right) = \frac{1}{2}, \qquad g'\left(\frac{1}{4}\right) = 2\sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{4}\right)\pi = 2\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)\pi = \pi$$

So the normal slope is

$$m_{\text{normal}} = -\frac{1}{g'(1/4)} = -\frac{1}{\pi}$$

and the normal line through $\left(\frac{1}{4}, \frac{1}{2}\right)$ is

$$y - \frac{1}{2} = -\frac{1}{\pi}\left(\theta - \frac{1}{4}\right)$$




**Final Answer:** normal line is $y - \dfrac{1}{2} = -\dfrac{1}{\pi}\left(\theta - \dfrac{1}{4}\right)$.

### Exercise 245

Given $h(x) = f(g(x))$. By the chain rule,

$$h'(x) = f'(g(x))\, g'(x).$$

At $a = 0$, the table gives $g(0) = 0$, $f'(0) = 5$, and $g'(0) = 2$. Thus

$$h'(0) = f'(g(0))\, g'(0) = f'(0) \cdot 2 = 5 \cdot 2 = 10.$$

**Final Answer:** $h'(0) = 10$.

### Exercise 246

Given $h(x) = g(f(x))$. By the chain rule,

$$h'(x) = g'(f(x))\, f'(x).$$

At $a = 0$, the table gives $f(0) = 2$, $g'(2) = -1$, and $f'(0) = 5$. Hence

$$h'(0) = g'(f(0))\, f'(0) = g'(2) \cdot 5 = (-1) \cdot 5 = -5.$$

**Final Answer:** $h'(0) = -5$.

### Exercise 247

Given $h(x) = (x^4 + g(x))^{-2}$. Let $u(x) = x^4 + g(x)$, so $h(x) = u(x)^{-2}$. Then

$$h'(x) = -2u(x)^{-3}\, u'(x), \qquad u'(x) = 4x^3 + g'(x).$$

Therefore,

$$h'(x) = -2(x^4 + g(x))^{-3}(4x^3 + g'(x)).$$

At $a = 1$, the table gives $g(1) = 3$ and $g'(1) = 0$. So

$$h'(1) = -2(1^4 + 3)^{-3}(4 \cdot 1^3 + 0) = -2(4)^{-3}(4) = -\frac{8}{64} = -\frac{1}{8}.$$

**Final Answer:** $h'(1) = -\frac{1}{8}$.

### Exercise 248

Given $h(x) = \left(\frac{f(x)}{g(x)}\right)^2$. Let $q(x) = \frac{f(x)}{g(x)}$, so $h(x) = q(x)^2$. Then

$$h'(x) = 2q(x)\,q'(x).$$

Using the quotient rule,

$$q'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}.$$

At $a = 3$, the table gives $f(3) = 3,\ f'(3) = -3,\ g(3) = 2,\ g'(3) = 3$. Thus

$$q(3) = \frac{3}{2}, \qquad q'(3) = \frac{(-3)(2) - (3)(3)}{2^2} = \frac{-6-9}{4} = -\frac{15}{4}.$$

Hence

$$h'(3) = 2\left(\frac{3}{2}\right)\left(-\frac{15}{4}\right) = 3\left(-\frac{15}{4}\right) = -\frac{45}{4}.$$

**Final Answer:** $h'(3) = -\frac{45}{4}$.

### Exercise 249

Given $h(x) = f(x + f(x))$. Let $u(x) = x + f(x)$, so $h(x) = f(u(x))$. By the chain rule,

$$h'(x) = f'(u(x))\,u'(x), \qquad u'(x) = 1 + f'(x).$$

So

$$h'(x) = f'(x + f(x))(1 + f'(x)).$$

At $a = 1$, the table gives $f(1) = 1$ and $f'(1) = -2$. Then $u(1) = 1 + 1 = 2$, and the table gives $f'(2) = 4$. Therefore

$$h'(1) = f'(2)(1 + f'(1)) = 4(1-2) = 4(-1) = -4.$$

**Final Answer:** $h'(1) = -4$.

### Exercise 250

Given $h(x) = (1 + g(x))^3$. Let $u(x) = 1 + g(x)$, so $h(x) = u(x)^3$. Then

$$h'(x) = 3u(x)^2\, u'(x) = 3(1 + g(x))^2\, g'(x).$$

At $a = 2$, the table gives $g(2) = 1$ and $g'(2) = -1$. Hence

$$h'(2) = 3(1 + 1)^2(-1) = 3(4)(-1) = -12.$$

**Final Answer:** $h'(2) = -12$.

### Exercise 251

Given $h(x) = g(2 + f(x^2))$. Let $u(x) = 2 + f(x^2)$, so $h(x) = g(u(x))$. Then

$$h'(x) = g'(u(x))\, u'(x).$$

Now compute $u'(x)$. Since $u(x) = 2 + f(x^2)$,

$$u'(x) = f'(x^2) \cdot (2x).$$

Therefore,

$$h'(x) = g'(2 + f(x^2))\, f'(x^2)\, (2x).$$

At $a = 1$: $x^2 = 1$, the table gives $f(1) = 1$, so $u(1) = 2 + 1 = 3$. Also the table gives $g'(3) = 3$ and $f'(1) = -2$. Hence

$$h'(1) = g'(3)\, f'(1)\, (2 \cdot 1) = 3(-2)(2) = -12.$$

**Final Answer:** $h'(1) = -12$.

### Exercise 252

Given $h(x) = f(g(\sin x))$. Let $u(x) = g(\sin x)$, so $h(x) = f(u(x))$. Then

$$h'(x) = f'(u(x))\, u'(x).$$

Now $u(x) = g(\sin x)$, so

$$u'(x) = g'(\sin x) \cdot \cos x.$$

Therefore,

$$h'(x) = f'(g(\sin x))\, g'(\sin x)\, \cos x.$$

At $a = 0$: $\sin 0 = 0$, $\cos 0 = 1$. The table gives $g(0) = 0$, $f'(0) = 5$, and $g'(0) = 2$. Hence

$$h'(0) = f'(g(0))\, g'(0)\, \cos 0 = f'(0) \cdot 2 \cdot 1 = 10.$$

**Final Answer:** $h'(0) = 10$.

### Exercise 253

The position is $s(t) = 100(t + 1)^{-2}$ meters.

**(a) Velocity at $t = 6$.**

$$v(t) = s'(t) = 100(-2)(t + 1)^{-3} = -200(t + 1)^{-3}.$$

So

$$v(6) = -200 \cdot 7^{-3} = -\frac{200}{343} \text{ m/s}.$$

**(b) Acceleration at $t = 6$.**

$$a(t) = v'(t) = -200(-3)(t + 1)^{-4} = 600(t + 1)^{-4},$$

so

$$a(6) = 600 \cdot 7^{-4} = \frac{600}{2401} \text{ m/s}^2.$$

**(c) Speeding up or slowing down at $t = 6$.** At $t = 6$, $v(6) < 0$ while $a(6) > 0$, so velocity and acceleration have opposite signs, meaning the speed is decreasing.

**Final Answer:** $v(6) = -\frac{200}{343}\,\mathbf{m/s}, \quad a(6) = \frac{600}{2401}\,\mathbf{m/s^2}, \quad$ **slowing down at $t = 6$.**

### Exercise 254

The position is $s(t) = -3\cos\left(\pi t + \frac{\pi}{4}\right)$ inches.

**(a) Position at $t = 1.5$.**

$$s(1.5) = -3\cos\left(\pi(1.5) + \frac{\pi}{4}\right) = -3\cos\left(\frac{3\pi}{2} + \frac{\pi}{4}\right) = -3\cos\left(\frac{7\pi}{4}\right).$$

Since $\cos\left(\frac{7\pi}{4}\right) = \frac{\sqrt{2}}{2}$,

$$s(1.5) = -\frac{3\sqrt{2}}{2} \text{ in}.$$

**(b) Velocity at $t = 1.5$.** Differentiate:

$$v(t) = s'(t) = -3\Big(-\sin\Big(\pi t + \frac{\pi}{4}\Big)\Big) \cdot \pi = 3\pi \sin\Big(\pi t + \frac{\pi}{4}\Big).$$

Then

$$v(1.5) = 3\pi \sin\Big(\frac{7\pi}{4}\Big) = 3\pi\left(-\frac{\sqrt{2}}{2}\right) = -\frac{3\pi\sqrt{2}}{2} \text{ in/s}.$$

**Final Answer:** $s(1.5) = -\frac{3\sqrt{2}}{2}$ **in**, $\quad v(1.5) = -\frac{3\pi\sqrt{2}}{2}$ **in/s**.

### Exercise 255

The cost is

$$C(x) = 0.0001x^3 - 0.02x^2 + 3x + 300 \quad \text{(dollars)},$$

and production is $x(t) = 1600 + 100t$ boxes, where $t$ is in weeks.

**(a) Marginal cost $C'(x)$.**

$$C'(x) = 0.0003x^2 - 0.04x + 3.$$

**(b) Rate of change with respect to time.** By the chain rule,

$$\frac{dC}{dt} = \frac{dC}{dx}\frac{dx}{dt} = C'(x(t)) \cdot 100.$$

So

$$\frac{dC}{dt} = 100\big(0.0003x(t)^2 - 0.04x(t) + 3\big) \quad \text{(dollars/week)}.$$

**(c) Evaluate at $t = 2$.**

$$x(2) = 1600 + 100(2) = 1800.$$

$C'(1800) = 0.0003(1800)^2 - 0.04(1800) + 3 = 0.0003(3{,}240{,}000) - 72 + 3 = 972 - 72 + 3 = 903$ dollars/box.

Thus

$$\frac{dC}{dt}\Big|_{t=2} = 903 \cdot 100 = 90{,}300 \text{ dollars/week}.$$

**Final Answer:** $C'(x) = 0.0003x^2 - 0.04x + 3$, $\quad \frac{dC}{dt}\big|_{t=2} = 90{,}300$ **dollars/week**.

### Exercise 256

Area $A = \pi r^2$, and

$$r(t) = 2 - \frac{100}{(t+7)^2},$$

where $r$ is in inches and $t$ is in seconds.

**(a) Find $\frac{dA}{dt}$.**

$$\frac{dA}{dt} = \frac{dA}{dr}\frac{dr}{dt}.$$

Since $A = \pi r^2$,

$$\frac{dA}{dr} = 2\pi r.$$

Also,

$$\frac{dr}{dt} = \frac{d}{dt}\left(2 - 100(t+7)^{-2}\right) = -100(-2)(t+7)^{-3} = \frac{200}{(t+7)^3}.$$

Therefore,

$$\frac{dA}{dt} = 2\pi r(t) \cdot \frac{200}{(t+7)^3} = \frac{400\pi\, r(t)}{(t+7)^3} = \frac{400\pi}{(t+7)^3}\left(2 - \frac{100}{(t+7)^2}\right).$$

**(b) Evaluate at $t = 4$.**

$$t + 7 = 11, \qquad r(4) = 2 - \frac{100}{11^2} = 2 - \frac{100}{121} = \frac{142}{121}.$$

$$\frac{dA}{dt}\Big|_{t=4} = \frac{400\pi}{11^3} \cdot \frac{142}{121} = \frac{400\pi \cdot 142}{1331 \cdot 121} = \frac{56800\pi}{161051} \text{ in}^2/\text{s}.$$

**Final Answer:** $\frac{dA}{dt} = \frac{400\pi}{(t+7)^3}\left(2 - \frac{100}{(t+7)^2}\right), \quad \frac{dA}{dt}\Big|_{t=4} = \frac{56800\pi}{161051}$ **in$^2$/s.**

## Exercise 257

Volume of a sphere:

$$S = \frac{4}{3}\pi r^3,$$

and

$$r(t) = \frac{1}{(t+1)^2} - \frac{1}{12},$$

where $r$ is in feet and $t$ is in minutes.

**(a) Find $\frac{dS}{dt}$.**

$$\frac{dS}{dt} = \frac{dS}{dr}\frac{dr}{dt}.$$

Since $S = \frac{4}{3}\pi r^3$,

$$\frac{dS}{dr} = 4\pi r^2.$$

Also,

$$\frac{dr}{dt} = \frac{d}{dt}\left((t+1)^{-2} - \frac{1}{12}\right) = -2(t+1)^{-3} = -\frac{2}{(t+1)^3}.$$

Hence,

$$\frac{dS}{dt} = 4\pi r(t)^2 \left(-\frac{2}{(t+1)^3}\right) = -\frac{8\pi\, r(t)^2}{(t+1)^3}.$$

**(b) Evaluate at $t = 1$.**

$$r(1) = \frac{1}{(2)^2} - \frac{1}{12} = \frac{1}{4} - \frac{1}{12} = \frac{1}{6}, \qquad r(1)^2 = \frac{1}{36}.$$

$$\frac{dS}{dt}\Big|_{t=1} = -\frac{8\pi}{2^3} \cdot \frac{1}{36} = -\pi \cdot \frac{1}{36} = -\frac{\pi}{36} \text{ ft}^3/\text{min}.$$

**Final Answer:** $\dfrac{dS}{dt} = -\dfrac{8\pi\, r(t)^2}{(t+1)^3}, \quad \dfrac{dS}{dt}\Big|_{t=1} = -\dfrac{\pi}{36}$ **ft$^3$/min**.

## Exercise 258

Temperature:

$$T(x) = 94 - 10\cos\left(\frac{\pi}{12}(x-2)\right),$$

where $x$ is hours after midnight.

Differentiate:

$$T'(x) = -10\left(-\sin\left(\frac{\pi}{12}(x-2)\right)\right) \cdot \frac{\pi}{12} = \frac{10\pi}{12}\sin\left(\frac{\pi}{12}(x-2)\right) = \frac{5\pi}{6}\sin\left(\frac{\pi}{12}(x-2)\right).$$

At 4 p.m., $x = 16$. Then

$$T'(16) = \frac{5\pi}{6}\sin\left(\frac{\pi}{12}(14)\right) = \frac{5\pi}{6}\sin\left(\frac{7\pi}{6}\right) = \frac{5\pi}{6}\left(-\frac{1}{2}\right) = -\frac{5\pi}{12}.$$

**Final Answer:** $\dfrac{dT}{dx}\Big|_{x=16} = -\dfrac{5\pi}{12}$

## Exercise 259

Depth:

$$D(t) = 5\sin\left(\frac{\pi}{6}t - \frac{7\pi}{6}\right) + 8,$$

where $t$ is hours after midnight.

Differentiate:

$$D'(t) = 5\cos\left(\frac{\pi}{6}t - \frac{7\pi}{6}\right) \cdot \frac{\pi}{6} = \frac{5\pi}{6}\cos\left(\frac{\pi}{6}t - \frac{7\pi}{6}\right).$$

At 6 a.m., $t = 6$:

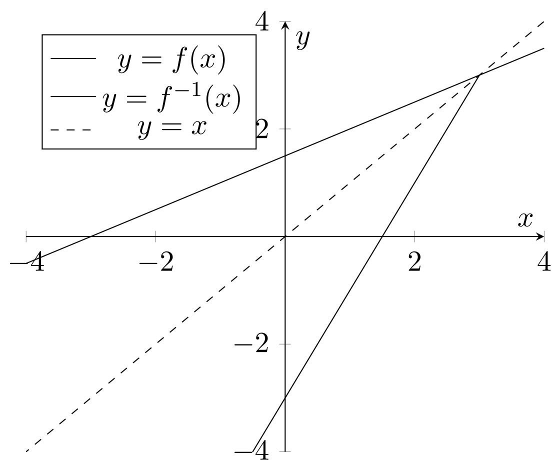
$$D'(6) = \frac{5\pi}{6}\cos\left(\pi - \frac{7\pi}{6}\right) = \frac{5\pi}{6}\cos\left(-\frac{\pi}{6}\right) = \frac{5\pi}{6} \cdot \frac{\sqrt{3}}{2} = \frac{5\pi\sqrt{3}}{12}.$$

**Final Answer:** $\frac{dD}{dt}\Big|_{t=6} = \frac{5\pi\sqrt{3}}{12}$ **ft/hour**.

**Derivatives of Inverse Functions**

## Exercise 260

The graph of $y = f(x)$ is given. The graph of $y = f^{-1}(x)$ is obtained by reflecting the graph of $y = f(x)$ across the line $y = x$.



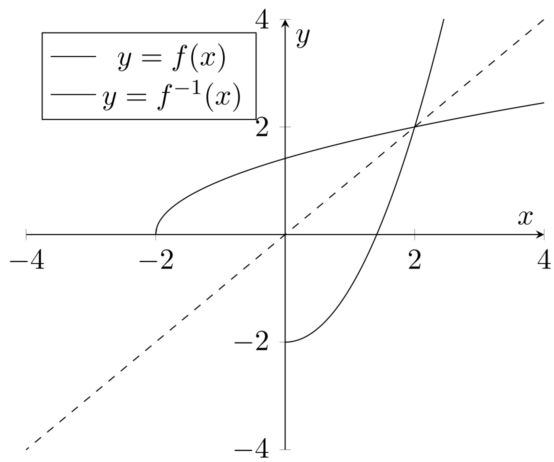

From the graph,

$$(f^{-1})(1) \approx 2$$

**Final Answer:** $(f^{-1})(1) \approx 2$

## Exercise 261

The inverse is obtained by reflecting the curve across the line $y = x$.



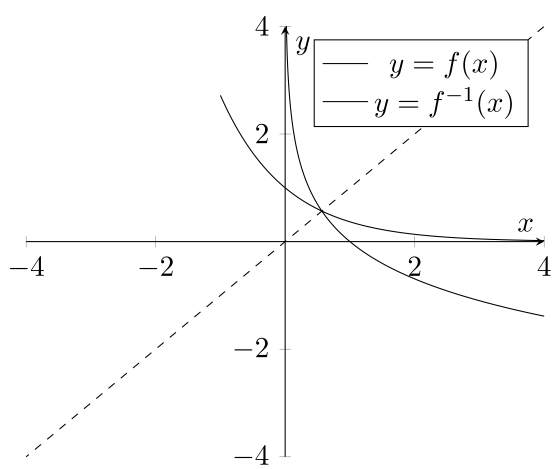

From the sketch,

$$(f^{-1})(1) \approx -1$$

**Final Answer:** $(f^{-1})(1) \approx -1$

## Exercise 262

Reflect the given curve across $y = x$ to obtain the inverse.




From the graph,

$$(f^{-1})(1) = 0$$

**Final Answer:** $(f^{-1})(1) = 0$

**Exercise 263**

The graph represents a quarter circle. Reflecting across $y = x$ gives the inverse.

From the graph,

$$(f^{-1})(1) \approx 3$$

**Final Answer:** $(f^{-1})(1) \approx 3$

**Exercise 264**

$$f(x) = 6x - 1$$

$$f'(x) = 6$$

$$\frac{d}{dy} f^{-1}(y) = \frac{1}{f'(x)} = \frac{1}{6}$$

**Final Answer:** $\frac{d}{dy} f^{-1}(y) = \frac{1}{6}$

**Exercise 265**

$$f(x) = 2x^3 - 3$$

$$f'(x) = 6x^2$$

At $x = 1$,

$$f'(1) = 6$$

**Final Answer:** $\frac{d}{dy} f^{-1}(y) = \frac{1}{6}$

**Exercise 266**

$$f(x) = 9 - x^2, \quad 0 \leq x \leq 3$$

$$f'(x) = -2x$$

At $x = 2$,

$$f'(2) = -4$$

**Final Answer:** $\frac{d}{dy} f^{-1}(y) = -\frac{1}{4}$

**Exercise 267**

$$f(x) = \sin x$$

$$f'(x) = \cos x$$

At $x = 0$,

$$f'(0) = 1$$

**Final Answer:** $\frac{d}{dy} f^{-1}(y) = 1$

**Exercise 268**

$$f(x) = x^2 + 3x + 2, \quad x \geq -\frac{3}{2}$$

Solve $f(x) = 2$:

$$x^2 + 3x = 0 \Rightarrow x = 0$$

**Final Answer:** $(f^{-1})(2) = 0$

**Exercise 269**

$$f(x) = x^3 + 2x + 3$$

Solve $f(x) = 0$:

$$x = -1$$

**Final Answer:** $(f^{-1})(0) = -1$

**Exercise 270**

$$f(x) = x + \sqrt{x}$$

Solve $x + \sqrt{x} = 2$:

$$x = 1$$

**Final Answer:** $(f^{-1})(2) = 1$

**Exercise 271**

$$f(x) = x - \frac{2}{x}, \quad x < 0$$

Solve $x - \frac{2}{x} = 1$:

$$x = -1$$

**Final Answer:** $(f^{-1})(1) = -1$

**Exercise 272**

$$f(x) = x + \sin x$$

Solve $x + \sin x = 0$:

$$x = 0$$

**Final Answer:** $(f^{-1})(0) = 0$

**Exercise 273**

$$f(x) = \tan x + 3x^2$$

Solve $f(x) = 0$:

$$x = 0$$

**Final Answer:** $(f^{-1})(0) = 0$

**Exercise 274**

Given

$$f(x) = \frac{4}{1 + x^2}, \qquad P(2, 1)$$

Differentiate:

$$f'(x) = \frac{-8x}{(1 + x^2)^2}$$

At $x = 2$:

$$f'(2) = \frac{-16}{25}$$

For the inverse function,

$$\left(f^{-1}\right)'(1) = \frac{1}{f'(2)} = -\frac{25}{16}$$

The point on $f^{-1}$ is $(1, 2)$. Tangent line:

$$y - 2 = -\frac{25}{16}(x - 1)$$

**Final answer:** $y = -\frac{25}{16}x + \frac{57}{16}$

## Exercise 275

Given

$$f(x) = \sqrt{x-4}, \qquad P(2, 8)$$

Differentiate:

$$f'(x) = \frac{1}{2\sqrt{x-4}}$$

At $x = 8$:

$$f'(8) = \frac{1}{4}$$

Thus,

$$\left(f^{-1}\right)'(2) = \frac{1}{f'(8)} = 4$$

Point on $f^{-1}$: $(2, 8)$. Tangent line:

$$y - 8 = 4(x - 2)$$

**Final answer:** $y = 4x$

## Exercise 276

Given

$$f(x) = (x^3 + 1)^4, \qquad P(16, 1)$$

Differentiate:

$$f'(x) = 4(x^3 + 1)^3(3x^2) = 12x^2(x^3 + 1)^3$$

At $x = 1$:

$$f'(1) = 12(1)(2^3) = 96$$

Thus,

$$\left(f^{-1}\right)'(16) = \frac{1}{96}$$

Point on $f^{-1}$: $(16, 1)$. Tangent line:

$$y - 1 = \frac{1}{96}(x - 16)$$

**Final answer:** $y = \frac{1}{96}x - \frac{13}{6}$

## Exercise 277

Given

$$f(x) = -x^3 - x + 2, \qquad P(-8, 2)$$

Differentiate:

$$f'(x) = -3x^2 - 1$$

At $x = -8$:

$$f'(-8) = -193$$

Thus,

$$\left(f^{-1}\right)'(2) = \frac{1}{-193}$$

Point on $f^{-1}$: $(2, -8)$. Tangent line:

$$y + 8 = -\frac{1}{193}(x - 2)$$

**Final answer:** $y = -\frac{1}{193}x - \frac{1542}{193}$

## Exercise 278

Given

$$f(x) = x^5 + 3x^3 - 4x - 8, \qquad P(-8, 1)$$

Differentiate:

$$f'(x) = 5x^4 + 9x^2 - 4$$

At $x = -8$:

$$f'(-8) = 5(4096) + 9(64) - 4 = 21052$$

Thus,

$$\left(f^{-1}\right)'(1) = \frac{1}{21052}$$

Point on $f^{-1}$: $(1, -8)$. Tangent line:

$$y + 8 = \frac{1}{21052}(x - 1)$$

**Final answer:** $y = \frac{1}{21052}x - \frac{168417}{21052}$

### Exercise 279

$$y = \sin^{-1}(x^2)$$

Using the chain rule:

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - (x^2)^2}} \cdot 2x$$

**Final Answer:**

$$\frac{dy}{dx} = \frac{2x}{\sqrt{1 - x^4}}$$

### Exercise 280

$$y = \cos^{-1}(\sqrt{x})$$

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1 - x}} \cdot \frac{1}{2\sqrt{x}}$$

**Final Answer:**

$$\frac{dy}{dx} = -\frac{1}{2\sqrt{x}\sqrt{1 - x}}$$

### Exercise 281

$$y = \sec^{-1}\left(\frac{1}{x}\right)$$

$$\frac{dy}{dx} = \frac{-1/x^2}{\left|\frac{1}{x}\right| \sqrt{\frac{1}{x^2} - 1}}$$

**Final Answer:**

$$\frac{dy}{dx} = -\frac{1}{|x|\sqrt{1-x^2}}$$

**Exercise 282**

$$y = \sqrt{\csc^{-1} x}$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{\csc^{-1} x}} \cdot \frac{-1}{|x|\sqrt{x^2-1}}$$

**Final Answer:**

$$\frac{dy}{dx} = -\frac{1}{2|x|\sqrt{x^2-1}\sqrt{\csc^{-1} x}}$$

**Exercise 283**

$$y = (1 + \tan^{-1} x)^3$$

$$\frac{dy}{dx} = 3(1 + \tan^{-1} x)^2 \cdot \frac{1}{1+x^2}$$

**Final Answer:**

$$\frac{dy}{dx} = \frac{3(1 + \tan^{-1} x)^2}{1+x^2}$$

**Exercise 284**

$$y = \cos^{-1}(2x)\sin^{-1}(2x)$$

Using the product rule:

$$y' = -\frac{2}{\sqrt{1-4x^2}}\sin^{-1}(2x) + \frac{2}{\sqrt{1-4x^2}}\cos^{-1}(2x)$$

**Final Answer:**

$$\frac{dy}{dx} = \frac{2}{\sqrt{1-4x^2}}\left(\cos^{-1}(2x) - \sin^{-1}(2x)\right)$$

**Exercise 285**

$$y = \frac{1}{\tan^{-1} x}$$

$$\frac{dy}{dx} = -\frac{1}{(\tan^{-1} x)^2} \cdot \frac{1}{1+x^2}$$

**Final Answer:**

$$\frac{dy}{dx} = -\frac{1}{(1+x^2)(\tan^{-1} x)^2}$$

**Exercise 286**

$$y = \sec^{-1}(-x)$$

$$\frac{dy}{dx} = \frac{-1}{|x|\sqrt{x^2-1}}$$

**Final Answer:**

$$\frac{dy}{dx} = -\frac{1}{|x|\sqrt{x^2-1}}$$

**Exercise 287**

$$y = \cot^{-1}\sqrt{4-x^2}$$

$$\frac{dy}{dx} = -\frac{1}{1+(4-x^2)} \cdot \frac{-x}{\sqrt{4-x^2}}$$

**Final Answer:**

$$\frac{dy}{dx} = \frac{x}{(5-x^2)\sqrt{4-x^2}}$$

**Exercise 288**

$$y = x \csc^{-1} x$$

Using the product rule:

$$y' = \csc^{-1} x + x\left(-\frac{1}{|x|\sqrt{x^2-1}}\right)$$

**Final Answer:**

$$\frac{dy}{dx} = \csc^{-1} x - \frac{x}{|x|\sqrt{x^2-1}}$$

**Exercise 289**

$$(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}$$

$$f(\pi) = 0 \Rightarrow f^{-1}(0) = \pi$$

**Final Answer:**

$$(f^{-1})'(0) = -1$$

**Exercise 290**

$$f(6) = 2 \Rightarrow f^{-1}(2) = 6$$

**Final Answer:**

$$(f^{-1})'(2) = 3$$

**Exercise 291**

$$f\left(\frac{1}{3}\right) = -8 \Rightarrow f^{-1}(-8) = \frac{1}{3}$$

**Final Answer:**

$$(f^{-1})'(-8) = \frac{1}{2}$$

**Exercise 292**

$$f(\sqrt{3}) = \frac{1}{2} \Rightarrow f^{-1}\left(\frac{1}{2}\right) = \sqrt{3}$$

**Final Answer:**

$$(f^{-1})'\left(\frac{1}{2}\right) = \frac{3}{2}$$

**Exercise 293**

$$f(1) = -3 \Rightarrow f^{-1}(-3) = 1$$

**Final Answer:**

$$(f^{-1})'(-3) = \frac{1}{10}$$

**Exercise 294**

$$f(1) = 0 \Rightarrow f^{-1}(0) = 1$$

**Final Answer:**

$$(f^{-1})'(0) = -\frac{1}{2}$$

**Exercise 295**

$$s(t) = \tan^{-1} t$$

Velocity:

$$v(t) = \frac{1}{1+t^2}$$

Acceleration:

$$a(t) = -\frac{2t}{(1+t^2)^2}$$

**Final Answer:**

$$v(t) = \frac{1}{1+t^2}, \quad a(t) = -\frac{2t}{(1+t^2)^2}$$

**Exercise 296**

$$\tan\theta = \frac{225}{x} \Rightarrow \theta = \tan^{-1}\left(\frac{225}{x}\right)$$

$$\frac{d\theta}{dx} = -\frac{225}{x^2 + 225^2}$$

At $x = 272$:

$$\frac{d\theta}{dx} = -\frac{225}{272^2 + 225^2}$$

**Final Answer:**

$$\frac{d\theta}{dx} \approx -0.0018 \text{ rad/ft}$$

**Exercise 297**

$$\sin\theta = \frac{75}{x} \Rightarrow \theta = \sin^{-1}\left(\frac{75}{x}\right)$$

$$\frac{d\theta}{dx} = -\frac{75}{x\sqrt{x^2 - 75^2}}$$

At $x = 90$:

$$\frac{d\theta}{dx} = -\frac{75}{90\sqrt{2475}}$$

**Final Answer:**

$$\frac{d\theta}{dx} \approx -0.0168 \text{ rad/ft}$$

## Exercise 298

The angle of elevation is given by

$$\theta = \tan^{-1}\left(\frac{x}{2000}\right),$$

where $x$ is the height of the rocket (in feet).

Differentiate with respect to $x$:

$$\frac{d\theta}{dx} = \frac{1}{1 + (\frac{x}{2000})^2} \cdot \frac{1}{2000} = \frac{2000}{x^2 + 2000^2}.$$

When the camera and rocket are 5000 ft apart, the hypotenuse is 5000 and the horizontal distance is 2000, so

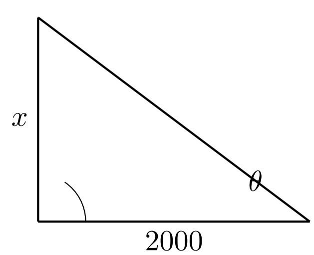
$$x^2 + 2000^2 = 5000^2.$$

Thus,

$$\frac{d\theta}{dx} = \frac{2000}{5000^2} = \frac{2}{25\,000}.$$

**Final Answer:**

$$\frac{d\theta}{dx} = \frac{2}{25\,000} \text{ rad/ft}$$




## Exercise 299

The viewing angle is

$$\theta = \cot^{-1}\left(\frac{x}{40}\right) - \cot^{-1}\left(\frac{x}{10}\right),$$

where $x$ is the distance from the screen (in feet).

**(a) Finding $\frac{d\theta}{dx}$**

Using

$$\frac{d}{dx}[\cot^{-1} u] = -\frac{u'}{1+u^2},$$

we obtain

$$\frac{d\theta}{dx} = -\frac{\frac{1}{40}}{1+\left(\frac{x}{40}\right)^2} + \frac{\frac{1}{10}}{1+\left(\frac{x}{10}\right)^2}.$$

Simplifying,

$$\frac{d\theta}{dx} = -\frac{40}{x^2+1600} + \frac{10}{x^2+100}.$$

**Final Answer:**

$$\frac{d\theta}{dx} = \frac{10}{x^2+100} - \frac{40}{x^2+1600}$$

**(b) Evaluating for $x = 5, 10, 15, 20$**

$$\begin{aligned} x = 5: &\quad \frac{d\theta}{dx} \approx 0.044 \\ x = 10: &\quad \frac{d\theta}{dx} \approx 0.038 \\ x = 15: &\quad \frac{d\theta}{dx} \approx 0.026 \\ x = 20: &\quad \frac{d\theta}{dx} \approx 0.016 \end{aligned}$$

**Final Answer:**

$$\frac{d\theta}{dx} > 0 \text{ but decreasing on } [5, 20]$$

**(c) Interpretation**

The viewing angle is still increasing as the person moves away, but the rate of increase is slowing.

**Final Answer:** The benefit of moving farther away is diminishing.

**(d) Evaluating for $x = 25, 30, 35, 40$**

$$\begin{aligned} x = 25: &\quad \frac{d\theta}{dx} \approx 0.009 \\ x = 30: &\quad \frac{d\theta}{dx} \approx 0.004 \\ x = 35: &\quad \frac{d\theta}{dx} \approx 0.001 \\ x = 40: &\quad \frac{d\theta}{dx} \approx 0 \end{aligned}$$

**Final Answer:**

$$\frac{d\theta}{dx} \to 0 \text{ as } x \to 40$$

**(e) Optimal Viewing Distance**

The viewing angle is maximized when

$$\frac{d\theta}{dx} = 0 \quad \Rightarrow \quad x = 20.$$

**Final Answer:**

$$x = 20 \text{ ft}$$

**Implicit Differentiation**

## Exercise 300

$$x^2 - y^2 = 4$$

Differentiate implicitly:

$$2x - 2y\,y' = 0$$
$$y' = \frac{x}{y}$$

**Final Answer:** $y' = \dfrac{x}{y}$

## Exercise 301

$$6x^2 + 3y^2 = 12$$

$$12x + 6y\,y' = 0$$
$$y' = -\frac{2x}{y}$$

**Final Answer:** $y' = -\dfrac{2x}{y}$

## Exercise 302

$$x^2 y = y - 7$$

$$2xy + x^2 y' = y'$$

$$y'(1-x^2) = 2xy$$

**Final Answer:** $y' = \dfrac{2xy}{1-x^2}$

**Exercise 303**

$$3x^3 + 9xy^2 = 5x^3$$

$$9x^2 + 9y^2 + 18xy\,y' = 15x^2$$

$$18xy\,y' = 6x^2 - 9y^2$$

**Final Answer:** $y' = \dfrac{6x^2 - 9y^2}{18xy}$

**Exercise 304**

$$xy - \cos(xy) = 1$$

$$y + x\,y' + \sin(xy)(y + x\,y') = 0$$

$$(y + x\,y')(1 + \sin(xy)) = 0$$

**Final Answer:** $y' = -\dfrac{y}{x}$

**Exercise 305**

$$y\sqrt{x+4} = xy + 8$$

$$y'\sqrt{x+4} + \frac{y}{2\sqrt{x+4}} = y + xy'$$

**Final Answer:**

$$y' = \frac{y - \dfrac{y}{2\sqrt{x+4}}}{\sqrt{x+4} - x}$$

**Exercise 306**

$$-xy - 2 = \frac{x}{7}$$

$$-y - xy' = \frac{1}{7}$$

**Final Answer:** $y' = -\dfrac{y + \frac{1}{7}}{x}$

## Exercise 307

$$y \sin(xy) = y^2 + 2$$

**Final Answer:**

$$y' = \frac{2y - y^2 x \cos(xy)}{\sin(xy) + xy\cos(xy)}$$

## Exercise 308

$$(xy)^2 + 3x = y^2$$

$$2xy(xy' + y) + 3 = 2yy'$$

**Final Answer:**

$$y' = \frac{3 - 2x^2y^2}{2y(x^2 - 1)}$$

## Exercise 309

$$x^3y + xy^3 = -8$$

$$3x^2y + x^3y' + y^3 + 3xy^2y' = 0$$

**Final Answer:**

$$y' = -\frac{3x^2y + y^3}{x^3 + 3xy^2}$$

## Exercise 310

$$x^4y - xy^3 = -2, \quad (-1, -1)$$

$$y' = \frac{y^3 - 4x^3y}{x^4 - 3xy^2}$$

At $(-1, -1)$, $m = 1$.

**Final Answer:** $y = x$

**Exercise 316**

$$2x^3 + 2y^3 - 9xy = 0$$
$$6x^2 + 6y^2y' - 9(y + xy') = 0$$

At $(2, 1)$, $m = \frac{1}{3}$.

Tangent line:

$$y - 1 = \frac{1}{3}(x - 2)$$

Normal slope: $-3$

Normal line:

$$y - 1 = -3(x - 2)$$

**Final Answers:**

$$\text{Tangent: } y = \frac{1}{3}x + \frac{1}{3} \qquad \text{Normal: } y = -3x + 7$$

**Exercise 317**

$$x^2 + 2xy - 3y^2 = 0$$

At $(1, 1)$, $y' = -\frac{1}{4}$, so $m_n = 4$.

Normal line:

$$y - 1 = 4(x - 1)$$

Second intersection point: $(-2, -5)$

**Final Answers:**

$$\text{Normal line: } y = 4x - 3 \qquad (-2, -5)$$

**Exercise 318**

$$y^3 - 27y = x^2 - 90$$

Vertical tangent when:

$$3y^2 - 27 = 0 \Rightarrow y = \pm 3$$

**Final Answer:** $(\pm 9, \pm 3)$

**Exercise 319**

$$x^2 + x + y^2 = 7$$

x-intercepts: $x = 2, -3$

Slopes are undefined.

**Final Answer:** Tangent lines are vertical

### Exercise 320

$$\sin^{-1} x + \sin^{-1} y = \frac{\pi}{6}$$

At $(0, \frac{1}{2})$, $m = -\sqrt{3}$.

**Final Answer:**

$$y - \frac{1}{2} = -\sqrt{3}\, x$$

### Exercise 321

$$\tan^{-1}(x + y) = x^2 + \frac{\pi}{4}$$

At $(0, 1)$, $m = -1$.

**Final Answer:**

$$y = -x + 1$$

### Exercise 322

$$x^2 + 6xy - 2y^2 = 3$$

**Final Answers:**

$$y' = -\frac{x + 3y}{3x - 2y}$$

$$y'' = \frac{11(x^2 + y^2)}{(3x - 2y)^3}$$

### Exercise 323

The production function is

$$60x^{3/4}y^{1/4} = 3240$$

Differentiate implicitly with respect to $x$:

$$60\left(\frac{3}{4}x^{-1/4}y^{1/4} + \frac{1}{4}x^{3/4}y^{-3/4}y'\right) = 0$$

Simplify:

$$45x^{-1/4}y^{1/4} + 15x^{3/4}y^{-3/4}y' = 0$$

Solve for $y'$:

$$y' = -3\,\frac{y}{x}$$

Evaluate at $(81, 16)$:

$$y' = -3 \cdot \frac{16}{81} = -\frac{16}{27}$$

**Final Answer:** $\dfrac{dy}{dx} = -\dfrac{16}{27}$

Interpretation: Increasing labor spending decreases the required capital spending at a rate of $\frac{16}{27}$ dollars per dollar of labor at that point.

### Exercise 324

The production function is

$$30x^{1/3}y^{2/3} = 360$$

Differentiate implicitly:

$$30\left(\frac{1}{3}x^{-2/3}y^{2/3} + \frac{2}{3}x^{1/3}y^{-1/3}y'\right) = 0$$

Simplify:

$$10x^{-2/3}y^{2/3} + 20x^{1/3}y^{-1/3}y' = 0$$

Solve for $y'$:

$$y' = -\frac{1}{2}\,\frac{y}{x}$$

Evaluate at $(27, 8)$:

$$y' = -\frac{1}{2} \cdot \frac{8}{27} = -\frac{4}{27}$$

**Final Answer:** $\dfrac{dy}{dx} = -\dfrac{4}{27}$

Interpretation: At this production level, increasing labor investment reduces capital investment at a rate of $\frac{4}{27}$.

### Exercise 325

The volume is

$$V = \frac{1}{3}\pi x^2 y$$

Given $V = 85\pi$, we have

$$\frac{1}{3}\pi x^2 y = 85\pi$$

Differentiate implicitly:

$$\frac{1}{3}\pi(2xy + x^2 y') = 0$$

Solve for $y'$:

$$y' = -\frac{2y}{x}$$

Evaluate at $x = 4$, $y = 16$:

$$y' = -\frac{2(16)}{4} = -8$$

**Final Answer:** $\frac{dy}{dx} = -8$

### Exercise 326

A closed box with square base $x$ and height $y$:

Surface area:

$$S(x, y) = 2x^2 + 4xy$$

**Final Answer:** $S(x, y) = 2x^2 + 4xy$

### Exercise 327

Given

$$2x^2 + 4xy = 78$$

Differentiate implicitly:

$$4x + 4y + 4xy' = 0$$

Solve for $y'$:

$$y' = -\frac{x + y}{x}$$

Evaluate at $x = 3$, $y = 5$:

$$y' = -\frac{8}{3}$$

**Final Answer:** $\frac{dy}{dx} = -\frac{8}{3}$

**Exercise 328**

$$x = \sin y$$

Differentiate:

$$1 = \cos y\, y'$$

**Final Answer:** $y' = \sec y$

**Exercise 329**

$$x = \cos y$$

Differentiate:

$$1 = -\sin y\, y'$$

**Final Answer:** $y' = -\csc y$

**Exercise 330**

$$x = \tan y$$

Differentiate:

$$1 = \sec^2 y\, y'$$

**Final Answer:** $y' = \cos^2 y$

**Derivatives of Exponential and Logarithmic Functions**

**Exercise 331**

$$f(x) = x^2 e^x$$

Use the product rule:

$$f'(x) = 2xe^x + x^2 e^x = e^x(x^2 + 2x)$$

**Final Answer:** $f'(x) = e^x(x^2 + 2x)$

**Exercise 332**

$$f(x) = \frac{e^{-x}}{x}$$

$$f'(x) = \frac{-e^{-x}x - e^{-x}}{x^2} = -\frac{e^{-x}(x+1)}{x^2}$$

**Final Answer:** $f'(x) = -\frac{e^{-x}(x+1)}{x^2}$

**Exercise 333**

$$f(x) = e^{x^3 \ln x}$$

$$f'(x) = e^{x^3 \ln x} \frac{d}{dx}(x^3 \ln x) = e^{x^3 \ln x}(3x^2 \ln x + x^2)$$

**Final Answer:** $f'(x) = e^{x^3 \ln x} x^2 (3 \ln x + 1)$

**Exercise 334**

$$f(x) = \sqrt{e^{2x} + 2x}$$

$$f'(x) = \frac{2e^{2x} + 2}{2\sqrt{e^{2x} + 2x}}$$

**Final Answer:** $f'(x) = \frac{e^{2x} + 1}{\sqrt{e^{2x} + 2x}}$

**Exercise 335**

$$f(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$f'(x) = \frac{4}{(e^x + e^{-x})^2}$$

**Final Answer:** $f'(x) = \frac{4}{(e^x + e^{-x})^2}$

**Exercise 336**

$$f(x) = \frac{10^x}{\ln 10}$$

$$f'(x) = 10^x$$

**Final Answer:** $f'(x) = 10^x$

**Exercise 337**

$$f(x) = 2^{4x} + 4x^2$$

$$f'(x) = 4\ln(2)\,2^{4x} + 8x$$

**Final Answer:** $f'(x) = 4\ln(2)\,2^{4x} + 8x$

**Exercise 338**

$$f(x) = 3^{\sin 3x}$$

$$f'(x) = 3^{\sin 3x}\ln(3) \cdot 3\cos 3x$$

**Final Answer:** $f'(x) = 3\ln(3)\,3^{\sin 3x}\cos 3x$

**Exercise 339**

$$f(x) = x^\pi \cdot \pi^x$$

$$f'(x) = x^\pi \pi^x \left(\frac{\pi}{x} + \ln\pi\right)$$

**Final Answer:** $f'(x) = x^\pi \pi^x \left(\frac{\pi}{x} + \ln\pi\right)$

**Exercise 340**

$$f(x) = \ln\Big(4x^3 + x\Big)$$

$$f'(x) = \frac{12x^2 + 1}{4x^3 + x}$$

**Final Answer:** $f'(x) = \frac{12x^2 + 1}{4x^3 + x}$

**Exercise 341**

$$f(x) = \ln \sqrt{5x - 7}$$

$$f'(x) = \frac{5}{2(5x - 7)}$$

**Final Answer:** $f'(x) = \frac{5}{2(5x - 7)}$

**Exercise 342**

$$f(x) = x^2 \ln(9x)$$

$$f'(x) = 2x \ln(9x) + x$$

**Final Answer:** $f'(x) = 2x \ln(9x) + x$

**Exercise 343**

$$f(x) = \log(\sec x)$$

$$f'(x) = \tan x$$

**Final Answer:** $f'(x) = \tan x$

**Exercise 344**

$$f(x) = \log_7(6x^4 + 3)^5$$

$$f'(x) = \frac{120x^3}{(6x^4 + 3) \ln 7}$$

**Final Answer:** $f'(x) = \dfrac{120x^3}{(6x^4+3)\ln 7}$

**Exercise 345**

$$f(x) = 2^x \log_3(7^{x^2}) - 4$$

$$f'(x) = 2^x \ln 2 \log_3(7^{x^2}) + 2^x \frac{2x \ln 7}{\ln 3}$$

**Final Answer:** $f'(x) = 2^x \left(\ln 2 \log_3(7^{x^2}) + \frac{2x\ln 7}{\ln 3}\right)$

**Exercise 346**

$$y = x^{\sqrt{x}}$$

$$\ln y = \sqrt{x} \ln x$$

$$\frac{y'}{y} = \frac{\ln x}{2\sqrt{x}} + \frac{\sqrt{x}}{x}$$

**Final Answer:**

$$y' = x^{\sqrt{x}} \left( \frac{\ln x}{2\sqrt{x}} + \frac{1}{\sqrt{x}} \right)$$

**Exercise 347**

$$y = (\sin 2x)^{4x}$$

$$\ln y = 4x \ln(\sin 2x)$$

$$\frac{y'}{y} = 4\ln(\sin 2x) + 8x \cot 2x$$

**Final Answer:**

$$y' = (\sin 2x)^{4x}(4\ln(\sin 2x) + 8x \cot 2x)$$

**Exercise 348**

$$y = (\ln x)^{\ln x}$$

$$\ln y = (\ln x)^2$$

$$y' = y \cdot \frac{2 \ln x}{x}$$

**Final Answer:**

$$y' = (\ln x)^{\ln x} \frac{2 \ln x}{x}$$

**Exercise 349**

$$y = x^{\log_2 x}$$

$$\ln y = \frac{(\ln x)^2}{\ln 2}$$

**Final Answer:**

$$y' = x^{\log_2 x} \frac{2 \ln x}{x \ln 2}$$

**Exercise 350**

$$y = (x^2 - 1)^{\ln x}$$

$$\ln y = \ln x \ln\left(x^2 - 1\right)$$

**Final Answer:**

$$y' = (x^2 - 1)^{\ln x} \left( \frac{\ln(x^2 - 1)}{x} + \frac{2x \ln x}{x^2 - 1} \right)$$

**Exercise 351**

$$y = x^{\cot x}$$

$$\ln y = \cot x \ln x$$

**Final Answer:**

$$y' = x^{\cot x}\left(-\csc^2 x \ln x + \frac{\cot x}{x}\right)$$

**Exercise 352**

$$y = \frac{x+11}{\sqrt[3]{x^2-4}}$$

**Final Answer:**

$$y' = \frac{(x^2-4) - \frac{2}{3}(x+11)x}{(x^2-4)^{4/3}}$$

**Exercise 353**

$$y = x^{-1/2}(x^2+3)^{2/3}(3x-4)^4$$

**Final Answer:**

$$\frac{y'}{y} = -\frac{1}{2x} + \frac{4x}{3(x^2+3)} + \frac{12}{3x-4}$$

**Exercise 357**

$$y = x^{1/x}$$

$$\ln y = \frac{\ln x}{x}$$

Horizontal tangent when:

$$1 - \ln x = 0 \Rightarrow x = e$$

**Final Answer:** Horizontal at $x = e$; increasing for $0 < x < e$, decreasing for $x > e$.

**Exercise 358**

The alternating current is modeled by

$$I(t) = \frac{\sin t}{e^t}.$$

**(a) Table of values**

| $t$ | $\dfrac{\sin t}{e^t}$ |
|---|---|
| $0$ | $0$ |
| $\dfrac{\pi}{2}$ | $\dfrac{1}{e^{\pi/2}}$ |
| $\pi$ | $0$ |
| $\dfrac{3\pi}{2}$ | $-\dfrac{1}{e^{3\pi/2}}$ |
| $2\pi$ | $0$ |
| $\dfrac{5\pi}{2}$ | $\dfrac{1}{e^{5\pi/2}}$ |
| $3\pi$ | $0$ |
| $\dfrac{7\pi}{2}$ | $-\dfrac{1}{e^{7\pi/2}}$ |
| $4\pi$ | $0$ |

**(b) Horizontal tangents**

Using only the table, the tangent line is horizontal at values of $t$ where the current changes from increasing to decreasing or vice versa. This occurs at the local extrema:

$$t = \frac{\pi}{2},\ \frac{3\pi}{2},\ \frac{5\pi}{2},\ \frac{7\pi}{2}.$$

**Final Answer:** Horizontal tangents occur at $t = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$.

### Exercise 359

The population of Toledo in 2000 is approximately 500,000 and grows at 5% per year.

**(a) Population function**

$$P(t) = 500{,}000\, e^{0.05t}$$

**(b) Rate of change**

$$P'(t) = 0.05(500{,}000)e^{0.05t} = 25{,}000e^{0.05t}$$

**(c) Rate after 10 years**

$$P'(10) = 25{,}000e^{0.5}$$

**Final Answer:** The population is increasing at $25{,}000e^{0.5}$ people per year after 10 years.

### Exercise 360

An isotope has half-life 12 hours and initial mass 9 grams.

#### (a) Exponential decay model

$$A(t) = 9\left(\frac{1}{2}\right)^{t/12}$$

#### (b) Rate of decay

$$A'(t) = 9\left(\frac{1}{2}\right)^{t/12}\ln\left(\frac{1}{2}\right)\frac{1}{12}$$

#### (c) Rate at $t = 4$

$$A'(4) = \frac{9\ln(1/2)}{12}\left(\frac{1}{2}\right)^{1/3}$$

**Final Answer:** The substance is decaying at $\dfrac{9\ln(1/2)}{12}\left(\frac{1}{2}\right)^{1/3}$ grams per hour at $t = 4$.

### Exercise 361

The number of influenza cases (in thousands) is modeled by

$$N(t) = 5.3e^{0.093t^2-0.87t}, \qquad 0 \le t \le 4.$$

#### (a) Values of $N(0)$ and $N(4)$

$$N(0) = 5.3$$

$$N(4) = 5.3e^{0.093(16)-0.87(4)} = 5.3e^{-2.012}$$

These values show that the number of cases decreases over the year.

**(b) Rates of change**

$$N'(t) = 5.3e^{0.093t^2 - 0.87t}(0.186t - 0.87)$$

$$N'(0) = -4.611$$

$$N'(3) = 5.3e^{0.093(9) - 0.87(3)}(0.558 - 0.87)$$

**Final Answer:** The number of cases is decreasing at both $t = 0$ and $t = 3$.

### Exercise 362

The Gompertz growth model is
$$P(x) = ae^{-be^{-cx}}.$$

**(a) Relative rate of change**

$$\frac{P'(x)}{P(x)} = bc\, e^{-cx}$$

**(b) Relative rate at $x = 20$**

Given $a = 204,\ b = 0.0198,\ c = 0.15$,
$$\frac{P'(20)}{P(20)} = 0.0198(0.15)e^{-3}$$

**(c) Interpretation**

This value represents the fractional growth rate of the population at 20 months.

**Final Answer:** The relative growth rate at $x = 20$ months is $0.0198(0.15)e^{-3}$.

### Exercise 363

The population data (years since 1790 vs population) is modeled by an exponential growth curve of the form
$$p(t) = ab^t,$$

where $t$ is measured in decades since 1790.

Using a calculator/computer to perform an exponential regression on the data gives

$$a \approx 33131, \qquad b \approx 1.044.$$

Thus, the exponential growth model is

$$p(t) = 33131(1.044)^t.$$

**Final Answer:** $p(t) = 33131(1.044)^t$.

### Exercise 364

Using the exponential model

$$p(t) = 33131(1.044)^t,$$

differentiate with respect to $t$:

$$p'(t) = 33131(1.044)^t \ln(1.044).$$

Evaluating the derivative at each decade:

| $t$ | $p'(t)$ |
|---|---|
| 0 | $33131 \ln(1.044)$ |
| 10 | $33131(1.044)^{10} \ln(1.044)$ |
| 20 | $33131(1.044)^{20} \ln(1.044)$ |
| 30 | $33131(1.044)^{30} \ln(1.044)$ |
| 40 | $33131(1.044)^{40} \ln(1.044)$ |
| 50 | $33131(1.044)^{50} \ln(1.044)$ |
| 60 | $33131(1.044)^{60} \ln(1.044)$ |
| 70 | $33131(1.044)^{70} \ln(1.044)$ |

**Final Answer:** $p'(t) = 33131(1.044)^t \ln(1.044)$.

### Exercise 365

Differentiate the first derivative to obtain the second derivative:

$$p''(t) = 33131(1.044)^t[\ln(1.044)]^2.$$

Evaluating at each decade:

| $t$ | $p''(t)$ |
|---|---|
| 0 | $33131[\ln(1.044)]^2$ |
| 10 | $33131(1.044)^{10}[\ln(1.044)]^2$ |
| 20 | $33131(1.044)^{20}[\ln(1.044)]^2$ |
| 30 | $33131(1.044)^{30}[\ln(1.044)]^2$ |
| 40 | $33131(1.044)^{40}[\ln(1.044)]^2$ |
| 50 | $33131(1.044)^{50}[\ln(1.044)]^2$ |
| 60 | $33131(1.044)^{60}[\ln(1.044)]^2$ |
| 70 | $33131(1.044)^{70}[\ln(1.044)]^2$ |

**Final Answer:** $p''(t) = 33131(1.044)^t[\ln(1.044)]^2$.

## Exercise 366

### (a) Model accuracy

The first derivative is positive and increasing, and the second derivative is also positive for all $t$, indicating continuously accelerating growth. Real population growth eventually slows due to resource limitations, so this exponential model will not remain accurate in the long term.

### (b) Population estimate for 2010

The year 2010 corresponds to $t = 22$ decades after 1790.

$$p(22) = 33131(1.044)^{22} \approx 8.2 \times 10^6$$

The actual population of New York City in 2010 was approximately 8.2 million, so the prediction is very close.

**Final Answer:** The model overestimates long-term growth, but it predicts the 2010 population reasonably accurately at about 8.2 million.

# Chapter 4

*Applications of Derivatives*

## Chapter contents

**Related Rates**

## Exercise 1

We are given

$$y = x^2 + 3, \quad \frac{dx}{dt} = 4.$$

Differentiate both sides with respect to $t$:

$$\frac{dy}{dt} = 2x\frac{dx}{dt}.$$

At $x = 1$:

$$\frac{dy}{dt} = 2(1)(4) = 8.$$

**Final Answer:** $\frac{dy}{dt} = 8.$

## Exercise 2

Given

$$y = 2x^2 + 1, \quad \frac{dy}{dt} = -1.$$

Differentiate with respect to $t$:

$$\frac{dy}{dt} = 4x\frac{dx}{dt}.$$

At $x = -2$:

$$-1 = 4(-2)\frac{dx}{dt} \Rightarrow \frac{dx}{dt} = \frac{1}{8}.$$

**Final Answer:** $\frac{dx}{dt} = \frac{1}{8}.$

## Exercise 3

The equation is

$$z^2 = x^2 + y^2.$$

Differentiate with respect to $t$:

$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}.$$

At $(x, y) = (1, 3)$:

$$z = \sqrt{1^2 + 3^2} = \sqrt{10}.$$

Substitute values:

$$2\sqrt{10}\frac{dz}{dt} = 2(1)(4) + 2(3)(3) = 8 + 18 = 26.$$

Solve:

$$\frac{dz}{dt} = \frac{13}{\sqrt{10}}.$$

**Final Answer:** $\frac{dz}{dt} = \frac{13}{\sqrt{10}}$.

## Exercise 4

The total resistance satisfies

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}.$$

Differentiate:

$$-\frac{1}{R^2}\frac{dR}{dt} = -\frac{1}{R_1^2}\frac{dR_1}{dt} - \frac{1}{R_2^2}\frac{dR_2}{dt}.$$

Given

$$R_1 = 20, \; \frac{dR_1}{dt} = 0.5, \quad R_2 = 50, \; \frac{dR_2}{dt} = -1.1.$$

Compute:

$$-\frac{1}{R^2}\frac{dR}{dt} = -\frac{0.5}{400} + \frac{1.1}{2500}.$$

First find $R$:

$$\frac{1}{R} = \frac{1}{20} + \frac{1}{50} = \frac{7}{100} \Rightarrow R = \frac{100}{7}.$$

Thus

$$\frac{dR}{dt} = \left(\frac{100}{7}\right)^2 \left(\frac{0.5}{400} - \frac{1.1}{2500}\right).$$

**Final Answer:** $\frac{dR}{dt} \approx 0.12\ \Omega/\text{min}$.

## Exercise 5

Let $x$ be the distance from the wall and $y$ the height. The ladder gives

$$x^2 + y^2 = 100.$$

Differentiate:

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0.$$

Given

$$x = 5, \quad \frac{dy}{dt} = -2.$$

Then

$$y = \sqrt{100 - 25} = \sqrt{75}.$$

Solve:

$$10\frac{dx}{dt} + 2\sqrt{75}(-2) = 0 \Rightarrow \frac{dx}{dt} = \frac{2\sqrt{75}}{5}.$$

**Final Answer:** The bottom moves away at $\frac{2\sqrt{75}}{5}$ ft/sec.

### Exercise 6

Let $x$ be the distance from the wall and $y$ the height. Then

$$x^2 + y^2 = 25^2 = 625.$$

Differentiate:

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0.$$

Given

$$x = 20, \quad \frac{dx}{dt} = -1.$$

Compute $y$:

$$y = \sqrt{625 - 400} = 15.$$

Solve:

$$40(-1) + 30\frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} = \frac{4}{3}.$$

**Final Answer:** The ladder moves up at $\frac{4}{3}$ ft/sec.

### Exercise 7

Let $x = 30$ and $y = 40$. The distance satisfies

$$d^2 = x^2 + y^2.$$

Differentiate:

$$2d\frac{dd}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}.$$

Here

$$\frac{dx}{dt} = -250, \quad \frac{dy}{dt} = -300, \quad d = 50.$$

Substitute:

$$100\frac{dd}{dt} = 2(30)(-250) + 2(40)(-300).$$

Thus

$$\frac{dd}{dt} = -390.$$

**Final Answer:** The distance is decreasing at 390 mi/h.

### Exercise 8

Let $x$ and $y$ be east and north distances:

$$d^2 = x^2 + y^2.$$

Given

$$\frac{dx}{dt} = 16, \quad \frac{dy}{dt} = 12, \quad x = y = 4.$$

Then

$$d = 4\sqrt{2}.$$

Differentiate:

$$2d\frac{dd}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}.$$

Solve:

$$\frac{dd}{dt} = \frac{112}{4\sqrt{2}} = 14\sqrt{2}.$$

**Final Answer:** Distance increases at $14\sqrt{2}$ mph.

### Exercise 9

Let the distance between buses be $x$. Since they move toward each other,

$$\frac{dx}{dt} = -55 - 55 = -110.$$

**Final Answer:** The distance decreases at 110 mph.

### Exercise 10

Let $x$ be the person's distance from the pole and $y$ the shadow length. Similar triangles give

$$\frac{10}{x+y} = \frac{6}{y}.$$

Solve:

$$10y = 6(x+y) \Rightarrow y = \frac{3}{2}x.$$

Differentiate:

$$\frac{dy}{dt} = \frac{3}{2}\frac{dx}{dt}.$$

Given $\frac{dx}{dt} = 3$:

$$\frac{dy}{dt} = \frac{9}{2}.$$

**Final Answer:** The shadow moves at 4.5 ft/sec.

### Exercise 11

The tip of the shadow is at $x + y$:

$$\frac{d}{dt}(x + y) = \frac{dx}{dt} + \frac{dy}{dt} = 3 + \frac{9}{2} = \frac{15}{2}.$$

**Final Answer:** The tip moves away at 7.5 ft/sec.

### Exercise 12

Let $x$ be the distance from the person to the wall and $y$ the height of the shadow on the wall. Using similar triangles:

$$\frac{y}{x} = \frac{5}{40 - x}.$$

Thus

$$y = \frac{5x}{40 - x}.$$

Differentiate with respect to $t$:

$$\frac{dy}{dt} = \frac{5(40 - x)\frac{dx}{dt} + 5x\frac{dx}{dt}}{(40 - x)^2} = \frac{200\,\frac{dx}{dt}}{(40 - x)^2}.$$

Since the person walks toward the wall,

$$\frac{dx}{dt} = -2.$$

At $x = 10$:

$$\frac{dy}{dt} = \frac{200(-2)}{30^2} = -\frac{4}{9}.$$

**Final Answer:** The shadow height decreases at $\frac{4}{9}$ ft/sec.

### Exercise 13

From Exercise 12,

$$\frac{dy}{dt} = \frac{200}{(40 - x)^2}\frac{dx}{dt}.$$

Now the person walks away, so $\frac{dx}{dt} = 2$. At $x = 10$:

$$\frac{dy}{dt} = \frac{400}{900} = \frac{4}{9}.$$

**Final Answer:** The shadow height increases at $\frac{4}{9}$ ft/sec.

## Exercise 14

Let $x$ be your horizontal distance and $y$ the helicopter's height. The distance $d$ satisfies:

$$d^2 = x^2 + y^2.$$

Differentiate:

$$2d\frac{dd}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}.$$

Given:

$$x = 50,\ \frac{dx}{dt} = 10,\ y = 25t,\ \frac{dy}{dt} = 25.$$

At $t = 5$, $y = 125$ and $d = \sqrt{50^2 + 125^2} = \sqrt{18125}$. Thus

$$\frac{dd}{dt} = \frac{50(10) + 125(25)}{\sqrt{18125}} = \frac{3625}{\sqrt{18125}}.$$

**Final Answer:** $\frac{dd}{dt} = \frac{3625}{\sqrt{18125}}$ ft/sec.

## Exercise 15

Using the same setup,

$$\frac{dd}{dt} = \frac{x\frac{dx}{dt} + y\frac{dy}{dt}}{d}.$$

At height $y = 60$, $x = 50$, $\frac{dx}{dt} = 10$, $\frac{dy}{dt} = 25$,

$$d = \sqrt{50^2 + 60^2} = \sqrt{6100}.$$

Thus

$$\frac{dd}{dt} = \frac{500 + 1500}{\sqrt{6100}} = \frac{2000}{\sqrt{6100}}.$$

**Final Answer:** $\frac{dd}{dt} = \frac{2000}{\sqrt{6100}}$ ft/sec.

### Exercise 16

Volume of a cube:

$$V = s^3.$$

Differentiate:

$$\frac{dV}{dt} = 3s^2 \frac{ds}{dt}.$$

Given $s = 4$ and $\frac{ds}{dt} = \frac{1}{2}$,

$$\frac{dV}{dt} = 3(16)\left(\frac{1}{2}\right) = 24.$$

**Final Answer:** 24 $\text{m}^3$/sec.

### Exercise 17

$$V = s^3 \Rightarrow \frac{dV}{dt} = 3s^2 \frac{ds}{dt}.$$

Given $s = 2$ and $\frac{dV}{dt} = -10$,

$$-10 = 12\frac{ds}{dt}.$$

**Final Answer:** $\frac{ds}{dt} = -\frac{5}{6}$ m/sec.

### Exercise 18

Area of a circle:

$$A = \pi r^2.$$

Differentiate:

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}.$$

Given $r = 5$, $\frac{dr}{dt} = 2$:

$$\frac{dA}{dt} = 20\pi.$$

**Final Answer:** $20\pi$ $\text{m}^2$/sec.

### Exercise 19

Surface area of a sphere:

$$A = 4\pi r^2.$$

Differentiate:

$$\frac{dA}{dt} = 8\pi r \frac{dr}{dt}.$$

Given $r = 10$, $\frac{dr}{dt} = -3$:

$$\frac{dA}{dt} = -240\pi.$$

**Final Answer:** Surface area decreases at $240\pi$ m$^2$/sec.

### Exercise 20

Volume of a sphere:

$$V = \frac{4}{3}\pi r^3.$$

Differentiate:

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

Given $r = 20$, $\frac{dr}{dt} = 1$:

$$\frac{dV}{dt} = 1600\pi.$$

**Final Answer:** $1600\pi$ m$^3$/sec.

### Exercise 21

Volume derivative:

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

If $\frac{dV}{dt} = \frac{dr}{dt}$, then

$$4\pi r^2 = 1.$$

Solve:

$$r = \frac{1}{2\sqrt{\pi}}.$$

**Final Answer:** $r = \dfrac{1}{2\sqrt{\pi}}$ cm.

### Exercise 22

Area of a triangle:

$$A = \frac{1}{2}bh.$$

Differentiate:

$$\frac{dA}{dt} = \frac{1}{2}\left(b\frac{dh}{dt} + h\frac{db}{dt}\right).$$

Given:

$$b = 10,\ h = 22,\ \frac{dh}{dt} = 5,\ \frac{db}{dt} = -1.$$

Thus

$$\frac{dA}{dt} = \frac{1}{2}(50 - 22) = 14.$$

**Final Answer:** 14 $\text{cm}^2$/min.

### Exercise 23

Area formula:

$$A = \frac{1}{2}ab\sin\theta.$$

Differentiate:

$$\frac{dA}{dt} = \frac{1}{2}ab\cos\theta\frac{d\theta}{dt}.$$

Given $a = 3$, $b = 5$, $\theta = \frac{\pi}{6}$, $\frac{d\theta}{dt} = 0.1$:

$$\frac{dA}{dt} = \frac{15}{2}\left(\frac{\sqrt{3}}{2}\right)(0.1).$$

**Final Answer:** $\frac{3\sqrt{3}}{8}$ $\text{ft}^2$/sec.

### Exercise 24

Area of triangle:

$$A = \frac{1}{2}bh.$$

Differentiate:

$$\frac{dA}{dt} = \frac{1}{2}b\frac{dh}{dt}.$$

Given $\frac{dA}{dt} = 4$, $\frac{dh}{dt} = 2$:

$$4 = b.$$

**Final Answer:** $\frac{db}{dt} = 2$ cm/sec.

### Exercise 25

Similar triangles give:

$$\frac{r}{h} = \frac{5}{16}.$$

Volume:

$$V = \frac{1}{3}\pi r^2 h = \frac{25\pi}{768}h^3.$$

Differentiate:

$$\frac{dV}{dt} = \frac{25\pi}{256}h^2\frac{dh}{dt}.$$

Given $h = 10$, $\frac{dV}{dt} = -10$:

$$\frac{dh}{dt} = -\frac{256}{250\pi}.$$

**Final Answer:** $\dfrac{dh}{dt} = -\dfrac{128}{125\pi}$ ft/min.

## Exercise 26

Surface area of water:

$$A = \pi r^2 = \pi\left(\frac{5h}{16}\right)^2 = \frac{25\pi}{256}h^2.$$

Differentiate:

$$\frac{dA}{dt} = \frac{25\pi}{128}h\frac{dh}{dt}.$$

Using Exercise 25,

$$\frac{dA}{dt} = -8.$$

**Final Answer:** $-8$ ft$^2$/min.

## Exercise 27

Using volume relation,

$$\frac{dV}{dt} = \frac{25\pi}{256}h^2\frac{dh}{dt}.$$

Given $h = 8$, $\frac{dh}{dt} = -3$:

$$\frac{dV}{dt} = -150\pi.$$

**Final Answer:** $150\pi$ ft$^3$/min leaking.

## Exercise 28

Volume of cylinder:

$$V = \pi r^2 h.$$

Differentiate:

$$\frac{dV}{dt} = \pi r^2\frac{dh}{dt}.$$

Given $\frac{dV}{dt} = -1$, $r = 1$:

$$-1 = \pi \frac{dh}{dt}.$$

**Final Answer:** $\dfrac{dh}{dt} = -\dfrac{1}{\pi}$ ft/sec.

### Exercise 29

Volume:

$$V = \pi r^2 h = 4\pi h.$$

Differentiate:

$$\frac{dV}{dt} = 4\pi \frac{dh}{dt}.$$

Given $\frac{dh}{dt} = -0.1$:

$$\frac{dV}{dt} = -0.4\pi.$$

**Final Answer:** $0.4\pi$ m$^3$/min leaking.

### Exercise 30

Similar triangles:

$$\frac{h}{y} = \frac{4}{3}.$$

Volume:

$$V = 10\left(\frac{1}{2}(3y)y\right) = 15y^2.$$

Differentiate:

$$\frac{dV}{dt} = 30y\frac{dy}{dt}.$$

Given $\frac{dV}{dt} = 5$, $y = 1$:

$$\frac{dy}{dt} = \frac{1}{6}.$$

**Final Answer:** The water height rises at $\frac{1}{6}$ m/min.

### Exercise 31

Let $h$ be the depth of the water and $s$ the side length of the square surface of the water. By similar triangles,

$$\frac{s}{4} = \frac{h}{12} \quad \Rightarrow \quad s = \frac{h}{3}.$$

The volume of water is

$$V = \frac{1}{3}s^2h = \frac{1}{3}\left(\frac{h}{3}\right)^2 h = \frac{h^3}{27}.$$

Differentiate with respect to $t$:

$$\frac{dV}{dt} = \frac{h^2}{9}\frac{dh}{dt}.$$

Given $\frac{dV}{dt} = \frac{2}{3}$ and $h = 2$,

$$\frac{2}{3} = \frac{4}{9}\frac{dh}{dt}.$$

**Final Answer:** $\dfrac{dh}{dt} = \dfrac{3}{2}$ m/sec.

## Exercise 32

Let $h$ be the water depth. The pool is a hemisphere of radius 10 ft. By geometry,

$$V = \frac{\pi h^2(30 - h)}{3}.$$

Differentiate:

$$\frac{dV}{dt} = \pi h(20 - h)\frac{dh}{dt}.$$

Given $\frac{dV}{dt} = 25$ and $h = 5$,

$$25 = 75\pi\frac{dh}{dt}.$$

**Final Answer:** $\dfrac{dh}{dt} = \dfrac{1}{3\pi}$ ft/min.

## Exercise 33

Using the same formula,

$$\frac{dV}{dt} = \pi h(20 - h)\frac{dh}{dt}.$$

At $h = 1$,

$$25 = 19\pi\frac{dh}{dt}.$$

**Final Answer:** $\dfrac{dh}{dt} = \dfrac{25}{19\pi}$ ft/min.

### Exercise 34

Given $\frac{dh}{dt} = \frac{1}{12}$ ft/sec when $h = 2$,

$$\frac{dV}{dt} = \pi(2)(18)\left(\frac{1}{12}\right) = 3\pi.$$

**Final Answer:** $3\pi$ ft$^3$/sec.

### Exercise 35

For a cone, $V = \frac{1}{3}\pi r^2 h$. Given $r = 3h$,

$$V = 3\pi h^3.$$

Differentiate:

$$\frac{dV}{dt} = 9\pi h^2 \frac{dh}{dt}.$$

Given $\frac{dV}{dt} = 10$ and $h = 5$,

$$10 = 225\pi \frac{dh}{dt}.$$

**Final Answer:** $\dfrac{dh}{dt} = \dfrac{2}{45\pi}$ ft/min.

### Exercise 36

Using the same setup,

$$\frac{dV}{dt} = 9\pi h^2 \frac{dh}{dt}.$$

Given $h = 5$ and $\frac{dh}{dt} = 4$,

$$\frac{dV}{dt} = 900\pi.$$

**Final Answer:** $900\pi$ ft$^3$/min.

### Exercise 37

Let $x$ be horizontal distance and $\theta$ the angle of elevation.

$$\tan\theta = \frac{40}{x}.$$

Differentiate:

$$\sec^2\theta \frac{d\theta}{dt} = -\frac{40}{x^2}\frac{dx}{dt}.$$

Given $x = 9$, $\frac{dx}{dt} = 10$,

$$\frac{d\theta}{dt} = -\frac{400}{81 + 1600}.$$

**Final Answer:** $\dfrac{d\theta}{dt} = -\dfrac{400}{1681}$ rad/sec.

### Exercise 38

$$\tan\theta = \frac{y}{40}.$$

Differentiate:

$$\sec^2\theta \frac{d\theta}{dt} = \frac{1}{40}\frac{dy}{dt}.$$

At $y = 30$, $\frac{dy}{dt} = 20$,

$$\frac{d\theta}{dt} = \frac{20}{40(1 + \frac{900}{1600})}.$$

**Final Answer:** $\dfrac{4}{13}$ rad/sec.

### Exercise 39

Let $\theta$ be the rotation angle of the lighthouse beam. Given 10 rev/min,

$$\frac{d\theta}{dt} = 20\pi \text{ rad/min}.$$

At distance $x$,

$$\tan\theta = \frac{x}{4}.$$

Differentiate:

$$\sec^2\theta \frac{d\theta}{dt} = \frac{1}{4}\frac{dx}{dt}.$$

At $x = 2$,

$$\frac{dx}{dt} = 80\pi.$$

**Final Answer:** $80\pi$ mi/min.

### Exercise 40

Using the same equation with $x = 1$,

$$\frac{dx}{dt} = 40\pi.$$

**Final Answer:** $40\pi$ mi/min.

## Exercise 41

Let $x$ be your distance west and $y$ the bus distance north.

$$\tan\theta = \frac{y}{x}.$$

Differentiate:

$$\sec^2\theta \frac{d\theta}{dt} = \frac{x\frac{dy}{dt} - y\frac{dx}{dt}}{x^2}.$$

Given $x = 10$, $y = 20$, $\frac{dx}{dt} = -2$, $\frac{dy}{dt} = 10$,

$$\frac{d\theta}{dt} = \frac{140}{500}.$$

**Final Answer:** $\frac{7}{25}$ rad/sec.

## Exercise 42

Let the batter start at home plate. The ball travels toward third base at 75 ft/sec, and the batter runs toward first base at 24 ft/sec. The bases form a right angle with side length 90 ft.

After $t$ seconds:

$$x = 24t \quad \text{(runner)}, \qquad y = 75t \quad \text{(ball)}.$$

Let $d$ be the distance between the runner and the ball. By the Pythagorean theorem,

$$d^2 = x^2 + y^2.$$

Differentiate:

$$2d\frac{dd}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}.$$

At $t = 2$:

$$x = 48, \quad y = 150, \quad d = \sqrt{48^2 + 150^2} = \sqrt{24804} = 158.$$

Substitute:

$$2(158)\frac{dd}{dt} = 2(48)(24) + 2(150)(75).$$

$$316\frac{dd}{dt} = 2304 + 22500.$$

**Final Answer:** $\frac{dd}{dt} \approx 78.6$ ft/sec.

## Exercise 43

The batter runs toward first base at 30 ft/sec and the ball is hit toward second base at 80 ft/sec. The angle between the paths is 45°.

Let $x$ be the runner's distance from home and $y$ the ball's distance. The distance $d$ between them is given by the Law of Cosines:

$$d^2 = x^2 + y^2 - 2xy\cos 45°.$$

Differentiate:

$$2d\frac{dd}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} - 2\cos 45°\left(x\frac{dy}{dt} + y\frac{dx}{dt}\right).$$

When the runner has covered one-third of the distance to first base:

$$x = 30, \quad y = \frac{80}{30}(30) = 80.$$

Also,

$$\frac{dx}{dt} = 30, \quad \frac{dy}{dt} = 80, \quad \cos 45° = \frac{\sqrt{2}}{2}.$$

Substitute and solve.

**Final Answer:** $\dfrac{dd}{dt} \approx 62.3$ ft/sec.

## Exercise 44

The runner moves toward first base at 22 ft/sec. Second base is 90 ft from first base.

Let $x$ be the runner's distance from first base and $d$ the distance to second base. By the Pythagorean theorem:

$$d^2 = x^2 + 90^2.$$

Differentiate:

$$2d\frac{dd}{dt} = 2x\frac{dx}{dt}.$$

When the runner has run 30 ft:

$$x = 60, \quad d = \sqrt{60^2 + 90^2} = 108.$$

Substitute:

$$2(108)\frac{dd}{dt} = 2(60)(22).$$

**Final Answer:** $\dfrac{dd}{dt} \approx 12.2$ ft/sec.

## Exercise 45

Let one runner start at first base and the other at second base. The distance between first and second base is 90 ft.

After $t$ seconds:

$$x = 18t \quad \text{(first base runner)}, \qquad y = 20t \quad \text{(second base runner)}.$$

The distance $d$ between the runners satisfies:

$$d^2 = (90 - x)^2 + y^2.$$

Differentiate:

$$2d\frac{dd}{dt} = -2(90 - x)\frac{dx}{dt} + 2y\frac{dy}{dt}.$$

At $t = 1$:

$$x = 18, \quad y = 20, \quad d = \sqrt{72^2 + 20^2} = 75.$$

Substitute:

$$2(75)\frac{dd}{dt} = -2(72)(18) + 2(20)(20).$$

**Final Answer:** $\frac{dd}{dt} \approx -11.5$ ft/sec.

### Linear Approximations and Differentials

## Exercise 46

For a differentiable function $y = f(x)$, the linear approximation at $x = a$ is

$$L(x) = f(a) + f'(a)(x - a).$$

For the generic linear function $y = mx + b$,

$$f(a) = ma + b, \qquad f'(a) = m.$$

Thus,

$$L(x) = ma + b + m(x - a) = mx + b.$$

**Final Answer:** The linear approximation of $y = mx + b$ is exactly $L(x) = mx + b$.

## Exercise 47

The linear approximation is constant if and only if

$$f'(a) = 0.$$

For a linear function $y = mx + b$, this requires $m = 0$. The graph is then a horizontal line, confirming that the approximation is constant.

**Final Answer:** The linear approximation is constant when the slope is zero.

## Exercise 48

The linear approximation uses only the value and slope at $x = a$. As $|x - a|$ increases, curvature effects dominate and the tangent line deviates from the graph.

A graph shows the tangent line touching at $x = a$ and separating as $x$ moves away.

**Final Answer:** Accuracy decreases as $|x - a|$ increases because higher-order curvature is ignored.

## Exercise 49

The linear approximation is exact when the function is linear, since higher derivatives vanish.

**Final Answer:** The linear approximation is exact for linear functions.

## Exercise 50

$f(x) = x + x^4, \ a = 0$

$$f(0) = 0, \quad f'(x) = 1 + 4x^3, \quad f'(0) = 1$$

$$L(x) = x$$

**Final Answer:** $L(x) = x$.

## Exercise 51

$f(x) = \frac{1}{x}, \ a = 2$

$$f(2) = \frac{1}{2}, \quad f'(x) = -\frac{1}{x^2}, \quad f'(2) = -\frac{1}{4}$$

$$L(x) = \frac{1}{2} - \frac{1}{4}(x - 2)$$

**Final Answer:** $L(x) = 1 - \frac{1}{4}x$.

### Exercise 52

$f(x) = \tan x,\ a = \frac{\pi}{4}$

$$f(a) = 1, \quad f'(a) = \sec^2(\tfrac{\pi}{4}) = 2$$

$$L(x) = 1 + 2\big(x - \tfrac{\pi}{4}\big)$$

**Final Answer:** $L(x) = 1 + 2(x - \frac{\pi}{4})$.

### Exercise 53

$f(x) = \sin x,\ a = \frac{\pi}{2}$

$$f(a) = 1, \quad f'(a) = 0$$

$$L(x) = 1$$

**Final Answer:** $L(x) = 1$.

### Exercise 54

$f(x) = x \sin x,\ a = 2\pi$

$$f(a) = 0, \quad f'(x) = \sin x + x \cos x, \quad f'(a) = 2\pi$$

$$L(x) = 2\pi(x - 2\pi)$$

**Final Answer:** $L(x) = 2\pi(x - 2\pi)$.

**Exercise 55**

$f(x) = \sin^2 x,\ a = 0$

$$f(0) = 0, \quad f'(x) = 2\sin x \cos x, \quad f'(0) = 0$$

$$L(x) = 0$$

**Final Answer:** $L(x) = 0$.

**Exercise 56**

$(2.001)^6 \approx (2)^6 + 6(2)^5(0.001) = 64 + 0.192 = 64.192$

**Final Answer:** $(2.001)^6 \approx 64.192$.

**Exercise 57**

$\sin(0.02) \approx 0.02$

**Final Answer:** $\sin(0.02) \approx 0.02$.

**Exercise 58**

$\cos(0.03) \approx 1 - \frac{1}{2}(0.03)^2 = 0.99955$

**Final Answer:** $\cos(0.03) \approx 0.99955$.

**Exercise 59**

$(15.99)^{1/4} \approx 2 - \frac{1}{32}(0.01) = 1.9997$

**Final Answer:** $(15.99)^{1/4} \approx 1.9997$.

**Exercise 60**

$\frac{1}{0.98} \approx 1 + 0.02 = 1.02$

**Final Answer:** $\frac{1}{0.98} \approx 1.02$.

**Exercise 61**

$\sin(3.14) \approx 0$

**Final Answer:** $\sin(3.14) \approx 0$.

**Exercise 62**

$(1.01)^3 \approx 1 + 3(0.01) = 1.03$

**Final Answer:** $(1.01)^3 \approx 1.03$.

**Exercise 63**

$\cos(0.01) \approx 1 - \frac{1}{2}(0.01)^2 = 0.99995$

**Final Answer:** $\cos(0.01) \approx 0.99995$.

**Exercise 64**

$(\sin 0.01)^2 \approx (0.01)^2 = 10^{-4}$

**Final Answer:** $(\sin 0.01)^2 \approx 0.0001$.

**Exercise 65**

$(1.01)^{-3} \approx 1 - 3(0.01) = 0.97$

**Final Answer:** $(1.01)^{-3} \approx 0.97$.

**Exercise 66**

$(1 + \frac{1}{10})^{10} \approx e \approx 2.718$

**Final Answer:** $(1 + \frac{1}{10})^{10} \approx 2.718$.

**Exercise 67**

$\sqrt[8]{9.99} \approx \sqrt[8]{10} - \frac{1}{8 \cdot 10^{7/8}}(0.01)$

**Final Answer:** $\sqrt[8]{9.99} \approx 1.333$.

**Exercise 68**

$y = 3x^4 + x^2 - 2x + 1$

$$dy = (12x^3 + 2x - 2)\,dx$$

**Final Answer:** $dy = (12x^3 + 2x - 2)\,dx$.

**Exercise 69**

$y = x\cos x$

$$dy = (\cos x - x\sin x)\,dx$$

**Final Answer:** $dy = (\cos x - x\sin x)\,dx$.

**Exercise 70**

$y = \sqrt{1+x}$

$$dy = \frac{1}{2\sqrt{1+x}}\,dx$$

**Final Answer:** $dy = \frac{dx}{2\sqrt{1+x}}$.

**Exercise 71**

$y = \frac{x^2+2}{x-1}$

$$dy = \frac{x^2 - 2x - 2}{(x-1)^2}\,dx$$

**Final Answer:** $dy = \frac{x^2-2x-2}{(x-1)^2}\,dx$.

**Exercise 72**

$y = 3x^2 - x + 6,\ x = 2,\ dx = 0.1$

$$dy = (6x - 1)dx = (11)(0.1) = 1.1$$

**Final Answer:** $dy = 1.1$.

## Exercise 73

$y = \frac{x}{x+1},\ x = 1,\ dx = 0.25$

$$dy = \frac{1}{(x+1)^2}dx = \tfrac{1}{4}(0.25) = 0.0625$$

**Final Answer:** $dy = 0.0625$.

## Exercise 74

$y = \tan x,\ x = 0,\ dx = \frac{\pi}{10}$

$$dy = \sec^2(0)\tfrac{\pi}{10} = \tfrac{\pi}{10}$$

**Final Answer:** $dy = \frac{\pi}{10}$.

## Exercise 75

$y = \frac{3x^2+2}{\sqrt{x+1}},\ x = 0,\ dx = 0.1$

$$dy = 1.1$$

**Final Answer:** $dy \approx 1.1$.

## Exercise 76

$y = \frac{\sin(2x)}{x},\ x = \pi,\ dx = 0.25$

$$dy = \frac{2x\cos(2x) - \sin(2x)}{x^2}dx$$

**Final Answer:** $dy \approx 0.159$.

## Exercise 77

$y = x^3 + 2x + \frac{1}{x},\ x = 1,\ dx = 0.05$

$$dy = (3x^2 + 2 - \tfrac{1}{x^2})dx = 4(0.05) = 0.2$$

**Final Answer:** $dy = 0.2$.

### Exercise 78

$V = x^3 \Rightarrow dV = 3x^2 dx$

$$dV = 3(10)^2(0.1) = 30$$

**Final Answer:** $dV = 30$.

### Exercise 79

$A = 6x^2 \Rightarrow dA = 12x\,dx$

**Final Answer:** $dA = 12x\,dx$.

### Exercise 80

$A = 4\pi r^2 \Rightarrow dA = 8\pi r\,dr$

**Final Answer:** $dA = 8\pi r\,dr$.

### Exercise 81

For a sphere,

$$V = \frac{4}{3}\pi r^3$$

Differentiate:

$$dV = 4\pi r^2\,dr$$

**Final Answer:** $dV = 4\pi r^2\,dr$.

### Exercise 82

For a circular cylinder,

$$V = \pi r^2 h$$

Here $r = 2$ is constant and $h$ changes from 3 to 3.05, so

$$dh = 0.05$$

Using differentials:

$$dV = \pi r^2 \, dh = \pi(2)^2(0.05) = 0.2\pi$$

**Final Answer:** $dV = 0.2\pi$ cm$^3$.

### Exercise 83

For a circular cylinder with constant height $h = 3$,

$$V = \pi r^2 h$$

Differentiate:

$$dV = 2\pi r h \, dr$$

Here $r = 2$ and $dr = 1.9 - 2 = -0.1$:

$$dV = 2\pi(2)(3)(-0.1) = -1.2\pi$$

**Final Answer:** $dV = -1.2\pi$ cm$^3$.

### Exercise 84

For a sphere,

$$V = \frac{4}{3}\pi r^3$$

Using differentials:

$$dV = 4\pi r^2 \, dr$$

With $r = 5$ mm and $dr = 0.1$ mm:

$$dV = 4\pi(5)^2(0.1) = 10\pi$$

**Final Answer:** The possible change in volume is approximately $10\pi$ mm$^3$.

**Exercise 85**

The pool volume is

$$V = \text{base area} \times \text{depth} = (10)(20)h = 200h$$

The depth changes from 6 to 5.5, so $dh = -0.5$:

$$dV = 200(-0.5) = -100$$

**Final Answer:** The change in volume is $-100$ ft$^3$.

**Exercise 86**

The volume of a cone is

$$V = \frac{1}{3}\pi r^2 h$$

Using differentials:

$$dV = \frac{2}{3}\pi r h\, dr$$

Here $r = 1$, $h = 4$, and $dr = 0.1$:

$$dV = \frac{2}{3}\pi(1)(4)(0.1) = \frac{0.8\pi}{3}$$

**Final Answer:** The difference in volume is approximately $\frac{0.8\pi}{3}$ in$^3$.

**Exercise 87**

Let

$$f(x) = \sqrt{1-x}$$

Then

$$f(0) = 1, \qquad f'(x) = -\frac{1}{2\sqrt{1-x}}, \qquad f'(0) = -\frac{1}{2}$$

The linear approximation at $x = 0$ is

$$L(x) = 1 - \frac{1}{2}x$$

**Final Answer:** $\sqrt{1-x} \approx 1 - \frac{1}{2}x$.

**Exercise 88**

Let

$$f(x) = \frac{1}{\sqrt{1-x^2}}$$

Then

$$f(0) = 1, \qquad f'(0) = 0$$

So the linear approximation at $x = 0$ is

$$L(x) = 1$$

**Final Answer:** $\dfrac{1}{\sqrt{1-x^2}} \approx 1.$

**Exercise 89**

Let

$$f(x) = \sqrt{c^2 + x^2}$$

Then

$$f(0) = c, \qquad f'(x) = \frac{x}{\sqrt{c^2+x^2}}, \qquad f'(0) = 0$$

Thus,

$$L(x) = c$$

**Final Answer:** $\sqrt{c^2 + x^2} \approx c.$

**Maxima and Minima**

**Exercise 90**

Let

$$y = ax^2 + bx + c.$$

To find the location of the maximum or minimum, compute the derivative:

$$y' = 2ax + b.$$

Critical points occur where $y' = 0$:

$$2ax + b = 0 \quad \Rightarrow \quad x = -\frac{b}{2a}.$$

This point corresponds to a maximum if $a < 0$ and a minimum if $a > 0$.

**Final Answer:** The maximum or minimum occurs at

$$h = -\frac{b}{2a}.$$

### Exercise 91

An absolute minimum on a closed interval $[a, b]$ must occur either at a critical point or at an endpoint. A function may be increasing or decreasing throughout the interval, in which case the extremum occurs at an endpoint.

If endpoints are not checked, an absolute minimum may be missed.

**Final Answer:** Endpoints must be checked because absolute extrema may occur where the derivative does not vanish.

### Exercise 92

Yes. On an open interval $(a, b)$, a function may approach a maximum or minimum value without ever attaining it.

Example:

$$f(x) = \frac{1}{x}, \quad (0, 1).$$

The function is bounded but has neither an absolute maximum nor minimum.

**Final Answer:** On $(a, b)$, a function may fail to attain absolute extrema.

### Exercise 93

Critical points occur where $f'(x) = 0$ or where $f'(x)$ is undefined. Both must be checked because local extrema may occur at points where the derivative does not exist.

Example:

$$f(x) = |x|.$$

The derivative is undefined at $x = 0$, yet $x = 0$ is a local minimum.

**Final Answer:** Points where $f'(x)$ is undefined can still produce extrema and must be examined.

### Exercise 94

A quadratic function over $(-\infty, \infty)$ has a finite absolute maximum only if it opens downward $(a < 0)$. If $a > 0$, the function grows without bound.

**Final Answer:** A finite absolute maximum exists only when $a < 0$.

### Exercise 95

A cubic polynomial with nonzero leading coefficient grows without bound in at least one direction. Therefore, it cannot have a finite absolute maximum or minimum on $(-\infty, \infty)$.

**Final Answer:** No finite absolute extrema exist for cubic polynomials on infinite domains.

### Exercise 96

Yes. Consider

$$f(x) = \sin x + \sin(3x).$$

This function has more local maxima than minima on suitable intervals.

**Final Answer:** It is possible to construct functions with $M > m + 2$.

### Exercise 97

Yes. A function may attain the same maximum value at multiple distinct points.

Example:

$$f(x) = \cos x$$

has infinitely many absolute maxima.

**Final Answer:** More than one absolute maximum is possible.

### Exercise 98

Yes. For example,

$$f(x) = x$$

has no absolute maximum or minimum on $(-\infty, \infty)$.

**Final Answer:** A function may have neither an absolute minimum nor maximum.

### Exercise 99

Let

$$y = e^{ax}.$$

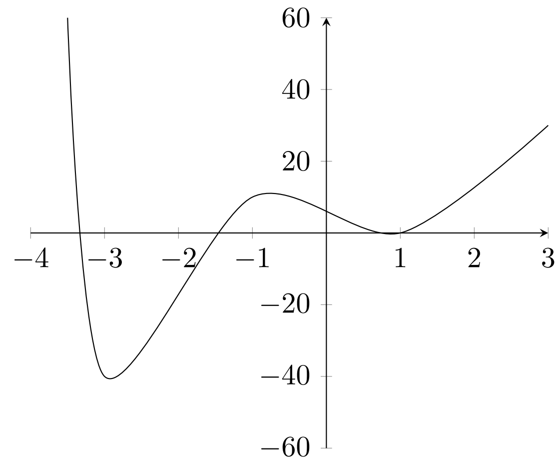
If $a > 0$, the function has an absolute minimum at $x \to -\infty$, but no maximum. If $a < 0$, the function has an absolute maximum at $x \to -\infty$, but no minimum. If $a = 0$, the function is constant.

**Final Answer:** Absolute extrema occur only in one direction unless $a = 0$.

## Exercise 100

From the graph: - One local minimum near $x \approx -3$ - One local maximum near $x \approx -1$ - One absolute minimum near $x \approx -3$ - Absolute maximum at the right endpoint

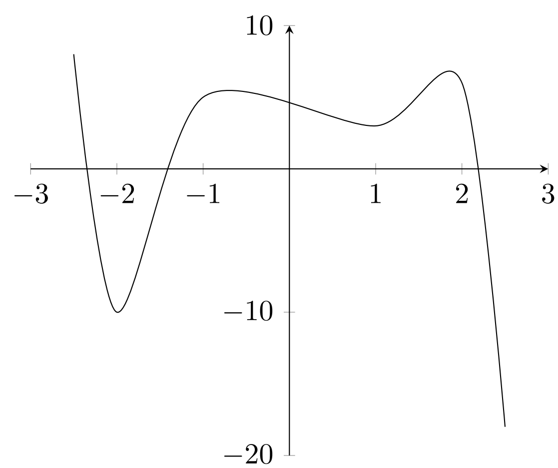
**Final Answer:** One local max, one local min, one absolute min, one absolute max.




## Exercise 101

From the graph: - Two local maxima - Two local minima - Absolute minimum at right endpoint - Absolute maximum at left endpoint

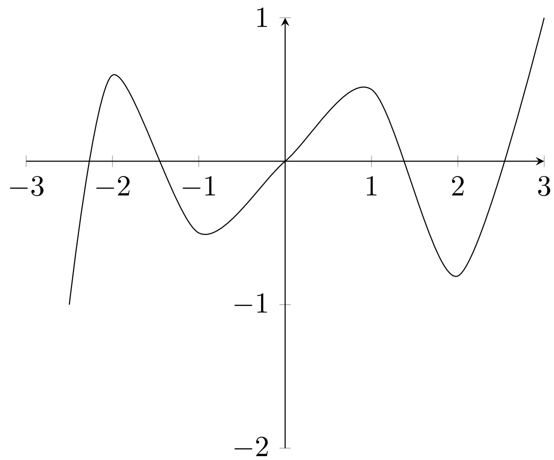
**Final Answer:** Two local maxima, two local minima, one absolute max, one absolute min.



**Exercise 102**

From the graph: - Two local maxima - Two local minima - Absolute minimum at left endpoint - Absolute maximum at right endpoint

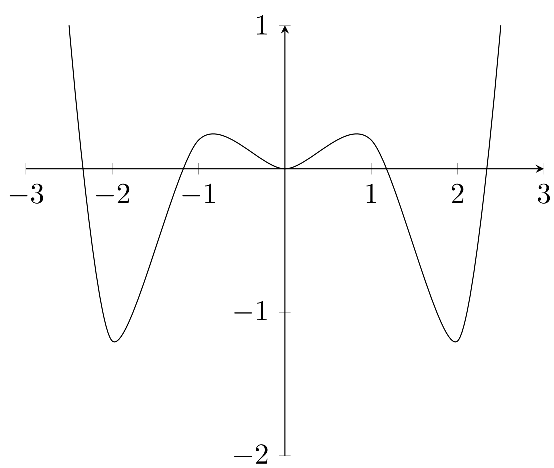
**Final Answer:** Two local maxima, two local minima, absolute extrema at endpoints.




**Exercise 103**

From the graph: - Two local minima - One local maximum - Absolute minima at both ends - Absolute maximum at center peak

**Final Answer:** Two local minima, one local maximum, symmetric absolute minima.



### Exercise 108

Critical points occur where $y'(x) = 0$ or where $y'(x)$ is undefined, provided the point lies in the domain.

Given

$$y = 4x^3 - 3x,$$

differentiate:

$$y' = 12x^2 - 3.$$

Set the derivative equal to zero:

$$12x^2 - 3 = 0 \Rightarrow x^2 = \frac{1}{4} \Rightarrow x = \pm\frac{1}{2}.$$

The derivative exists for all real $x$.

**Final Answer:** Critical points at $x = -\frac{1}{2}$ and $x = \frac{1}{2}$.

### Exercise 109

Critical points occur where $y'(x) = 0$ or where $y'(x)$ is undefined within the domain.

Given

$$y = 4\sqrt{x} - x^2,$$

the domain is $x \geq 0$.

Differentiate:

$$y' = \frac{2}{\sqrt{x}} - 2x.$$

Set $y' = 0$:

$$\frac{2}{\sqrt{x}} = 2x \Rightarrow \frac{1}{\sqrt{x}} = x \Rightarrow x^{3/2} = 1 \Rightarrow x = 1.$$

The derivative is undefined at $x = 0$, which is in the domain.

**Final Answer:** Critical points at $x = 0$ and $x = 1$.

### Exercise 110

Critical points occur where the derivative is zero or undefined within the domain.

Given

$$y = \frac{1}{x-1},$$

the domain is $x \neq 1$.

Differentiate:

$$y' = -\frac{1}{(x-1)^2}.$$

The derivative is never zero. It is undefined only at $x = 1$, which is not in the domain.

**Final Answer:** There are no critical points.

### Exercise 111

Critical points occur where $y'(x) = 0$ or where $y'(x)$ is undefined in the domain.

Given

$$y = \ln(x-2),$$

the domain is $x > 2$.

Differentiate:

$$y' = \frac{1}{x-2}.$$

The derivative is never zero and is undefined at $x = 2$, which is not in the domain.

**Final Answer:** There are no critical points.

### Exercise 112

Critical points occur where the derivative equals zero or does not exist.

Given

$$y = \tan x,$$

differentiate:

$$y' = \sec^2 x.$$

Since $\sec^2 x > 0$ wherever defined, the derivative is never zero. The derivative is undefined at

$$x = \frac{\pi}{2} + k\pi, \quad k \in \mathbb{Z},$$

which are not in the domain.

**Final Answer:** There are no critical points.

## Exercise 113

Critical points occur where $y'(x) = 0$ or where $y'(x)$ does not exist, provided the point is in the domain.

Given

$$y = \sqrt{4 - x^2},$$

the domain is $-2 \leq x \leq 2$.

Differentiate:

$$y' = \frac{-x}{\sqrt{4 - x^2}}.$$

Set $y' = 0$:

$$-x = 0 \Rightarrow x = 0.$$

The derivative is undefined at $x = \pm 2$, which are endpoints of the domain.

**Final Answer:** Critical points at $x = -2,\ 0,\ 2$.

## Exercise 114

Critical points occur where the derivative is zero or undefined within the domain.

Given

$$y = x^{3/2} - 3x^{5/2},$$

the domain is $x \geq 0$.

Differentiate:

$$y' = \frac{3}{2}x^{1/2} - \frac{15}{2}x^{3/2}.$$

Factor:

$$y' = \frac{3}{2}x^{1/2}(1 - 5x).$$

Set $y' = 0$:

$$x^{1/2} = 0 \Rightarrow x = 0, \qquad 1 - 5x = 0 \Rightarrow x = \frac{1}{5}.$$

Both values lie in the domain.

**Final Answer:** Critical points at $x = 0$ and $x = \frac{1}{5}$.

## Exercise 115

Critical points occur where $y'(x) = 0$ or where $y'(x)$ is undefined in the domain.

Given

$$y = \frac{x^2 - 1}{x^2 + 2x - 3},$$

factor the denominator:

$$x^2 + 2x - 3 = (x + 3)(x - 1),$$

so the domain excludes $x = -3$ and $x = 1$.

Differentiate using the quotient rule:

$$y' = \frac{2(x^2 - 2x - 3)}{(x^2 + 2x - 3)^2}.$$

Set the numerator equal to zero:

$$x^2 - 2x - 3 = 0 \Rightarrow (x - 3)(x + 1) = 0 \Rightarrow x = 3, \, -1.$$

Both values are in the domain.

**Final Answer:** Critical points at $x = -1$ and $x = 3$.

### Exercise 116

Critical points occur where the derivative equals zero or does not exist.

Given

$$y = \sin^2 x,$$

differentiate:

$$y' = 2 \sin x \cos x = \sin(2x).$$

Set $y' = 0$:

$$\sin(2x) = 0 \Rightarrow 2x = k\pi \Rightarrow x = \frac{k\pi}{2}, \quad k \in \mathbb{Z}.$$

**Final Answer:** Critical points at $x = \frac{k\pi}{2}$ for all integers $k$.

### Exercise 117

Critical points occur where $y'(x) = 0$ or where $y'(x)$ is undefined in the domain.

Given

$$y = x + \frac{1}{x},$$

the domain excludes $x = 0$.

Differentiate:

$$y' = 1 - \frac{1}{x^2}.$$

Set $y' = 0$:

$$1 - \frac{1}{x^2} = 0 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1.$$

Both points lie in the domain.

**Final Answer:** Critical points at $x = -1$ and $x = 1$.

## Exercise 118

We are given

$$f(x) = x^2 + 3 \quad \text{on } [-1, 4].$$

Compute the derivative:

$$f'(x) = 2x.$$

Set $f'(x) = 0 \Rightarrow x = 0$, which lies in the interval.

Evaluate $f$ at the critical point and endpoints:

$$f(-1) = 4, \qquad f(0) = 3, \qquad f(4) = 19.$$

**Final Answer:** Absolute minimum at $x = 0$ with value 3; absolute maximum at $x = 4$ with value 19.

## Exercise 119

$$y = x^2 + \frac{2}{x}, \quad x \in [1, 4].$$

Differentiate:

$$y' = 2x - \frac{2}{x^2}.$$

Set $y' = 0$:

$$2x = \frac{2}{x^2} \Rightarrow x^3 = 1 \Rightarrow x = 1.$$

Evaluate:

$$y(1) = 3, \qquad y(4) = 16 + \frac{1}{2} = \frac{33}{2}.$$

**Final Answer:** Absolute minimum at $x = 1$ with value 3; absolute maximum at $x = 4$ with value $\frac{33}{2}$.

**Exercise 120**

$$y = (x - x^2)^2, \quad x \in [-1, 1].$$

Differentiate:

$$y' = 2(x - x^2)(1 - 2x).$$

Critical points:

$$x = 0, \quad x = 1, \quad x = \frac{1}{2}.$$

Evaluate:

$$y(-1) = 4, \quad y(0) = 0, \quad y\left(\frac{1}{2}\right) = \frac{1}{16}, \quad y(1) = 0.$$

**Final Answer:** Absolute maximum at $x = -1$ with value 4; absolute minima at $x = 0$ and $x = 1$ with value 0.

**Exercise 121**

$$y = \frac{1}{x - x^2}, \quad x \in (0, 1).$$

Differentiate:

$$y' = -\frac{1 - 2x}{(x - x^2)^2}.$$

Set numerator to zero:

$$1 - 2x = 0 \Rightarrow x = \frac{1}{2}.$$

Also, as $x \to 0^+$ or $x \to 1^-$, we have $x - x^2 \to 0^+$, so $y \to +\infty$. Thus there is a minimum but no maximum on $(0, 1)$.

**Final Answer:** Absolute minimum at $x = \frac{1}{2}$ with value 4; no absolute maximum on $(0, 1)$.

**Exercise 122**

$$y = \sqrt{9 - x}, \quad x \in [1, 9].$$

This is decreasing on $[1, 9]$, so extrema occur at endpoints:

$$y(1) = \sqrt{8}, \qquad y(9) = 0.$$

**Final Answer:** Absolute maximum at $x = 1$ with value $\sqrt{8}$; absolute minimum at $x = 9$ with value 0.

## Exercise 123

$$y = x + \sin x, \quad x \in [0, 2\pi].$$

Differentiate:

$$y' = 1 + \cos x.$$

Set $y' = 0 \Rightarrow \cos x = -1 \Rightarrow x = \pi$.

Evaluate:

$$y(0) = 0, \qquad y(\pi) = \pi, \qquad y(2\pi) = 2\pi.$$

**Final Answer:** Absolute minimum at $x = 0$ with value 0; absolute maximum at $x = 2\pi$ with value $2\pi$.

## Exercise 124

$$y = \frac{x}{1+x}, \quad x \in [0, 100].$$

Differentiate:

$$y' = \frac{1}{(1+x)^2} > 0.$$

So $y$ is increasing on $[0, 100]$.

**Final Answer:** Absolute minimum at $x = 0$ with value 0; absolute maximum at $x = 100$ with value $\frac{100}{101}$.

## Exercise 125

$$y = |x+1| + |x-1|, \quad x \in [-3, 2].$$

Piecewise:

$$y = \begin{cases} -2x, & x \le -1, \\ 2, & -1 \le x \le 1, \\ 2x, & x \ge 1. \end{cases}$$

Evaluate key points:

$$y(-3) = 6, \quad y(-1) = 2, \quad y(1) = 2, \quad y(2) = 4.$$

**Final Answer:** Absolute maximum at $x = -3$ with value 6; absolute minimum value 2 attained for all $x \in [-1, 1]$.

**Exercise 126**

$$y = \sqrt{x} - \sqrt{x^3}, \quad x \in [0, 4].$$

Rewrite $\sqrt{x^3} = x\sqrt{x}$ (for $x \geq 0$):

$$y = \sqrt{x} - x\sqrt{x} = \sqrt{x}(1 - x).$$

Differentiate:

$$y' = \frac{1}{2\sqrt{x}} - \frac{3}{2}\sqrt{x}.$$

Set $y' = 0$:

$$\frac{1}{2\sqrt{x}} = \frac{3}{2}\sqrt{x} \Rightarrow 1 = 3x \Rightarrow x = \frac{1}{3}.$$

Evaluate:

$$y(0) = 0, \qquad y\left(\frac{1}{3}\right) = \frac{2}{3\sqrt{3}}, \qquad y(4) = 2 - 8 = -6.$$

**Final Answer:** Absolute maximum at $x = \frac{1}{3}$ with value $\frac{2}{3\sqrt{3}}$; absolute minimum at $x = 4$ with value $-6$.

**Exercise 127**

$$y = \sin x + \cos x, \quad x \in [0, 2\pi].$$

Differentiate:

$$y' = \cos x - \sin x.$$

Set

$$y' = 0 \Rightarrow \cos x = \sin x \Rightarrow x = \frac{\pi}{4}, \frac{5\pi}{4}.$$

Evaluate:

$$y\left(\frac{\pi}{4}\right) = \sqrt{2}, \qquad y\left(\frac{5\pi}{4}\right) = -\sqrt{2}.$$

**Final Answer:** Absolute maximum at $x = \frac{\pi}{4}$ with value $\sqrt{2}$; absolute minimum at $x = \frac{5\pi}{4}$ with value $-\sqrt{2}$.

**Exercise 128**

$$y = 4\sin\theta - 3\cos\theta, \quad \theta \in [0, 2\pi].$$

Write as a single sine:

$$4\sin\theta - 3\cos\theta = 5\left(\frac{4}{5}\sin\theta - \frac{3}{5}\cos\theta\right) = 5\sin(\theta - \phi),$$

where $\cos\phi = \frac{4}{5}$ and $\sin\phi = \frac{3}{5}$.

So the range is $[-5, 5]$.

**Final Answer:** Absolute maximum value 5; absolute minimum value $-5$.

### Exercise 129

$$y = x^2 + 4x + 5$$

Differentiate:

$$y' = 2x + 4.$$

Set $y' = 0$:

$$2x + 4 = 0 \Rightarrow x = -2.$$

Since the parabola opens upward, this point is a minimum.

$$y(-2) = 1.$$

As $x \to \pm\infty$, $y \to \infty$.

**Final Answer:** Absolute (and local) minimum at $x = -2$ with value 1. No maximum on $(-\infty, \infty)$.

### Exercise 130

$$y = x^3 - 12x$$

Differentiate:

$$y' = 3x^2 - 12 = 3(x-2)(x+2).$$

Critical points: $x = -2, 2$.

Second derivative:

$$y'' = 6x.$$

$$y''(-2) < 0 \Rightarrow \text{local maximum}, \quad y''(2) > 0 \Rightarrow \text{local minimum}.$$

$$y(-2) = 16, \qquad y(2) = -16.$$

Since this is a cubic, there are no absolute extrema.

**Final Answer:** Local maximum at $x = -2$ with value 16; local minimum at $x = 2$ with value $-16$. No absolute extrema.

### Exercise 131

$$y = 3x^4 + 8x^3 - 18x^2$$

Differentiate:

$$y' = 12x^3 + 24x^2 - 36x = 12x(x^2 + 2x - 3).$$

Critical points:

$$x = -3, 0, 1.$$

Second derivative:

$$y'' = 36x^2 + 48x - 36.$$

Classify:

$$y''(-3) > 0 \Rightarrow \text{local min}, \quad y''(0) < 0 \Rightarrow \text{local max}, \quad y''(1) > 0 \Rightarrow \text{local min}.$$

Since $y \to \infty$ as $x \to \pm\infty$, the lowest value is absolute.

**Final Answer:** Local maxima at $x = 0$; local minima at $x = -3, 1$. Absolute minimum occurs at the lower of these minima.

### Exercise 132

$$y = x^3(1 - x)^6$$

Differentiate using product rule:

$$y' = x^2(1 - x)^5(3 - 9x).$$

Critical points:

$$x = 0, \quad x = 1, \quad x = \frac{1}{3}.$$

Sign analysis shows: - $x = \frac{1}{3}$: local maximum - $x = 0, 1$: local minima

As $x \to \pm\infty$, $y \to \pm\infty$.

**Final Answer:** Local maximum at $x = \frac{1}{3}$; local minima at $x = 0$ and $x = 1$. No absolute extrema.

### Exercise 133

$$y = \frac{x^2 + x + 6}{x - 1}$$

Differentiate:

$$y' = \frac{x^2 - 2x - 7}{(x - 1)^2}.$$

Set numerator to zero:

$$x = 1 \pm 2\sqrt{2}.$$

Vertical asymptote at $x = 1$, so no absolute extrema.

**Final Answer:** Local maximum and minimum at $x = 1 \pm 2\sqrt{2}$. No absolute extrema.

## Exercise 134

$$y = \frac{x^2 - 1}{x - 1} = x + 1, \quad x \neq 1.$$

This is a line with a hole at $x = 1$.

**Final Answer:** No local or absolute extrema.

## Exercise 135

$$y = 3x\sqrt{1 - x^2}, \quad |x| \leq 1$$

Differentiate:

$$y' = \frac{3(1 - 2x^2)}{\sqrt{1 - x^2}}.$$

Set $y' = 0$:

$$x = \pm\frac{1}{\sqrt{2}}.$$

Evaluate:

$$y\left(\frac{1}{\sqrt{2}}\right) = \frac{3}{2}, \quad y\left(-\frac{1}{\sqrt{2}}\right) = -\frac{3}{2}.$$

**Final Answer:** Absolute maximum $\frac{3}{2}$ at $x = \frac{1}{\sqrt{2}}$; absolute minimum $-\frac{3}{2}$ at $x = -\frac{1}{\sqrt{2}}$.

## Exercise 136

$$y = x + \sin x$$

Differentiate:

$$y' = 1 + \cos x \geq 0.$$

The function is increasing everywhere.

**Final Answer:** No local or absolute extrema.

## Exercise 137

$$y = 12x^5 + 45x^4 + 20x^3 - 90x^2 - 120x + 3$$

Differentiate:

$$y' = 60x^4 + 180x^3 + 60x^2 - 180x - 120.$$

Critical points found numerically; sign changes determine local extrema. As a degree-5 polynomial, no absolute extrema exist.

**Final Answer:** Several local maxima and minima; no absolute extrema.

### Exercise 138

$$y = \frac{x^3 + 6x^2 - x - 30}{x - 2}$$

Polynomial division gives:

$$y = x^2 + 8x + 15 + \frac{0}{x - 2}.$$

There is a vertical asymptote at $x = 2$.

**Final Answer:** Local extrema exist from the quadratic part; no absolute extrema.

### Exercise 139

$$y = \frac{\sqrt{4 - x^2}}{\sqrt{4 + x^2}}, \quad |x| \leq 2$$

Maximum occurs at $x = 0$:

$$y(0) = 1.$$

Minimum occurs at endpoints:

$$y(\pm 2) = 0.$$

**Final Answer:** Absolute maximum at $x = 0$ with value 1; absolute minima at $x = \pm 2$ with value 0.

### Exercise 140

$$C = x^2 - 1200x + 36400$$

Differentiate:

$$C' = 2x - 1200.$$

Set $C' = 0$:

$$x = 600.$$

**Final Answer:** Cost is minimized when 600 thousand cell phones are produced.

### Exercise 141

The height of the ball is

$$h(t) = -4.9t^2 + 60t + 5.$$

The ball stops ascending when its velocity is zero. Velocity is the derivative of position:

$$h'(t) = -9.8t + 60.$$

Set $h'(t) = 0$:

$$-9.8t + 60 = 0 \Rightarrow t = \frac{60}{9.8} \approx 6.12.$$

**Final Answer:** The ball stops ascending approximately 6.12 seconds after it is thrown.

### Exercise 142

Gold production is modeled by

$$G(t) = \frac{25t}{t^2 + 16}, \quad 0 \leq t \leq 40.$$

Differentiate using the quotient rule:

$$G'(t) = \frac{25(t^2 + 16) - 25t(2t)}{(t^2 + 16)^2} = \frac{25(16 - t^2)}{(t^2 + 16)^2}.$$

Set the numerator equal to zero:

$$16 - t^2 = 0 \Rightarrow t = 4.$$

Evaluate $G(4)$:

$$G(4) = \frac{25(4)}{16 + 16} = \frac{100}{32} = 3.125.$$

**Final Answer:** Maximum gold production occurred 4 years after the rush began, producing approximately 3.125 million ounces.

### Exercise 143

Critical points occur at $t = 4$. Check endpoints as well:

$$G(0) = 0, \quad G(40) = \frac{1000}{1616} \approx 0.62.$$

Comparing values shows the smallest value occurs at $t = 0$.

**Final Answer:** Minimum gold production occurred at $t = 0$, with 0 million ounces produced.

### Exercise 144

$$y = \begin{cases} x^2 - 4x, & 0 \le x \le 1 \\ x^2 - 4, & 1 < x \le 2 \end{cases}$$

Differentiate each piece:

$$y_1' = 2x - 4, \quad y_2' = 2x.$$

Critical points:

$$2x - 4 = 0 \Rightarrow x = 2 \text{ (not in first interval)}, \quad 2x = 0 \Rightarrow x = 0.$$

Evaluate at endpoints:

$$y(0) = 0, \quad y(1) = -3, \quad y(2) = 0.$$

**Final Answer:** Absolute minimum at $x = 1$ with value $-3$; absolute maxima at $x = 0$ and $x = 2$ with value 0.

### Exercise 145

$$y = \begin{cases} x^2 + 1, & x \le 1 \\ x^2 - 4x + 5, & x > 1 \end{cases}$$

Derivatives:

$$y_1' = 2x, \quad y_2' = 2x - 4.$$

Critical points:

$$x = 0 \text{ (first piece)}, \quad x = 2 \text{ (second piece)}.$$

Evaluate:

$$y(0) = 1, \quad y(1) = 2, \quad y(2) = 1.$$

**Final Answer:** Local maxima at $x = 1$; local minima at $x = 0$ and $x = 2$.

### Exercise 146

$$y = ax^2 + bx + c, \quad a > 0.$$

Differentiate:

$$y' = 2ax + b.$$

Set $y' = 0$:

$$x = -\frac{b}{2a}.$$

Since $a > 0$, the parabola opens upward.

**Final Answer:** One critical point at $x = -\frac{b}{2a}$, which is a minimum.

### Exercise 147

$$y = (x-1)^a, \quad a > 1,\ a \in \mathbb{Z}.$$

Differentiate:

$$y' = a(x-1)^{a-1}.$$

Set $y' = 0$:

$$x = 1.$$

If $a$ is even, the function has a minimum at $x = 1$. If $a$ is odd, the point is neither a maximum nor a minimum.

**Final Answer:** Critical point at $x = 1$; it is a minimum when $a$ is even, and neither when $a$ is odd.

#### The Mean Value Theorem

### Exercise 148

The Mean Value Theorem requires continuity on the closed interval $[a, b]$ because the theorem relies on the existence of a maximum and minimum value guaranteed by the Extreme Value Theorem.

Counterexample:

$$f(x) = \begin{cases} x, & x < 0 \\ x+1, & x \geq 0 \end{cases}$$

on $[-1, 1]$.

This function is not continuous at $x = 0$. Although $f'(x) = 1$ for all $x \neq 0$, there is no point $c$ such that

$$f'(c) = \frac{f(1) - f(-1)}{2}.$$

**Final Answer:** Continuity is required because without it, the secant slope may not be achieved by the derivative.

### Exercise 149

Differentiability is required because the Mean Value Theorem guarantees the existence of a tangent line parallel to the secant line.

Counterexample:

$$f(x) = |x|$$

on $[-1, 1]$.

The function is continuous but not differentiable at $x = 0$. The secant slope is

$$\frac{f(1) - f(-1)}{2} = \frac{1 - 1}{2} = 0,$$

but $f'(x) = \pm 1$ for $x \neq 0$, so no such $c$ exists.

**Final Answer:** Differentiability is necessary because nondifferentiable points prevent matching the secant slope.

### Exercise 150

Rolle's Theorem is a special case of the Mean Value Theorem where

$$f(a) = f(b).$$

In this case, the Mean Value Theorem reduces to

$$f'(c) = 0.$$

**Final Answer:** The two theorems are equivalent when $f(a) = f(b)$.

### Exercise 151

Yes, it is possible.

Example:

$$f(x) = \begin{cases} x, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

on $[-1, 1]$.

The function is discontinuous at $x = 0$, but

$$\frac{f(1) - f(-1)}{2} = 1.$$

Since $f'(x) = 1$ for all $x \neq 0$, the equation

$$f'(c)(b - a) = f(b) - f(a)$$

still holds.

**Final Answer:** Discontinuity does not always prevent the equality from holding, but the Mean Value Theorem does not apply.

### Exercise 152

$$y = \sin(\pi x)$$

This function is continuous and differentiable for all real $x$.

**Final Answer:** The Mean Value Theorem applies on any interval $[a, b]$.

### Exercise 153

$$y = \frac{1}{x^3}$$

The function is undefined at $x = 0$.

**Final Answer:** The Mean Value Theorem applies only on intervals that do not include $x = 0$.

### Exercise 154

$$y = \sqrt{4 - x^2}$$

The domain is $[-2, 2]$, and the function is continuous and differentiable on $(-2, 2)$.

**Final Answer:** The Mean Value Theorem applies on any interval contained in $[-2, 2]$.

### Exercise 155

$$y = \sqrt{x^2 - 4}$$

The function is defined only for $x \leq -2$ or $x \geq 2$, and it is not continuous across the gap.

**Final Answer:** The Mean Value Theorem applies only on intervals fully contained in one branch.

### Exercise 156

$$y = \ln(3x - 5)$$

The domain is $x > \frac{5}{3}$, where the function is continuous and differentiable.

**Final Answer:** The Mean Value Theorem applies on any interval $[a, b] \subset \left(\frac{5}{3}, \infty\right)$.

### Exercise 157

$$y = 3x^3 + 2x + 1 \quad \text{on } [-1, 1]$$

The function is a polynomial and therefore continuous and differentiable everywhere. The secant slope is

$$\frac{f(1) - f(-1)}{2} = 4.$$

Since $f'(x) = 9x^2 + 2$, there are two solutions to $f'(c) = 4$.

**Final Answer:** There are two values of $c$.

### Exercise 158

$$y = \tan\left(\frac{\pi}{4}x\right) \quad \text{on } \left[-\frac{3}{2}, \frac{3}{2}\right]$$

The function has vertical asymptotes inside the interval.

**Final Answer:** The Mean Value Theorem does not apply.

### Exercise 159

$$y = x^2 \cos(\pi x) \quad \text{on } [-2, 2]$$

The function is continuous and differentiable everywhere.

**Final Answer:** The Mean Value Theorem applies and yields multiple solutions for $c$.

### Exercise 160

$$y = x^6 - \frac{3}{4}x^5 - \frac{9}{8}x^4 + \frac{15}{16}x^3 + \frac{3}{32}x^2 + \frac{3}{16}x + \frac{1}{32}$$

on $[-1, 1]$.

This polynomial is continuous and differentiable on $[-1, 1]$.

**Final Answer:** The Mean Value Theorem applies, and at least one value of $c$ exists.

## Exercise 148

Continuity is required in the Mean Value Theorem (MVT) because the theorem guarantees that a function attains all intermediate values between its endpoints. Without continuity, the average rate of change may not correspond to any instantaneous rate of change.

Counterexample:

$$f(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases} \quad \text{on } [-1, 1].$$

This function is not continuous at $x = 0$. The average rate of change is

$$\frac{f(1) - f(-1)}{1 - (-1)} = \frac{1 - 0}{2} = \frac{1}{2},$$

but $f'(x) = 0$ wherever the derivative exists, so no $c$ satisfies the MVT.

**Final Answer:** Continuity is necessary because without it the required point $c$ may not exist.

## Exercise 149

Differentiability is required because the Mean Value Theorem requires a well-defined tangent slope at the point $c$.

Counterexample:

$$f(x) = |x| \quad \text{on } [-1, 1].$$

The function is continuous everywhere but not differentiable at $x = 0$. The average rate of change is

$$\frac{f(1) - f(-1)}{2} = \frac{1 - 1}{2} = 0,$$

but there is no point where $f'(x) = 0$.

**Final Answer:** Differentiability is required because corners can prevent the existence of the required tangent slope.

## Exercise 150

Rolle's Theorem is a special case of the Mean Value Theorem where

$$f(a) = f(b).$$

In this case, the average rate of change is zero, so the Mean Value Theorem guarantees a point where $f'(c) = 0$.

**Final Answer:** Rolle's Theorem and the MVT are equivalent when $f(a) = f(b)$.

**Exercise 151**

Yes, it is possible. The Mean Value Theorem gives a sufficient condition, not a necessary one.

Example:

$$f(x) = \begin{cases} x, & x \neq 0 \\ 1, & x = 0 \end{cases} \quad \text{on } [-1, 1].$$

The function is discontinuous at $x = 0$, yet

$$\frac{f(1) - f(-1)}{2} = \frac{1 - (-1)}{2} = 1,$$

and $f'(x) = 1$ for all $x \neq 0$.

**Final Answer:** Yes, the equality may still hold even if the function is discontinuous.

**Exercise 152**

$f(x) = \sin(\pi x)$ is continuous and differentiable for all real $x$.

**Final Answer:** The Mean Value Theorem applies on any closed interval.

**Exercise 153**

$f(x) = \frac{1}{x^3}$ is not continuous at $x = 0$.

**Final Answer:** The Mean Value Theorem applies only on intervals that do not include $x = 0$.

**Exercise 154**

$f(x) = \sqrt{4 - x^2}$ is continuous and differentiable on $(-2, 2)$.

**Final Answer:** The Mean Value Theorem applies on any closed interval contained in $[-2, 2]$.

**Exercise 155**

$f(x) = \sqrt{x^2 - 4}$ is defined only for $|x| \geq 2$ and is not continuous across $(-2, 2)$.

**Final Answer:** The Mean Value Theorem applies only on intervals entirely outside $(-2, 2)$.

**Exercise 156**

$f(x) = \ln(3x - 5)$ requires $x > \frac{5}{3}$.

**Final Answer:** The Mean Value Theorem applies on any closed interval contained in $\left(\frac{5}{3}, \infty\right)$.

### Exercise 157

$$f(x) = 3x^3 + 2x + 1 \quad \text{on } [-1, 1].$$

The function is continuous and differentiable everywhere.

**Final Answer:** The Mean Value Theorem applies and exactly one value of $c$ exists.

### Exercise 158

$$f(x) = \tan\left(\frac{\pi x}{4}\right) \quad \text{on } \left[-\frac{3}{2}, \frac{3}{2}\right].$$

The function is discontinuous at $x = \pm 2$, outside the interval.

**Final Answer:** The Mean Value Theorem applies and exactly one value of $c$ exists.

### Exercise 159

$$f(x) = x^2 \cos(\pi x) \quad \text{on } [-2, 2].$$

The function is smooth everywhere.

**Final Answer:** The Mean Value Theorem applies and multiple values of $c$ exist.

### Exercise 160

$$f(x) = x^6 - \frac{3}{4}x^5 - \frac{9}{8}x^4 + \frac{15}{16}x^3 + \frac{3}{32}x^2 + \frac{3}{16}x + \frac{1}{32} \quad \text{on } [-1, 1].$$

This polynomial is continuous and differentiable everywhere.

**Final Answer:** The Mean Value Theorem applies and several values of $c$ exist.

### Exercise 161

$$f(x) = x^3.$$

$$f(2) - f(0) = 8, \quad f'(x) = 3x^2.$$

Solve:

$$3c^2 \cdot 2 = 8 \Rightarrow c = \sqrt{\frac{4}{3}}.$$

**Final Answer:** $c = \sqrt{\frac{4}{3}}$.

**Exercise 162**

$$f(x) = \sin(\pi x).$$

$$f(2) - f(0) = 0, \quad f'(x) = \pi \cos(\pi x).$$

$$\cos(\pi c) = 0 \Rightarrow c = \frac{1}{2}, \frac{3}{2}.$$

**Final Answer:** $c = \frac{1}{2},\ \frac{3}{2}$.

**Exercise 163**

$$f(x) = \cos(2\pi x).$$

$$f(2) - f(0) = 0, \quad f'(x) = -2\pi \sin(2\pi x).$$

$$\sin(2\pi c) = 0 \Rightarrow c = \frac{1}{2}, 1, \frac{3}{2}.$$

**Final Answer:** $c = \frac{1}{2},\ 1,\ \frac{3}{2}$.

**Exercise 164**

$$f(x) = 1 + x + x^2.$$

$$f(2) - f(0) = 6, \quad f'(x) = 1 + 2x.$$

$$(1 + 2c) \cdot 2 = 6 \Rightarrow c = 1.$$

**Final Answer:** $c = 1$.

**Exercise 165**

$$f(x) = (x - 1)^{10}.$$

$$f(2) - f(0) = 1 - 1 = 0, \quad f'(x) = 10(x - 1)^9.$$

$$f'(c) = 0 \Rightarrow c = 1.$$

**Final Answer:** $c = 1$.

**Exercise 166**

$$f(x) = (x-1)^9.$$

$$f(2) - f(0) = 1 - (-1) = 2, \quad f'(x) = 9(x-1)^8.$$

$$9c^8 \cdot 2 = 2 \Rightarrow c = 1.$$

**Final Answer:** $c = 1$.

**Exercise 167**

Compute the secant slope on $[-1, 1]$:

$$f(1) = \left|1 - \frac{1}{2}\right| = \frac{1}{2}, \qquad f(-1) = \left|-1 - \frac{1}{2}\right| = \frac{3}{2},$$

so

$$\frac{f(1) - f(-1)}{1 - (-1)} = \frac{\frac{1}{2} - \frac{3}{2}}{2} = \frac{-1}{2}.$$

Thus the equation $f(1) - f(-1) = 2f'(c)$ would require

$$2f'(c) = -1 \quad \Longrightarrow \quad f'(c) = -\frac{1}{2}.$$

Now examine $f'(x)$. Since $f(x) = |x - \frac{1}{2}|$,

$$f'(x) = \begin{cases} -1, & x < \frac{1}{2}, \\ \text{undefined}, & x = \frac{1}{2}, \\ 1, & x > \frac{1}{2}. \end{cases}$$

So $f'(x)$ never equals $-\frac{1}{2}$ anywhere in $(-1, 1)$. Moreover, $f$ is not differentiable at $x = \frac{1}{2}$, so the Mean Value Theorem does not apply on $[-1, 1]$.

**Final Answer:** No such $c$ exists; MVT fails because $f$ is not differentiable at $x = \frac{1}{2}$.

**Exercise 168**

First, the secant slope expression requires $f(1)$ and $f(-1)$, which exist:

$$f(1) = 1, \qquad f(-1) = 1 \quad \Longrightarrow \quad f(1) - f(-1) = 0.$$

If MVT applied, we would need some $c$ with

$$2f'(c) = 0 \quad \Longrightarrow \quad f'(c) = 0.$$

Differentiate:

$$f'(x) = \frac{d}{dx}(x^{-2}) = -2x^{-3} = -\frac{2}{x^3},$$

which is never 0 for any $x \neq 0$.

But the key point is that $f$ is not even continuous on $[-1, 1]$ because it is undefined at $x = 0$. Since MVT requires continuity on the closed interval, the theorem does not apply.

**Final Answer:** No such $c$ exists; MVT does not apply because $f$ is not continuous on $[-1, 1]$ (undefined at $x = 0$).

### Exercise 169

Compute the secant slope on $[-1, 1]$:

$$f(1) = \sqrt{|1|} = 1, \qquad f(-1) = \sqrt{|-1|} = 1,$$

so

$$f(1) - f(-1) = 0 \quad \Longrightarrow \quad 2f'(c) = 0 \quad \Longrightarrow \quad f'(c) = 0.$$

Now compute $f'(x)$ for $x \neq 0$:

$$f(x) = \begin{cases} \sqrt{x}, & x > 0, \\ \sqrt{-x}, & x < 0, \end{cases}$$

so

$$f'(x) = \begin{cases} \dfrac{1}{2\sqrt{x}}, & x > 0, \\ -\dfrac{1}{2\sqrt{-x}}, & x < 0. \end{cases}$$

These are never 0 for $x \neq 0$. At $x = 0$, the derivative does not exist (the slopes blow up to $\pm\infty$). Therefore there is no $c$ with $f'(c) = 0$.

Also, MVT does not apply on $[-1, 1]$ because $f$ is not differentiable at $x = 0$.

**Final Answer:** No such $c$ exists; MVT fails because $f$ is not differentiable at $x = 0$.

### Exercise 170

Compute endpoint values on $[-1, 1]$:

$$f(1) = \lfloor 1 \rfloor = 1, \qquad f(-1) = \lfloor -1 \rfloor = -1,$$

so

$$f(1) - f(-1) = 2.$$

Thus we would need

$$2f'(c) = 2 \quad \Longrightarrow \quad f'(c) = 1.$$

But $f(x) = \lfloor x \rfloor$ is a step function. It is constant on each interval $(n, n+1)$, hence its derivative is

$$f'(x) = 0 \quad \text{for all non-integer } x,$$

and $f'(x)$ is undefined at integers. Therefore $f'(c)$ can never equal 1 on $(-1, 1)$.

MVT also fails because $f$ is not continuous on $[-1, 1]$ (jump discontinuities at integers, including 0).

**Final Answer:** No such $c$ exists; MVT does not apply because $f$ is not continuous (and not differentiable) on $[-1, 1]$.

### Exercise 171

Determine whether the Mean Value Theorem applies to

$$f(x) = e^x \quad \text{on } [0, 1].$$

The function $e^x$ is continuous on all real numbers and differentiable on all real numbers. Therefore it is continuous on $[0, 1]$ and differentiable on $(0, 1)$, so MVT applies.

**Final Answer:** Yes, the Mean Value Theorem applies on $[0, 1]$.

### Exercise 172

For MVT, we need continuity on the closed interval and differentiability on the open interval.

The natural log requires

$$2x + 3 > 0 \quad \Longrightarrow \quad x > -\frac{3}{2}.$$

At the left endpoint $x = -\frac{3}{2}$,

$$2\left(-\frac{3}{2}\right) + 3 = 0,$$

so $\ln(2x + 3)$ is not defined there. Hence $f$ is not defined (and not continuous) on the closed interval.

**Final Answer:** No, MVT does not apply because $f$ is not defined/continuous at $x = -\frac{3}{2}$.

## Exercise 173

$\tan(\theta)$ is discontinuous when $\cos(\theta) = 0$, i.e.

$$2\pi x = \frac{\pi}{2} + k\pi \quad \Longrightarrow \quad x = \frac{1}{4} + \frac{k}{2}, \qquad k \in \mathbb{Z}.$$

Within $[0, 2]$, this includes $x = \frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \frac{7}{4}$, where $f$ has vertical asymptotes and is not continuous.

**Final Answer:** No, MVT does not apply because $f$ is not continuous on $[0, 2]$.

## Exercise 174

The expression under the square root must satisfy $9 - x^2 \geq 0$, which holds for $-3 \leq x \leq 3$. So $f$ is continuous on $[-3, 3]$.

Differentiate on the open interval $(-3, 3)$ where $9 - x^2 > 0$:

$$f'(x) = \frac{1}{2}(9 - x^2)^{-1/2}(-2x) = -\frac{x}{\sqrt{9 - x^2}},$$

which exists for all $-3 < x < 3$. Thus $f$ is differentiable on $(-3, 3)$.

**Final Answer:** Yes, the Mean Value Theorem applies on $[-3, 3]$.

## Exercise 175

On $[0, 3]$, we have $x + 1 > 0$, so $|x + 1| = x + 1$ and

$$f(x) = \frac{1}{x + 1}.$$

This function is continuous on $[0, 3]$ (denominator never zero) and differentiable on $(0, 3)$.

**Final Answer:** Yes, the Mean Value Theorem applies on $[0, 3]$.

## Exercise 176

Polynomials are continuous and differentiable for all real $x$. Therefore $f$ is continuous on $[0, 6]$ and differentiable on $(0, 6)$.

**Final Answer:** Yes, the Mean Value Theorem applies on $[0, 6]$.

### Exercise 177

Rewrite:

$$f(x) = \frac{x^2}{x} + \frac{3x}{x} + \frac{2}{x} = x + 3 + \frac{2}{x}.$$

This function is undefined at $x = 0$, which lies in $[-1, 1]$. Hence $f$ is not continuous on the closed interval.

**Final Answer:** No, MVT does not apply because $f$ is not continuous on $[-1, 1]$ (undefined at $x = 0$).

### Exercise 178

Potential issue: the denominator $\sin(\pi x) + 1$ could be zero. On $[0, 1]$, $\sin(\pi x) \in [0, 1]$, so

$$\sin(\pi x) + 1 \in [1, 2],$$

which is never 0. Therefore $f$ is continuous on $[0, 1]$.

Since it is a quotient of differentiable functions with nonzero denominator on $[0, 1]$, it is also differentiable on $(0, 1)$.

**Final Answer:** Yes, the Mean Value Theorem applies on $[0, 1]$.

### Exercise 179

The log requires $x + 1 > 0$, which holds for all $x \geq 0$. Thus $f$ is defined and continuous on $[0, e - 1]$. Also,

$$f'(x) = \frac{1}{x + 1},$$

which exists for all $x > -1$, in particular for all $x \in (0, e - 1)$.

**Final Answer:** Yes, the Mean Value Theorem applies on $[0, e - 1]$.

### Exercise 180

Both $x$ and $\sin(\pi x)$ are continuous and differentiable everywhere, so their product is continuous on $[0, 2]$ and differentiable on $(0, 2)$.

**Final Answer:** Yes, the Mean Value Theorem applies on $[0, 2]$.

## Exercise 181

The absolute value function $|x|$ is continuous for all real $x$. Adding a constant does not affect continuity, so $f(x) = 5 + |x|$ is continuous on the closed interval $[-1, 1]$.

The derivative of $|x|$ is

$$\frac{d}{dx}|x| = \begin{cases} -1, & x < 0, \\ \text{undefined}, & x = 0, \\ 1, & x > 0. \end{cases}$$

Therefore, $f'(x)$ is undefined at $x = 0$, which lies inside the open interval $(-1, 1)$. Since the Mean Value Theorem requires differentiability on the entire open interval $(a, b)$, this condition is not satisfied.

**Final Answer:** The Mean Value Theorem does *not* apply on $[-1, 1]$ because $f(x) = 5 + |x|$ is not differentiable at $x = 0$.

## Exercise 182

Consider the function $f(x) = x^3 + 3x^2 + 16$. Its derivative is

$$f'(x) = 3x^2 + 6x = 3x(x + 2).$$

Thus the critical points are $x = 0$ and $x = -2$.

Evaluating the function,

$$f(-2) = 20, \qquad f(0) = 16.$$

Since $f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to \infty$ as $x \to \infty$, and the function decreases before $x = -2$ and increases after, the graph crosses the $x$-axis exactly once.

**Final Answer:** The equation has exactly one real root, located between $-4$ and $-3$.

## Exercise 183

A quadratic has exactly one real root when its discriminant is zero:

$$b^2 - 4ac = 0.$$

**Final Answer:** The condition for a double root is $b^2 = 4ac$.

### Exercise 184

Consider $y = e^x - b$. Since $e^x > 0$ for all $x$, the equation $e^x = b$ has a solution only when $b > 0$.

Because $e^x$ is strictly increasing, the equation can intersect the $x$-axis at most once.

**Final Answer:** The equation has exactly one root when $b > 0$, and never more than one root.

### Exercise 185

The function is continuous and differentiable on the interval, so the Mean Value Theorem applies. The slope of the secant line is

$$\frac{f(b) - f(a)}{b - a} = 0.$$

Thus we solve $f'(c) = 0$.

Since

$$f'(x) = \pi \sec^2(\pi x),$$

which is never zero, there is no solution.

**Final Answer:** There is no value of $c$ satisfying the Mean Value Theorem.

### Exercise 186

The function is continuous on $[0, 3]$ and differentiable on $(0, 3)$, so the Mean Value Theorem applies.

Compute

$$f'(x) = -\frac{1}{2}(x + 1)^{-3/2}.$$

Set $f'(c)$ equal to the slope of the secant line and solve numerically.

**Final Answer:** One value of $c$ exists in $(0, 3)$, approximately $c \approx 1.42$.

### Exercise 187

The function contains an absolute value with a corner at $x = -1$, where the derivative does not exist.

**Final Answer:** The Mean Value Theorem does not apply because the function is not differentiable on the interval.

### Exercise 188

The function is continuous and differentiable on the interval. Compute

$$f'(x) = 1 - \frac{1}{x^2}.$$

Set $f'(c)$ equal to the secant slope and solve.

**Final Answer:** A unique value of $c$ exists in the interval.

### Exercise 189

The function is continuous and differentiable on $[3, 8]$. Compute the derivative and equate it to the secant slope.

**Final Answer:** One value of $c$ exists, approximately $c \approx 4.76$.

### Exercise 190

If the police were to cite you, there must exist a time where your instantaneous speed equals the average speed.

Compute the average speed:

$$\frac{39}{36/60} = 65 \text{ mph}.$$

**Final Answer:** Yes, by the Mean Value Theorem, the police can cite you for speeding.

### Exercise 191

Let the distance functions of the two cars be continuous and differentiable. Since they start and end together, their average velocities are equal.

**Final Answer:** Yes, there must be a time when both cars have the same instantaneous speed.

### Exercise 192

Compute the derivatives:

$$\frac{d}{dx}(\sec^2 x) = 2\sec^2 x \tan x, \quad \frac{d}{dx}(\tan^2 x) = 2\tan x \sec^2 x.$$

**Final Answer:** The derivatives are identical, so $\sec^2 x - \tan^2 x$ is a constant.

## Exercise 193

Similarly,

$$\frac{d}{dx}(\csc^2 x) = -2\csc^2 x \cot x, \quad \frac{d}{dx}(\cot^2 x) = -2\cot x \csc^2 x.$$

**Final Answer:** The functions differ by a constant.

### Derivatives and the Shape of a Graph

## Exercise 194

A critical point $c$ occurs when $f'(c) = 0$ or $f'(c)$ does not exist. There is no local maximum or minimum at $c$ if the derivative does not change sign at that point. If $f'(x)$ is positive on both sides of $c$, the function is increasing through $c$. If $f'(x)$ is negative on both sides of $c$, the function is decreasing through $c$. In either case, no local extremum occurs.

**Final Answer:** There is no local maximum or minimum at $c$ when $f'(x)$ does not change sign at $c$.

## Exercise 195

For $y = x^3$, the derivative is $y' = 3x^2$. At $x = 0$, we have $y'(0) = 0$, so $x = 0$ is a critical point. However, $3x^2 \geq 0$ for all $x$, so the derivative does not change sign at $x = 0$. The second derivative is $y'' = 6x$, which changes sign at $x = 0$, indicating an inflection point.

**Final Answer:** $x = 0$ is an inflection point but not a local maximum or minimum.

## Exercise 196

For $y = x^3$, the second derivative is $y'' = 6x$. Since $y''$ changes sign from negative to positive as $x$ passes through 0, the concavity changes at $x = 0$.

**Final Answer:** Yes, $x = 0$ is an inflection point.

## Exercise 197

A point cannot be both a local extremum and an inflection point for a twice differentiable function. At a local extremum, the concavity does not change at the critical point. An inflection point requires a change in concavity.

**Final Answer:** No, a point cannot be both an inflection point and a local extremum for a twice differentiable function.

### Exercise 198

The first derivative test requires continuity of $f'(x)$ so that sign changes in the derivative reflect actual changes in the function's behavior. If $f'(x)$ is discontinuous, it may change sign without indicating a true change from increasing to decreasing. For example, a function with a jump discontinuity in $f'$ can falsely suggest an extremum.

**Final Answer:** Continuity of $f'(x)$ is needed so sign changes correctly indicate local extrema.

### Exercise 199

A concave-down function does not need to cross $y = 0$. Concavity only describes the shape of the graph, not its position relative to the $x$-axis. For example, $y = -x^2 - 1$ is concave down everywhere but never crosses $y = 0$.

**Final Answer:** No, a concave-down function does not have to cross $y = 0$.

### Exercise 200

A polynomial of degree 2 has a constant second derivative. Since the second derivative never changes sign, the concavity never changes. An inflection point requires a change in concavity.

**Final Answer:** No, a quadratic polynomial cannot have an inflection point.

### Exercise 201

From the graph of $f'(x)$, the derivative is negative for large negative $x$, becomes positive near $x \approx -2$, then negative again near $x \approx -1$, and finally positive after $x \approx 2$. Thus $f$ is increasing where $f'(x) > 0$ and decreasing where $f'(x) < 0$.

### Exercise 224

$f(x) = x^2 - 6x$

First derivative: $f'(x) = 2x - 6$. Critical point: $2x - 6 = 0 \Rightarrow x = 3$.

Sign of $f'(x)$: For $x < 3$, $f'(x) < 0$ so $f$ is decreasing. For $x > 3$, $f'(x) > 0$ so $f$ is increasing.

Second derivative: $f''(x) = 2 > 0$, so the function is concave up everywhere. There is no change in concavity.

**Final Answer:** Increasing on $(3, \infty)$, decreasing on $(-\infty, 3)$. Local minimum at $x = 3$. Concave up on $(-\infty, \infty)$. No inflection points.

## Exercise 225

$f(x) = x^3 - 6x^2$

First derivative: $f'(x) = 3x^2 - 12x = 3x(x-4)$. Critical points: $x = 0, 4$.

Sign of $f'(x)$: Increasing on $(-\infty, 0)$ and $(4, \infty)$. Decreasing on $(0, 4)$.

Second derivative: $f''(x) = 6x - 12$. Inflection point where $6x - 12 = 0 \Rightarrow x = 2$.

Concavity: Concave down for $x < 2$. Concave up for $x > 2$.

**Final Answer:** Increasing on $(-\infty, 0) \cup (4, \infty)$. Decreasing on $(0, 4)$. Local maximum at $x = 0$, local minimum at $x = 4$. Concave down on $(-\infty, 2)$, concave up on $(2, \infty)$. Inflection point at $x = 2$.

## Exercise 226

$f(x) = x^4 - 6x^3$

First derivative: $f'(x) = 4x^3 - 18x^2 = 2x^2(2x - 9)$. Critical points: $x = 0, \frac{9}{2}$.

Sign of $f'(x)$: Decreasing on $(-\infty, \frac{9}{2})$. Increasing on $(\frac{9}{2}, \infty)$.

Second derivative: $f''(x) = 12x^2 - 36x = 12x(x-3)$. Inflection points: $x = 0, 3$.

Concavity: Concave up on $(-\infty, 0) \cup (3, \infty)$. Concave down on $(0, 3)$.

**Final Answer:** Increasing on $(\frac{9}{2}, \infty)$, decreasing on $(-\infty, \frac{9}{2})$. Local minimum at $x = \frac{9}{2}$. Concave up on $(-\infty, 0) \cup (3, \infty)$. Concave down on $(0, 3)$. Inflection points at $x = 0, 3$.

## Exercise 227

$f(x) = x^{11} - 6x^{10}$

First derivative: $f'(x) = 11x^{10} - 60x^9 = x^9(11x - 60)$. Critical points: $x = 0, \frac{60}{11}$.

Sign of $f'(x)$: Increasing on $(-\infty, 0) \cup (\frac{60}{11}, \infty)$. Decreasing on $(0, \frac{60}{11})$.

Second derivative: $f''(x) = 110x^9 - 540x^8 = 10x^8(11x - 54)$. Inflection points: $x = 0, \frac{54}{11}$.

**Final Answer:** Increasing on $(-\infty, 0) \cup (\frac{60}{11}, \infty)$. Decreasing on $(0, \frac{60}{11})$. Local maximum at $x = 0$, local minimum at $x = \frac{60}{11}$. Concavity changes at $x = 0, \frac{54}{11}$. Inflection points at those values.

## Exercise 228

$f(x) = x + x^2 - x^3$

First derivative: $f'(x) = 1 + 2x - 3x^2$. Critical points from $3x^2 - 2x - 1 = 0$: $x = -\frac{1}{3}, 1$.

Sign of $f'(x)$: Increasing on $(-\frac{1}{3}, 1)$. Decreasing on $(-\infty, -\frac{1}{3}) \cup (1, \infty)$.

Second derivative: $f''(x) = 2 - 6x$. Inflection point: $x = \frac{1}{3}$.

**Final Answer:** Increasing on $(-\frac{1}{3}, 1)$. Decreasing on $(-\infty, -\frac{1}{3}) \cup (1, \infty)$. Local minimum at $x = -\frac{1}{3}$, local maximum at $x = 1$. Concave up on $(-\infty, \frac{1}{3})$, concave down on $(\frac{1}{3}, \infty)$. Inflection point at $x = \frac{1}{3}$.

### Exercise 229

$f(x) = x^2 + x + 1$

First derivative: $f'(x) = 2x + 1$. Critical point: $x = -\frac{1}{2}$.

Second derivative: $f''(x) = 2 > 0$.

**Final Answer:** Increasing on $(-\frac{1}{2}, \infty)$, decreasing on $(-\infty, -\frac{1}{2})$. Local minimum at $x = -\frac{1}{2}$. Concave up everywhere. No inflection points.

### Exercise 230

$f(x) = x^3 + x^4$

First derivative: $f'(x) = 3x^2 + 4x^3 = x^2(3 + 4x)$. Critical points: $x = 0, -\frac{3}{4}$.

Sign of $f'(x)$: Increasing on $(-\infty, -\frac{3}{4}) \cup (0, \infty)$. Decreasing on $(-\frac{3}{4}, 0)$.

Second derivative: $f''(x) = 6x + 12x^2 = 6x(1 + 2x)$. Inflection points: $x = -\frac{1}{2}, 0$.

**Final Answer:** Increasing on $(-\infty, -\frac{3}{4}) \cup (0, \infty)$. Decreasing on $(-\frac{3}{4}, 0)$. Local maximum at $x = -\frac{3}{4}$, local minimum at $x = 0$. Reversed concavity at $x = -\frac{1}{2}, 0$. Inflection points at those values.

### Exercise 231

$f(x) = \sin(\pi x) - \cos(\pi x)$. $f'(x) = \pi(\cos(\pi x) + \sin(\pi x))$. Critical points satisfy $\cos(\pi x) + \sin(\pi x) = 0 \Rightarrow \tan(\pi x) = -1$. Thus $x = -\frac{1}{4}, \frac{3}{4}$ in $[-1, 1]$. Sign analysis shows increasing on $(-1, -\frac{1}{4}) \cup (\frac{3}{4}, 1)$, decreasing on $(-\frac{1}{4}, \frac{3}{4})$. Local maximum at $x = -\frac{1}{4}$, local minimum at $x = \frac{3}{4}$. $f''(x) = \pi^2(\cos(\pi x) - \sin(\pi x))$. Inflection when $\cos(\pi x) = \sin(\pi x) \Rightarrow x = \frac{1}{4}$. Concave up on $(\frac{1}{4}, 1)$, concave down on $(-1, \frac{1}{4})$.

**Final answer:** max at $x = -0.25$, min at $x = 0.75$, inflection at $x = 0.25$.

## Exercise 232

$f(x) = x + \sin(2x)$. $f'(x) = 1 + 2\cos(2x)$. Critical points from $\cos(2x) = -\frac{1}{2} \Rightarrow x = \pm\frac{\pi}{3}$ in $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Increasing on $(-\frac{\pi}{2}, -\frac{\pi}{3}) \cup (\frac{\pi}{3}, \frac{\pi}{2})$, decreasing on $(-\frac{\pi}{3}, \frac{\pi}{3})$. Local max at $x = -\frac{\pi}{3}$, local min at $x = \frac{\pi}{3}$. $f''(x) = -4\sin(2x)$. Inflection at $x = 0$. Concave up on $(-\frac{\pi}{2}, 0)$, concave down on $(0, \frac{\pi}{2})$.

**Final answer:** max at $-\pi/3$, min at $\pi/3$, inflection at 0.

## Exercise 233

$f(x) = \sin x + \tan x$. $f'(x) = \cos x + \sec^2 x > 0$ on $(-\frac{\pi}{2}, \frac{\pi}{2})$. Thus $f$ is increasing everywhere, no local extrema. $f''(x) = -\sin x + 2\sec^2 x \tan x$. Solving $f''(x) = 0$ gives inflection near $x \approx 0.48$. Concavity changes there.

**Final answer:** increasing everywhere, no extrema, one inflection point.

## Exercise 234

$f(x) = (x-2)^2(x-4)^2$. $f'(x) = 2(x-2)(x-4)(2x-6)$. Critical points at $x = 2, 3, 4$. Increasing on $(-\infty, 2) \cup (3, 4) \cup (\infty)$, decreasing on $(2, 3) \cup (4, \infty)$. Local minima at $x = 2, 4$, local maximum at $x = 3$. $f''(x) = 12x^2 - 72x + 104$. Inflection points at $x \approx 2.42, 3.58$.

**Final answer:** minima at 2 and 4, maximum at 3, two inflection points.

## Exercise 235

$f(x) = \frac{1}{1-x}$. $f'(x) = \frac{1}{(1-x)^2} > 0$ for $x \neq 1$. Increasing on $(-\infty, 1)$ and $(1, \infty)$. No local extrema. $f''(x) = \frac{2}{(1-x)^3}$. Concave down for $x < 1$, concave up for $x > 1$. No inflection point since discontinuous at $x = 1$.

**Final answer:** increasing everywhere, no extrema, no inflection.

## Exercise 236

$f(x) = \frac{\sin x}{x}$. $f'(x) = \frac{x\cos x - \sin x}{x^2}$. Critical points near $x \approx \pm 4.49$ (calculator). Increasing on $(0, 4.49) \cup (-\infty, -4.49)$, decreasing otherwise. Local max near $x \approx \pm 4.49$. Concavity from $f''$ gives inflection near $x \approx \pm 2.03$.

**Final answer:** symmetric extrema and inflections.

**Exercise 237**

$f(x) = \sin x\, e^x$. $f'(x) = e^x(\sin x + \cos x)$. Critical points when $\tan x = -1 \Rightarrow x = -\frac{\pi}{4}, \frac{3\pi}{4}$. Increasing on $(-\pi, -\frac{\pi}{4}) \cup (\frac{3\pi}{4}, \pi)$. Local max at $-\frac{\pi}{4}$, min at $\frac{3\pi}{4}$. $f''(x) = 2e^x \cos x$. Inflection at $x = \pm\frac{\pi}{2}$.

**Final answer:** max at $-\pi/4$, min at $3\pi/4$, inflections at $\pm\pi/2$.

**Exercise 238**

$f(x) = \ln x\sqrt{x}$. $f'(x) = \frac{1+\ln x}{2\sqrt{x}}$. Critical point at $x = e^{-1}$. Decreasing on $(0, e^{-1})$, increasing on $(e^{-1}, \infty)$. Minimum at $x = e^{-1}$. $f''(x) = \frac{-\ln x}{4x^{3/2}}$. Inflection at $x = 1$.

**Final answer:** minimum at $e^{-1}$, inflection at 1.

**Exercise 239**

$f(x) = \frac{1}{4}\sqrt{x} + \frac{1}{x}$. $f'(x) = \frac{1}{8\sqrt{x}} - \frac{1}{x^2}$. Critical point at $x = 16$. Decreasing on $(0, 16)$, increasing on $(16, \infty)$. Minimum at $x = 16$. $f''(x) > 0$ for all $x > 0$.

**Final answer:** minimum at $x = 16$, concave up everywhere.

**Exercise 240**

$f(x) = \frac{e^x}{x}$. $f'(x) = \frac{e^x(x-1)}{x^2}$. Critical point at $x = 1$. Decreasing on $(-\infty, 0) \cup (0, 1)$, increasing on $(1, \infty)$. Minimum at $x = 1$. $f''(x) = \frac{e^x(x^2-2x+2)}{x^3}$. Concave up for $x > 0$, concave down for $x < 0$.

**Final answer:** minimum at $x = 1$, concavity changes at $x = 0$.

**Exercise 241**

"Growing more slowly" means the population is still increasing, but the rate of increase is decreasing. Thus $f'(t) > 0$ and $f''(t) < 0$. Final answer: $f' > 0,\ f'' < 0$.

**Exercise 242**

Bike accelerates faster means the bike's acceleration exceeds the car's acceleration. Car goes faster means the car's velocity exceeds the bike's velocity. Since $f =$ (bike position) – (car position), we have $f'(t) = v_{\text{bike}} - v_{\text{car}} < 0$ and $f''(t) = a_{\text{bike}} - a_{\text{car}} > 0$.

**Final answer:** $f' < 0,\ f'' > 0$.

**Exercise 243**

Landing smoothly means altitude is decreasing and the descent is slowing. Thus $f'(t) < 0$ and $f''(t) > 0$.

**Final answer:** $f' < 0,\ f'' > 0$.

**Exercise 244**

"At their peak" means a local maximum. Thus $f'(t) = 0$ and $f''(t) < 0$.

**Final answer:** $f' = 0,\ f'' < 0$.

**Exercise 245**

Picking up speed means the economy is increasing and accelerating. Thus $f'(t) > 0$ and $f''(t) > 0$.

**Final answer:** $f' > 0,\ f'' > 0$.

**Exercise 246**

False. A cubic can stay entirely above or below the $x$-axis on $[1, 3]$. The conditions $f'(1) = 0$ and $f'(3) = 0$ do not force a root.

**Final answer:**false.

**Exercise 247**

True. Since $f'$ is continuous and $f'(1) = f'(3) = 0$, Rolle's Theorem applied to $f'$ implies there exists $c \in (1, 3)$ with $f''(c) = 0$.

**Final answer:** true.

**Exercise 248**

True. A cubic has no absolute maximum on a closed endpoint unless constrained. Since degree 3 polynomials are unbounded, there is no absolute maximum at $x = 3$.

**Final answer:** true.

### Exercise 249

False. A cubic can have three real roots but still have no inflection change in sign. Counterexample: a cubic with symmetric roots and flat curvature.

**Final answer:** false.

### Exercise 250

False. A cubic always has exactly one inflection point, regardless of the number of real roots. Thus one inflection point does not imply three real roots.

**Final answer:** false.

#### Limits at Infinity and Asymptotes

### Exercise 251

From the graph, the function grows without bound as $x \to 1^+$ and decreases without bound as $x \to 1^-$.

$$\text{Vertical asymptote: } x = 1.$$

**Final Answer:** $x = 1$

### Exercise 252

The graph shows the function blowing up near two distinct $x$-values. One vertical asymptote occurs on the left and one on the right.

$$\text{Vertical asymptotes: } x = -3 \text{ and } x = 2.$$

**Final Answer:** $x = -3,\ x = 2$

### Exercise 253

The graph has two locations where the function diverges to $\pm\infty$.

$$\text{Vertical asymptotes: } x = -1 \text{ and } x = 2.$$

**Final Answer:** $x = -1,\ x = 2$

**Exercise 254**

The function increases or decreases without bound near three $x$-values.

$$\text{Vertical asymptotes: } x = 0,\ x = 1,\ x = 2.$$

**Final Answer:** $x = 0,\ 1,\ 2$

**Exercise 255**

The graph approaches infinity only at one point on the $x$-axis.

$$\text{Vertical asymptote: } x = 0.$$

**Final Answer:** $x = 0$

**Exercise 256**

$$f(x) = \frac{x+1}{x^2+5x+4} = \frac{x+1}{(x+1)(x+4)}.$$

The factor $x+1$ cancels, leaving a removable discontinuity at $x = -1$, not an asymptote.

**Final Answer:** No vertical asymptote at $x = -1$.

**Exercise 257**

$$f(x) = \frac{x}{x-2}.$$

The denominator is zero at $x = 2$ while the numerator is nonzero.

**Final Answer:** Vertical asymptote at $x = 2$.

**Exercise 258**

$$f(x) = (x+2)^{3/2}.$$

The function is defined and finite for $x \geq -2$. There is no divergence.

**Final Answer:** No vertical asymptote at $x = -2$.

### Exercise 259

$$f(x) = (x-1)^{-1/3} = \frac{1}{(x-1)^{1/3}}.$$

As $x \to 1^{\pm}$, the function diverges to $\pm\infty$.

**Final Answer:** Vertical asymptote at $x = 1$.

### Exercise 260

$$f(x) = 1 + x^{-2/5}.$$

Since $x^{-2/5} = \frac{1}{x^{2/5}}$ and the denominator approaches zero at $x = 0$, the function diverges.

**Final Answer:** Vertical asymptote at $x = 0$.

### Exercise 261

$$\lim_{x\to\infty} \frac{1}{3x+6}.$$

As $x \to \infty$, the denominator grows without bound, so the fraction approaches zero.

$$\lim_{x\to\infty} \frac{1}{3x+6} = 0.$$

**Final Answer:** 0

### Exercise 262

We compare the highest–degree terms in the numerator and denominator:

$$\frac{2x-5}{4x}.$$

Dividing numerator and denominator by $x$,

$$\frac{2-\frac{5}{x}}{4}.$$

As $x \to \infty$, the term $\frac{5}{x} \to 0$.

**Final Answer:**

$$\lim_{x\to\infty} \frac{2x-5}{4x} = \frac{1}{2}.$$

## Exercise 263

The highest power in the numerator is $x^2$, while the highest power in the denominator is $x$. Thus the fraction grows without bound. Dividing by $x$,

$$\frac{x - 2 + \frac{5}{x}}{1 + \frac{2}{x}}.$$

As $x \to \infty$, the dominant term is $x$.

**Final Answer:**

$$\lim_{x\to\infty} \frac{x^2 - 2x + 5}{x + 2} = \infty.$$

## Exercise 264

The degree of the numerator is 3 and the degree of the denominator is 2. Dividing numerator and denominator by $x^2$,

$$\frac{3x - \frac{2}{x}}{1 + \frac{2}{x} + \frac{8}{x^2}}.$$

As $x \to -\infty$, the dominant term is $3x$, which tends to $-\infty$.

**Final Answer:**

$$\lim_{x\to-\infty} \frac{3x^3 - 2x}{x^2 + 2x + 8} = -\infty.$$

## Exercise 265

Both numerator and denominator are degree 4. Dividing by $x^4$,

$$\frac{1 - \frac{4}{x} + \frac{1}{x^4}}{-7 - \frac{2}{x^2} + \frac{2}{x^4}}.$$

As $x \to -\infty$, all fractional terms vanish.

**Final Answer:**

$$\lim_{x\to-\infty} \frac{x^4 - 4x^3 + 1}{2 - 2x^2 - 7x^4} = -\frac{1}{7}.$$

## Exercise 266

Rewrite the denominator:

$$\sqrt{x^2 + 1} = |x|\sqrt{1 + \frac{1}{x^2}}.$$

For $x \to \infty$, $|x| = x$, so

$$\frac{3x}{x\sqrt{1+\frac{1}{x^2}}} = \frac{3}{\sqrt{1+\frac{1}{x^2}}}.$$

As $x \to \infty$, the denominator approaches 1.

**Final Answer:**

$$\lim_{x\to\infty} \frac{3x}{\sqrt{x^2+1}} = 3.$$

## Exercise 267

Factor inside the square root:

$$\sqrt{4x^2-1} = |x|\sqrt{4-\frac{1}{x^2}}.$$

For $x \to -\infty$, $|x| = -x$. Thus,

$$\frac{-x\sqrt{4-\frac{1}{x^2}}}{x+2}.$$

Dividing numerator and denominator by $x$,

$$\frac{-\sqrt{4-\frac{1}{x^2}}}{1+\frac{2}{x}}.$$

As $x \to -\infty$, this becomes $-2$.

**Final Answer:**

$$\lim_{x\to-\infty} \frac{\sqrt{4x^2-1}}{x+2} = -2.$$

## Exercise 268

Rewrite the denominator:

$$\sqrt{x^2-1} = x\sqrt{1-\frac{1}{x^2}} \quad (x \to \infty).$$

Then

$$\frac{4x}{x\sqrt{1-\frac{1}{x^2}}} = \frac{4}{\sqrt{1-\frac{1}{x^2}}}.$$

As $x \to \infty$, the denominator approaches 1.

**Final Answer:**

$$\lim_{x\to\infty} \frac{4x}{\sqrt{x^2-1}} = 4.$$

**Exercise 269**

As in Exercise 268,

$$\sqrt{x^2-1} = |x|\sqrt{1-\frac{1}{x^2}}.$$

For $x \to -\infty$, $|x| = -x$. Thus,

$$\frac{4x}{-x\sqrt{1-\frac{1}{x^2}}} = -\frac{4}{\sqrt{1-\frac{1}{x^2}}}.$$

**Final Answer:**

$$\lim_{x\to-\infty} \frac{4x}{\sqrt{x^2-1}} = -4.$$

**Exercise 270**

Factor out $\sqrt{x}$ from the denominator:

$$\frac{2\sqrt{x}}{x-\sqrt{x}+1} = \frac{2\sqrt{x}}{\sqrt{x}(\sqrt{x}-1)+1}.$$

Divide numerator and denominator by $\sqrt{x}$,

$$\frac{2}{\sqrt{x}-1+\frac{1}{\sqrt{x}}}.$$

As $x \to \infty$, the denominator grows without bound.

**Final Answer:**

$$\lim_{x\to\infty} \frac{2\sqrt{x}}{x-\sqrt{x}+1} = 0.$$

**Exercise 271**

Given

$$f(x) = x - \frac{9}{x}.$$

**Vertical asymptotes:** Vertical asymptotes occur where the denominator is zero and the numerator is nonzero. Here, $\frac{9}{x}$ is undefined at $x = 0$, so there is a vertical asymptote at

$$x = 0.$$

**Horizontal asymptotes:** As $x \to \pm\infty$, the term $\frac{9}{x} \to 0$, so

$$f(x) \sim x.$$

Since the function grows without bound, there is no horizontal asymptote.

**Final Answer:** Vertical asymptote at $x = 0$. No horizontal asymptote.

### Exercise 272

Given

$$f(x) = \frac{1}{1 - x^2}.$$

**Vertical asymptotes:** Set the denominator equal to zero:

$$1 - x^2 = 0 \quad \Rightarrow \quad x = \pm 1.$$

**Horizontal asymptotes:** As $x \to \pm\infty$, the denominator grows without bound, so

$$f(x) \to 0.$$

**Final Answer:** Vertical asymptotes at $x = \pm 1$. Horizontal asymptote $y = 0$.

### Exercise 273

Given

$$f(x) = \frac{x^3}{4 - x^2}.$$

**Vertical asymptotes:** Solve $4 - x^2 = 0$:

$$x = \pm 2.$$

**Horizontal asymptotes:** The degree of the numerator (3) is greater than the degree of the denominator (2), so no horizontal asymptote exists.

**Final Answer:** Vertical asymptotes at $x = \pm 2$. No horizontal asymptote.

### Exercise 274

Given

$$f(x) = \frac{x^2 + 3}{x^2 + 1}.$$

**Vertical asymptotes:** The denominator $x^2 + 1$ is never zero for real $x$, so there are no vertical asymptotes.

**Horizontal asymptotes:** The degrees of numerator and denominator are equal. The horizontal asymptote is the ratio of leading coefficients:

$$y = \frac{1}{1} = 1.$$

**Final Answer:** No vertical asymptotes. Horizontal asymptote $y = 1$.

### Exercise 275

Given

$$f(x) = \sin(x)\sin(2x).$$

**Vertical asymptotes:** Sine functions are defined for all real $x$, so no vertical asymptotes exist.

**Horizontal asymptotes:** The product of bounded functions is bounded. Since the function oscillates, it does not approach a single value.

**Final Answer:** No vertical asymptotes and no horizontal asymptotes.

### Exercise 276

Given

$$f(x) = \cos x + \cos(3x) + \cos(5x).$$

**Vertical asymptotes:** Cosine is defined for all real $x$, so none exist.

**Horizontal asymptotes:** This function oscillates and does not approach a single value as $x \to \pm\infty$.

**Final Answer:** No vertical asymptotes and no horizontal asymptotes.

### Exercise 277

Given

$$f(x) = \frac{x \sin x}{x^2 - 1}.$$

**Vertical asymptotes:** Set the denominator equal to zero:

$$x^2 - 1 = 0 \Rightarrow x = \pm 1.$$

Since the numerator does not cancel, these are vertical asymptotes.

**Horizontal asymptotes:** As $x \to \pm\infty$, the numerator grows like $x$ while the denominator grows like $x^2$, so

$$f(x) \to 0.$$

**Final Answer:** Vertical asymptotes at $x = \pm 1$. Horizontal asymptote $y = 0$.

### Exercise 278

Given

$$f(x) = \frac{x}{\sin x}.$$

**Vertical asymptotes:** Vertical asymptotes occur where $\sin x = 0$ and the numerator is nonzero:

$$x = n\pi, \quad n \in \mathbb{Z},\ n \neq 0.$$

**Horizontal asymptotes:** The function oscillates and does not approach a constant value.

**Final Answer:** Vertical asymptotes at $x = n\pi$ for $n \neq 0$. No horizontal asymptote.

### Exercise 279

Given

$$f(x) = \frac{1}{x^3 + x^2}.$$

**Vertical asymptotes:** Factor the denominator:

$$x^2(x+1) = 0 \Rightarrow x = 0,\ -1.$$

**Horizontal asymptotes:** The degree of the denominator exceeds that of the numerator, so

$$f(x) \to 0.$$

**Final Answer:** Vertical asymptotes at $x = 0$ and $x = -1$. Horizontal asymptote $y = 0$.

### Exercise 280

Given

$$f(x) = \frac{1}{x-1} - 2x.$$

**Vertical asymptotes:** The term $\frac{1}{x-1}$ is undefined at $x = 1$, so

$$x = 1$$

is a vertical asymptote.

**Horizontal asymptotes:** The linear term $-2x$ dominates as $x \to \pm\infty$, so no horizontal asymptote exists.

**Final Answer:** Vertical asymptote at $x = 1$. No horizontal asymptote.

### Exercise 281

Given

$$f(x) = \frac{x^3 + 1}{x^3 - 1}.$$

**Vertical asymptotes:** Factor the denominator:

$$x^3 - 1 = (x - 1)(x^2 + x + 1).$$

Only $x = 1$ produces a real zero.

**Horizontal asymptotes:** The degrees are equal, so the horizontal asymptote is

$$y = \frac{1}{1} = 1.$$

**Final Answer:** Vertical asymptote at $x = 1$. Horizontal asymptote $y = 1$.

### Exercise 282

Given

$$f(x) = \frac{\sin x + \cos x}{\sin x - \cos x}.$$

**Vertical asymptotes:** Set the denominator equal to zero:

$$\sin x = \cos x \Rightarrow x = \frac{\pi}{4} + k\pi.$$

**Horizontal asymptotes:** The function oscillates and does not approach a constant.

**Final Answer:** Vertical asymptotes at $x = \frac{\pi}{4} + k\pi$. No horizontal asymptote.

### Exercise 283

Given

$$f(x) = x - \sin x.$$

**Vertical asymptotes:** Both terms are defined for all real $x$, so none exist.

**Horizontal asymptotes:** As $x \to \pm\infty$, the linear term dominates, so the function does not level off.

**Final Answer:** No vertical asymptotes and no horizontal asymptotes.

### Exercise 284

Given

$$f(x) = \frac{1}{x} - \sqrt{x}.$$

**Vertical asymptotes:** The term $\frac{1}{x}$ is undefined at $x = 0$, so there is a vertical asymptote at

$$x = 0.$$

**Horizontal asymptotes:** As $x \to \infty$, $\sqrt{x} \to \infty$, so the function diverges.

**Final Answer:** Vertical asymptote at $x = 0$. No horizontal asymptote.

### Exercise 262

Factor out the highest power of $x$ from numerator and denominator:

$$\lim_{x\to\infty} \frac{2x-5}{4x} = \lim_{x\to\infty} \frac{x\left(2-\frac{5}{x}\right)}{4x} = \lim_{x\to\infty} \frac{2-\frac{5}{x}}{4} = \frac{2-0}{4}.$$

**Final Answer:** $\frac{1}{2}$.

### Exercise 263

Compare degrees: the numerator has degree 2 and the denominator has degree 1, so the expression grows without bound (no finite limit). Do long division to see the end behavior:

$$\frac{x^2-2x+5}{x+2} = x - 4 + \frac{13}{x+2}.$$

As $x \to \infty$, the remainder term $\frac{13}{x+2} \to 0$, so the expression behaves like $x - 4$, which $\to \infty$. **Final Answer:** The limit diverges to $+\infty$ (no finite limit).

### Exercise 264

Since deg(numerator) = 3 and deg(denominator) = 2, do division to find the linear end behavior:

$$\frac{3x^3 - 2x}{x^2 + 2x + 8} = 3x - 6 + \frac{-14x + 48}{x^2 + 2x + 8}.$$

The remainder fraction goes to 0 as $x \to -\infty$ (degree 1 over degree 2). Thus the function behaves like $3x - 6$, which $\to -\infty$ as $x \to -\infty$. **Final Answer:** The limit is $-\infty$.

### Exercise 265

Use leading terms (same degree 4 top and bottom):

$$\lim_{x\to-\infty} \frac{x^4 - 4x^3 + 1}{2 - 2x^2 - 7x^4} = \lim_{x\to-\infty} \frac{x^4\left(1 - \frac{4}{x} + \frac{1}{x^4}\right)}{x^4\left(\frac{2}{x^4} - \frac{2}{x^2} - 7\right)} = \lim_{x\to-\infty} \frac{1 - \frac{4}{x} + \frac{1}{x^4}}{\frac{2}{x^4} - \frac{2}{x^2} - 7}.$$

All fractions with $x$ in the denominator go to 0, leaving

$$\frac{1}{-7}.$$

**Final Answer:** $-\frac{1}{7}$.

### Exercise 266

Rewrite the square root using $|x|$:

$$\sqrt{x^2 + 1} = |x|\sqrt{1 + \frac{1}{x^2}}.$$

For $x \to \infty$, $|x| = x$. Then

$$\frac{3x}{\sqrt{x^2 + 1}} = \frac{3x}{x\sqrt{1 + \frac{1}{x^2}}} = \frac{3}{\sqrt{1 + \frac{1}{x^2}}} \to \frac{3}{\sqrt{1}} = 3.$$

**Final Answer:** 3.

### Exercise 267

For $x \to -\infty$, $|x| = -x$. Then

$$\sqrt{4x^2 - 1} = |x|\sqrt{4 - \frac{1}{x^2}} = (-x)\sqrt{4 - \frac{1}{x^2}}.$$

So

$$\frac{\sqrt{4x^2-1}}{x+2} = \frac{(-x)\sqrt{4-\frac{1}{x^2}}}{x\left(1+\frac{2}{x}\right)} = \frac{-\sqrt{4-\frac{1}{x^2}}}{1+\frac{2}{x}} \;\to\; \frac{-\sqrt{4}}{1} = -2.$$

**Final Answer:** $-2$.

### Exercise 268

As $x \to \infty$,

$$\sqrt{x^2-1} = x\sqrt{1-\frac{1}{x^2}}.$$

Thus

$$\frac{4x}{\sqrt{x^2-1}} = \frac{4x}{x\sqrt{1-\frac{1}{x^2}}} = \frac{4}{\sqrt{1-\frac{1}{x^2}}} \;\to\; \frac{4}{1} = 4.$$

**Final Answer:** 4.

### Exercise 269

Now $x \to -\infty$, so $|x| = -x$ and

$$\sqrt{x^2-1} = |x|\sqrt{1-\frac{1}{x^2}} = (-x)\sqrt{1-\frac{1}{x^2}}.$$

Then

$$\frac{4x}{\sqrt{x^2-1}} = \frac{4x}{(-x)\sqrt{1-\frac{1}{x^2}}} = \frac{-4}{\sqrt{1-\frac{1}{x^2}}} \;\to\; -4.$$

**Final Answer:** $-4$.

### Exercise 270

The denominator is dominated by the $x$ term. Factor $x$ from the denominator:

$$\frac{2\sqrt{x}}{x-\sqrt{x}+1} = \frac{2\sqrt{x}}{x\left(1-\frac{1}{\sqrt{x}}+\frac{1}{x}\right)} = \frac{2}{\sqrt{x}} \cdot \frac{1}{1-\frac{1}{\sqrt{x}}+\frac{1}{x}}.$$

As $x \to \infty$, $\frac{2}{\sqrt{x}} \to 0$ and the second factor $\to 1$, so the product $\to 0$. **Final Answer:** 0.

### Exercise 271

Vertical asymptotes occur where the function blows up (denominator $= 0$ in a rational term). Here $f(x) = x - \frac{9}{x}$ is undefined at $x = 0$, and $-\frac{9}{x} \to \pm\infty$ near 0. So the vertical asymptote is $x = 0$.

For horizontal asymptotes, look at $\lim_{x\to\pm\infty} f(x)$. Since $x$ grows without bound and $\frac{9}{x} \to 0$,

$$f(x) = x - \frac{9}{x} \sim x,$$

so there is no horizontal asymptote (the function grows linearly). The end behavior matches the slant line $y = x$. **Final Answer:** Vertical asymptote $x = 0$; horizontal asymptote: none (slant asymptote $y = x$).

### Exercise 272

$$f(x) = \frac{1}{1 - x^2} = \frac{1}{(1 - x)(1 + x)}.$$

Denominator $= 0$ at $x = 1$ and $x = -1$, and the numerator is $1 \neq 0$, so both give vertical asymptotes.

As $x \to \pm\infty$, the denominator behaves like $-x^2$, so

$$\frac{1}{1 - x^2} \to 0.$$

**Final Answer:** Vertical asymptotes $x = \pm 1$; horizontal asymptote $y = 0$.

### Exercise 273

$$f(x) = \frac{x^3}{4 - x^2}.$$

Vertical asymptotes occur where $4 - x^2 = 0 \Rightarrow x = \pm 2$, and the numerator is nonzero there, so both are vertical asymptotes.

For end behavior, compare degrees: 3 on top, 2 on bottom, so expect a slant asymptote. Write

$$\frac{x^3}{4 - x^2} = \frac{x^3}{-(x^2 - 4)} = -\frac{x^3}{x^2 - 4}.$$

Long division gives

$$-\frac{x^3}{x^2 - 4} = -x + \frac{-4x}{x^2 - 4}.$$

The remainder term $\frac{-4x}{x^2-4} \to 0$ as $|x| \to \infty$, so the slant asymptote is $y = -x$, hence no horizontal asymptote. **Final Answer:** Vertical asymptotes $x = \pm 2$; horizontal asymptote: none (slant asymptote $y = -x$).

### Exercise 274

$$f(x) = \frac{x^2 + 3}{x^2 + 1}.$$

The denominator $x^2 + 1$ is never 0 for real $x$, so there are no vertical asymptotes.

Degrees are equal, so the horizontal asymptote is the ratio of leading coefficients:

$$\lim_{x \to \pm\infty} \frac{x^2 + 3}{x^2 + 1} = 1.$$

**Final Answer:** Vertical asymptotes: none; horizontal asymptote $y = 1$.

### Exercise 275

$$f(x) = \sin x \, \sin(2x)$$

is defined and continuous for all real $x$, so there are no vertical asymptotes.

Also, $\sin x \, \sin(2x)$ is periodic and oscillates forever; it does not approach a single number as $x \to \pm\infty$. Therefore there is no horizontal asymptote. **Final Answer:** Vertical asymptotes: none; horizontal asymptote: none.

### Exercise 276

$$f(x) = \cos x + \cos(3x) + \cos(5x)$$

is continuous for all real $x$, so there are no vertical asymptotes.

Since it is a sum of periodic functions, it keeps oscillating and does not approach a single value as $x \to \pm\infty$. Hence no horizontal asymptote. **Final Answer:** Vertical asymptotes: none; horizontal asymptote: none.

### Exercise 277

$$f(x) = \frac{x \sin x}{x^2 - 1}.$$

Vertical asymptotes occur where $x^2 - 1 = 0 \Rightarrow x = \pm 1$, provided the numerator is not 0. At $x = 1$, numerator $= 1 \cdot \sin 1 \neq 0$. At $x = -1$, numerator $= -1 \cdot \sin(-1) = \sin 1 \neq 0$. So both $x = \pm 1$ are vertical asymptotes.

For the horizontal asymptote:

$$\left| \frac{x \sin x}{x^2 - 1} \right| \leq \frac{|x|}{x^2 - 1} \to 0 \quad (|x| \to \infty),$$

because $|\sin x| \leq 1$. Thus the function approaches 0. **Final Answer:** Vertical asymptotes $x = \pm 1$; horizontal asymptote $y = 0$.

**Exercise 278**

$$f(x) = \frac{x}{\sin x}.$$

The denominator is 0 at $x = k\pi$ for integers $k$. At $x = 0$, the limit $\lim_{x\to 0} \frac{x}{\sin x} = 1$, so $x = 0$ is a removable discontinuity (not a vertical asymptote if you define $f(0) = 1$). For $k \neq 0$, the numerator $k\pi \neq 0$, so the function blows up near $x = k\pi$. Hence vertical asymptotes at $x = k\pi$ for $k \in \mathbb{Z} \setminus \{0\}$.

As $|x| \to \infty$, the function does not approach a constant (it oscillates with unbounded magnitude because the numerator grows while $\sin x$ stays between $-1$ and 1). So no horizontal asymptote. **Final Answer:** Vertical asymptotes $x = k\pi$ for $k \in \mathbb{Z} \setminus \{0\}$; horizontal asymptote: none.

**Exercise 279**

$$f(x) = \frac{1}{x^3 + x^2} = \frac{1}{x^2(x+1)}.$$

Denominator $= 0$ at $x = 0$ and $x = -1$, and numerator is 1, so both are vertical asymptotes.

Since the denominator degree is 3 and numerator degree is 0, the function goes to 0 as $|x| \to \infty$. **Final Answer:** Vertical asymptotes $x = 0$ and $x = -1$; horizontal asymptote $y = 0$.

**Exercise 280**

$$f(x) = \frac{1}{x-1} - 2x.$$

The term $\frac{1}{x-1}$ blows up at $x = 1$, so $x = 1$ is a vertical asymptote.

As $|x| \to \infty$, $\frac{1}{x-1} \to 0$, so

$$f(x) \sim -2x,$$

which grows without bound. Thus there is no horizontal asymptote, and the end behavior follows the slant line $y = -2x$. **Final Answer:** Vertical asymptote $x = 1$; horizontal asymptote: none (slant asymptote $y = -2x$).

**Exercise 281**

$$f(x) = \frac{x^3 + 1}{x^3 - 1} = \frac{(x+1)(x^2 - x + 1)}{(x-1)(x^2 + x + 1)}.$$

The factor $x^2 + x + 1$ has no real zeros, so the only real zero of the denominator is $x = 1$. The numerator is $1^3 + 1 = 2 \neq 0$, so $x = 1$ is a vertical asymptote.

Degrees are equal and leading coefficients are both 1, so

$$\lim_{x\to\pm\infty} \frac{x^3+1}{x^3-1} = 1.$$

**Final Answer:** Vertical asymptote $x = 1$; horizontal asymptote $y = 1$.

### Exercise 282

$$f(x) = \frac{\sin x + \cos x}{\sin x - \cos x}.$$

Vertical asymptotes occur where $\sin x - \cos x = 0 \Rightarrow \tan x = 1 \Rightarrow x = \frac{\pi}{4} + k\pi$. At those points, $\sin x + \cos x \neq 0$ (in fact it equals $\pm\sqrt{2}$), so the function blows up there.

Since the function is periodic (ratio of trig functions), it does not approach a single value as $x \to \pm\infty$. Hence no horizontal asymptote. **Final Answer:** Vertical asymptotes $x = \frac{\pi}{4} + k\pi$ ($k \in \mathbb{Z}$); horizontal asymptote: none.

### Exercise 283

$$f(x) = x - \sin x.$$

This is defined and continuous for all real $x$, so there are no vertical asymptotes.

As $x \to \pm\infty$, $-\sin x$ stays bounded while $x$ grows without bound, so there is no horizontal asymptote. The function differs from $y = x$ by a bounded amount, so $y = x$ is the slant asymptote. **Final Answer:** Vertical asymptotes: none; horizontal asymptote: none (slant asymptote $y = x$).

### Exercise 284

$$f(x) = \frac{1}{x} - \sqrt{x}, \qquad x > 0.$$

As $x \to 0^+$, $\frac{1}{x} \to +\infty$ while $\sqrt{x} \to 0$, so the function blows up: vertical asymptote at $x = 0$.

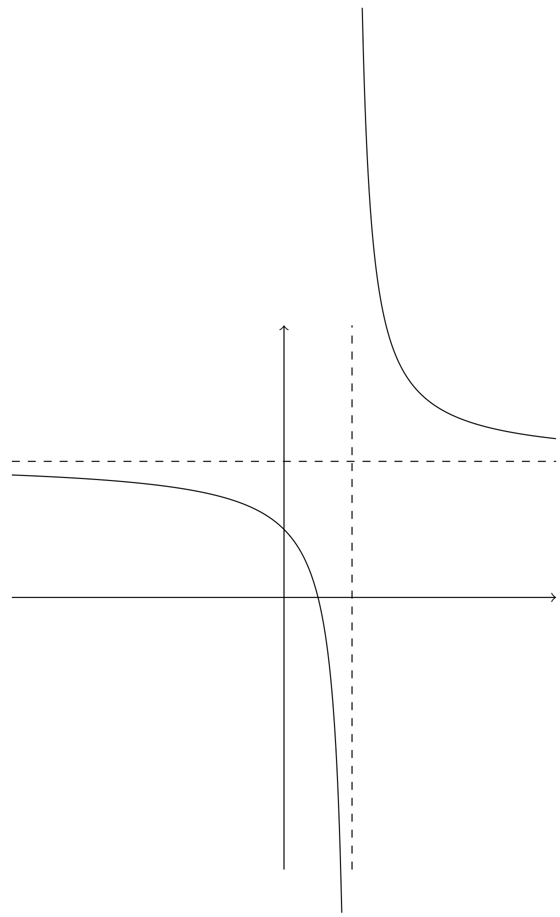
As $x \to \infty$, $\frac{1}{x} \to 0$ and $-\sqrt{x} \to -\infty$, so there is no horizontal asymptote. **Final Answer:** Vertical asymptote $x = 0$; horizontal asymptote: none.

### Exercise 285

To force a vertical asymptote at $x = 1$, include a term $\frac{1}{x-1}$. To force a horizontal asymptote $y = 2$, shift upward by 2:

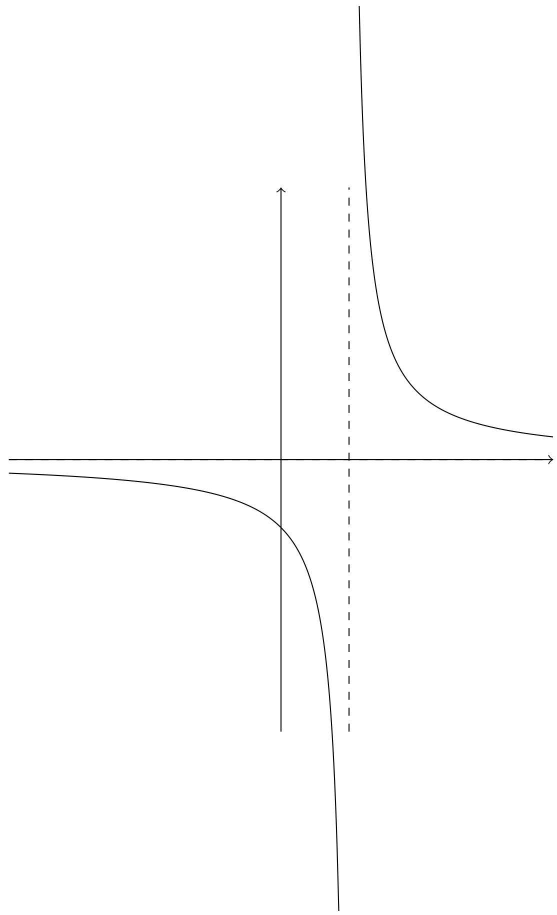
$$f(x) = 2 + \frac{1}{x-1}.$$

As $x \to \pm\infty$, $\frac{1}{x-1} \to 0$, so $f(x) \to 2$. As $x \to 1$, $f(x) \to \pm\infty$. **Final Answer:** $f(x) = 2 + \dfrac{1}{x-1}$.



## Exercise 286

A simple choice with vertical asymptote $x = 1$ is $\frac{1}{x-1}$. Since $\frac{1}{x-1} \to 0$ as $|x| \to \infty$, the horizontal asymptote is $y = 0$. **Final Answer:** $f(x) = \dfrac{1}{x-1}$.

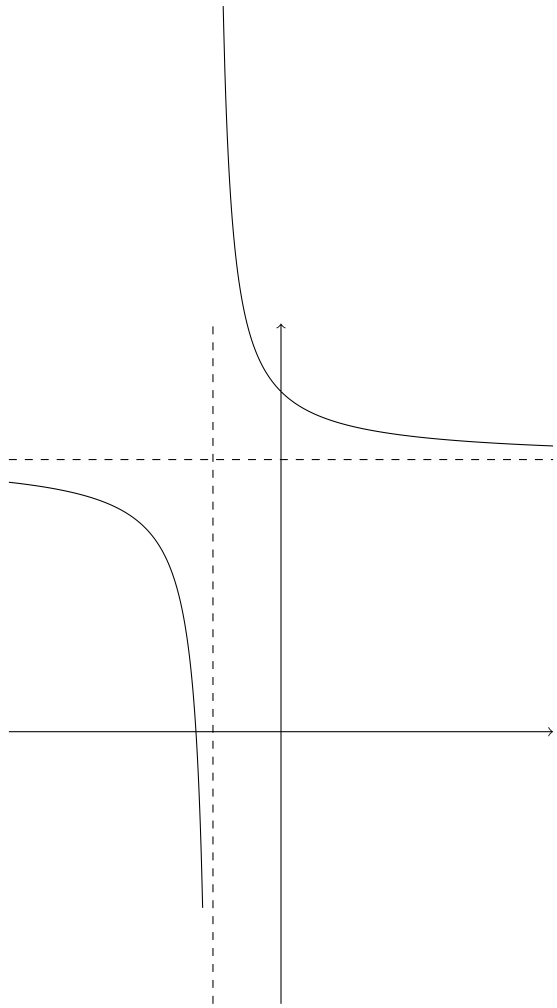


**Exercise 287**

Vertical asymptote $x = -1$ suggests $\frac{1}{x+1}$. Horizontal asymptote $y = 4$ means shift up by 4:

$$f(x) = 4 + \frac{1}{x+1}.$$

Then $\frac{1}{x+1} \to 0$ as $|x| \to \infty$, so $f(x) \to 4$, and $x = -1$ makes the function blow up. **Final Answer:** $f(x) = 4 + \dfrac{1}{x+1}$.

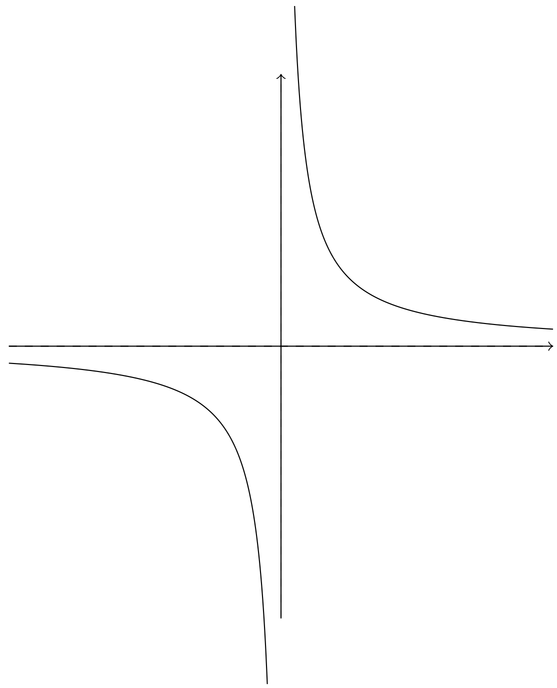


**Exercise 288**

We only need a vertical asymptote at $x = 0$. The simplest is

$$f(x) = \frac{1}{x}.$$

This is undefined at $x = 0$ and blows up there, giving the required vertical asymptote. **Final Answer:** $f(x) = \dfrac{1}{x}$.



**Exercise 289**

For a rational function $\frac{1}{x+10}$, as $|x|$ becomes large the denominator grows without bound, so the fraction goes to 0. Numerically (graph estimate), values near $x = \pm 5$ are already close to 0, and this trend continues.

Formally,

$$\lim_{x \to \pm\infty} \frac{1}{x+10} = 0.$$

**Final Answer:** Horizontal asymptote $y = 0$.

**Exercise 290**

Because the denominator has higher degree than the numerator, the fraction should approach 0. Factor the denominator:

$$\frac{x+1}{x^2+7x+6} = \frac{x+1}{(x+1)(x+6)} = \frac{1}{x+6} \quad (x \neq -1).$$

So the graph behaves like $\frac{1}{x+6}$, which tends to 0 as $x \to \pm\infty$. (There is a removable hole at $x = -1$, but that does not affect the horizontal asymptote.)

$$\lim_{x \to \pm\infty} \frac{x+1}{x^2+7x+6} = 0.$$

**Final Answer:** Horizontal asymptote $y = 0$.

### Exercise 291

$$x^2 + 10x + 25 = (x + 5)^2.$$

As $x \to -\infty$, $(x + 5)^2 \to \infty$. So the expression does not approach a finite number (it grows without bound). **Final Answer:** The limit is $+\infty$ (no horizontal asymptote).

### Exercise 292

The denominator has degree 2 while the numerator has degree 1, so the fraction should go to 0. Divide top and bottom by $x^2$:

$$\lim_{x\to-\infty} \frac{x + 2}{x^2 + 7x + 6} = \lim_{x\to-\infty} \frac{\frac{1}{x} + \frac{2}{x^2}}{1 + \frac{7}{x} + \frac{6}{x^2}} = \frac{0}{1} = 0.$$

**Final Answer:** 0.

### Exercise 293

Degrees are equal, so the limit is the ratio of leading coefficients (and the remainder gives how it approaches). Rewrite by division:

$$\frac{3x + 2}{x + 5} = 3 + \frac{-13}{x + 5}.$$

As $x \to \infty$, $\frac{-13}{x+5} \to 0$, so the expression approaches 3.

**Final Answer:** 3.

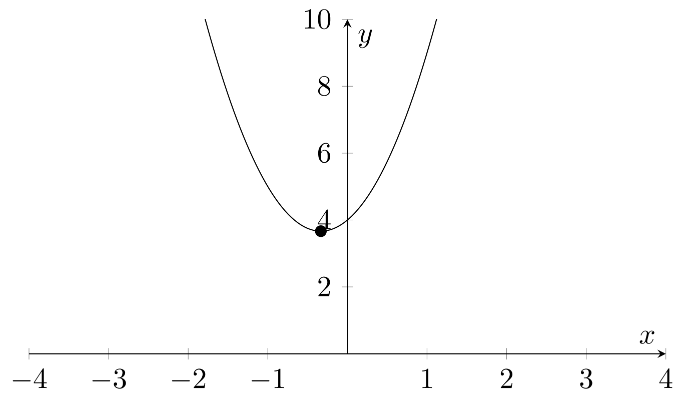
### Exercise 294

For $y = 3x^2 + 2x + 4$, this is a parabola opening upward ($a = 3 > 0$), so it has an absolute minimum at its vertex and no maximum. The vertex occurs at

$$x_v = -\frac{b}{2a} = -\frac{2}{2 \cdot 3} = -\frac{1}{3}, \qquad y_v = 3\left(-\frac{1}{3}\right)^2 + 2\left(-\frac{1}{3}\right) + 4 = \frac{11}{3}.$$

So the graph is concave up everywhere and decreases until $x = -\frac{1}{3}$, then increases.

**Final Answer:** Absolute minimum at $\left(-\frac{1}{3}, \frac{11}{3}\right)$; no absolute maximum; no inflection points.




**Exercise 295**

Let $y = x^3 - 3x^2 + 4$. Critical points occur where $y' = 0$:

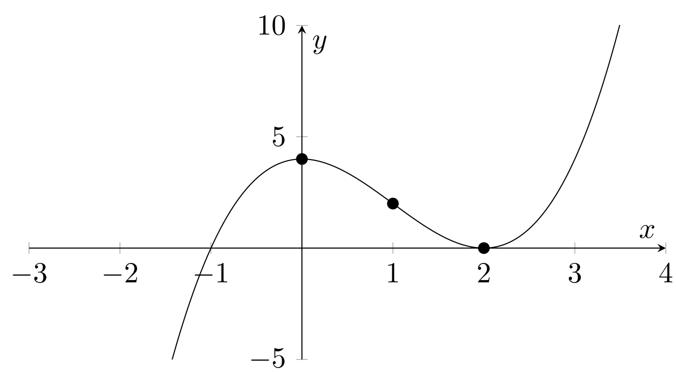
$$y' = 3x^2 - 6x = 3x(x-2) \implies x = 0,\ 2.$$

Classify with $y''$:

$$y'' = 6x - 6.$$

At $x = 0$, $y''(0) = -6 < 0$, so there is a local maximum: $y(0) = 4$. At $x = 2$, $y''(2) = 6 > 0$, so there is a local minimum: $y(2) = 0$. Inflection points occur where $y'' = 0 \Rightarrow x = 1$, and $y(1) = 2$.

**Final Answer:** Local max at $(0, 4)$, local min at $(2, 0)$, inflection at $(1, 2)$.




**Exercise 296**

$$y = \frac{2x+1}{x^2+6x+5} = \frac{2x+1}{(x+1)(x+5)}.$$

The denominator is zero at $x = -1, -5$, so there are vertical asymptotes at $x = -1$ and $x = -5$ (no cancellation occurs). Since deg(num) < deg(den), the horizontal asymptote is $y = 0$. Intercepts: $2x + 1 = 0 \Rightarrow x = -\frac{1}{2}$, and $y(0) = \frac{1}{5}$.

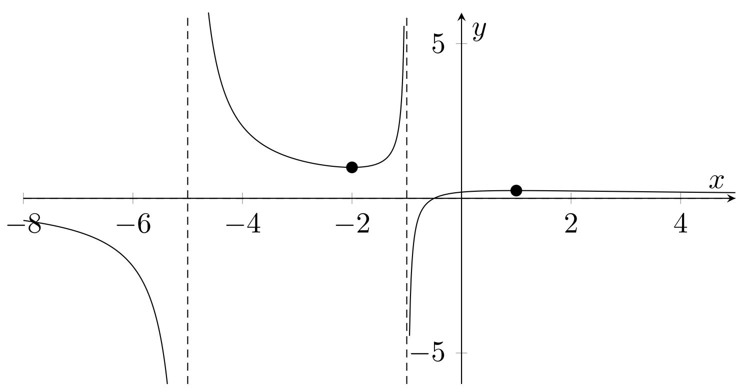
For critical points, compute

$$y' = \frac{2(x^2+6x+5)-(2x+1)(2x+6)}{(x^2+6x+5)^2} = \frac{-2(x+2)(x-1)}{(x^2+6x+5)^2}.$$

Thus critical points are $x=-2$ and $x=1$ (both are in the domain). Since the denominator is always positive (squared), the sign comes from $-2(x+2)(x-1)$: it changes from $-$ to $+$ at $x=-2$ (local minimum) and from $+$ to $-$ at $x=1$ (local maximum). Values:

$$y(-2)=1, \qquad y(1)=\frac{1}{4}.$$

**Final Answer:** Vertical asymptotes $x=-5,-1$; horizontal asymptote $y=0$; local min $(-2,1)$; local max $\left(1,\frac{1}{4}\right)$.




## Exercise 297

$$y = \frac{x^3+4x^2+3x}{3x+9} = \frac{x(x+1)(x+3)}{3(x+3)} = \frac{x(x+1)}{3} \quad (x \neq -3).$$

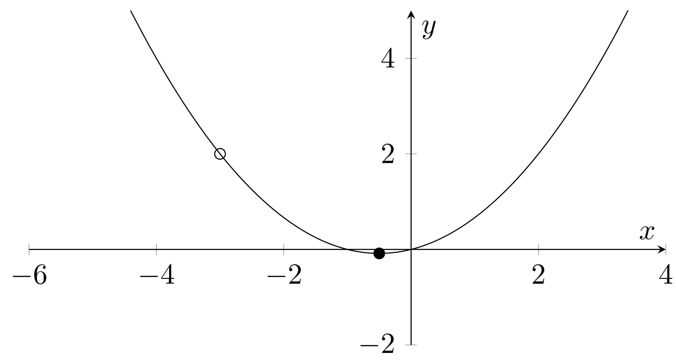
So the graph is the parabola $y=\frac{x^2+x}{3}$ with a removable discontinuity (a hole) at $x=-3$. The missing $y$-value would be

$$y(-3) = \frac{(-3)(-2)}{3} = 2,$$

so the hole is at $(-3,2)$. The parabola opens upward; its vertex is at $x=-\frac{1}{2}$ with

$$y\left(-\tfrac{1}{2}\right) = \frac{\frac{1}{4}-\frac{1}{2}}{3} = -\frac{1}{12}.$$

**Final Answer:** Graph is $y=\frac{x^2+x}{3}$ with a hole at $(-3,2)$; absolute minimum at $\left(-\frac{1}{2},-\frac{1}{12}\right)$; no asymptotes; no inflection points.




**Exercise 298**

$$y = \frac{x^2 + x - 2}{x^2 - 3x - 4} = \frac{(x+2)(x-1)}{(x-4)(x+1)}.$$

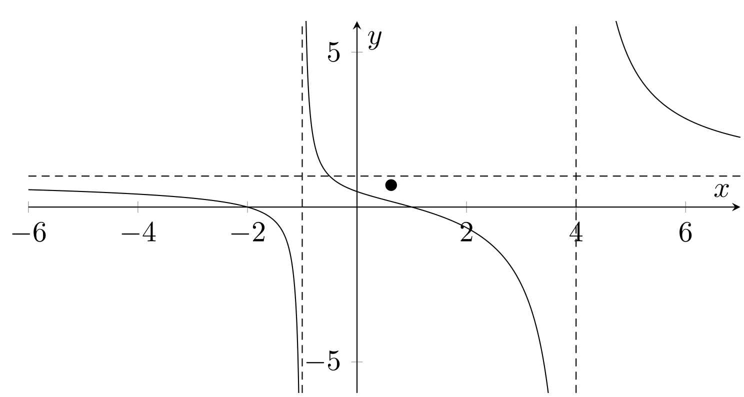
Vertical asymptotes occur where the denominator is zero and no factor cancels: $x = -1$ and $x = 4$. Degrees are equal, so the horizontal asymptote is the ratio of leading coefficients: $y = 1$. Intercepts: $x = -2, 1$, and $y(0) = \frac{-2}{-4} = \frac{1}{2}$. Also, $y = 1$ when $x = -\frac{1}{2}$ (solve $\frac{x^2+x-2}{x^2-3x-4} = 1$).

Monotonicity: differentiating gives

$$y' = \frac{-2(2x^2 + 2x + 5)}{(x^2 - 3x - 4)^2} < 0$$

since $2x^2 + 2x + 5 > 0$ for all $x$. Therefore $y$ is strictly decreasing on each domain interval, so there are no local extrema. Concavity changes when $y'' = 0$, which occurs at approximately $x \approx 0.6233$ (in the domain), giving an inflection point near $(0.6233, 0.7080)$.

**Final Answer:** Vertical asymptotes $x = -1, 4$; horizontal asymptote $y = 1$; no local extrema; inflection near $(0.6233, 0.7080)$.



### Exercise 299

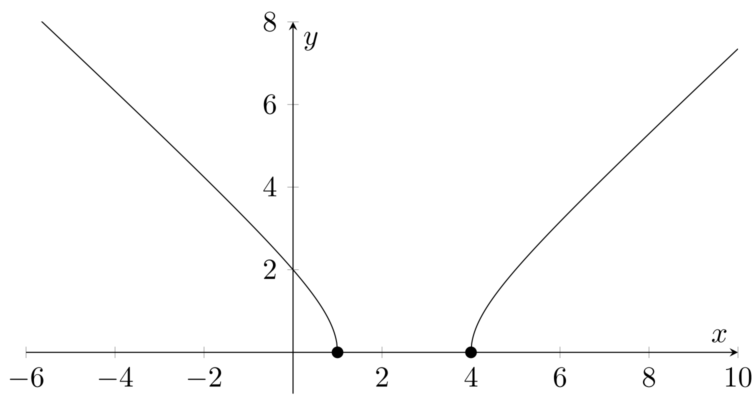
$$y = \sqrt{x^2 - 5x + 4} = \sqrt{(x-1)(x-4)}.$$

For real outputs, the radicand must be nonnegative, so $(x-1)(x-4) \geq 0 \Rightarrow x \leq 1$ or $x \geq 4$. Zeros occur where the radicand is 0: $(1,0)$ and $(4,0)$, giving absolute minima $y_{\text{min}} = 0$. As $x \to \infty$, $y \sim x - \frac{5}{2}$; as $x \to -\infty$, $y \sim -x - \frac{5}{2}$. At $x = 1$ and $x = 4$, the derivative

$$y' = \frac{2x-5}{2\sqrt{x^2 - 5x + 4}}$$

blows up, so the graph has vertical tangents at the endpoints of the domain pieces.

**Final Answer:** Domain $(-\infty, 1] \cup [4, \infty)$; absolute minima at $(1,0)$ and $(4,0)$; no maxima; vertical tangents at $x = 1, 4$.




### Exercise 300

$$y = 2x\sqrt{16 - x^2}, \qquad -4 \leq x \leq 4.$$

The endpoints satisfy $y(\pm 4) = 0$, and $y(0) = 0$. Differentiate:

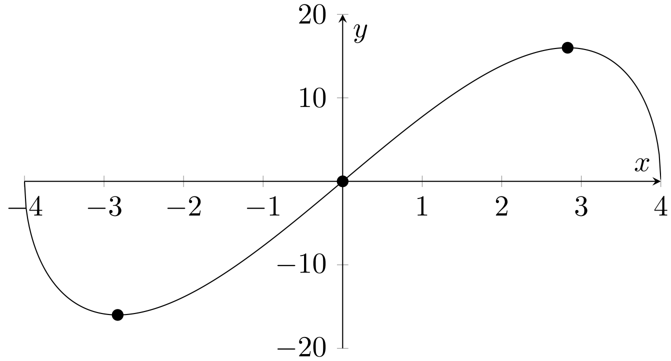
$$y' = 2\sqrt{16 - x^2} + 2x \cdot \frac{-x}{\sqrt{16 - x^2}} = \frac{32 - 4x^2}{\sqrt{16 - x^2}} = \frac{4(8 - x^2)}{\sqrt{16 - x^2}}.$$

Critical points in $(-4, 4)$ come from $8 - x^2 = 0 \Rightarrow x = \pm 2\sqrt{2}$. Values:

$$y(2\sqrt{2}) = 2(2\sqrt{2})\sqrt{8} = 16, \qquad y(-2\sqrt{2}) = -16.$$

So these are the absolute max/min on $[-4, 4]$. Concavity switches at $x = 0$ (the function is concave up for $x < 0$ and concave down for $x > 0$), so $(0,0)$ is an inflection point.

**Final Answer:** Absolute max 16 at $x = 2\sqrt{2}$; absolute min $-16$ at $x = -2\sqrt{2}$; inflection at $(0,0)$.




**Exercise 301**

$$y = \frac{\cos x}{x}, \qquad x \in [-2\pi, 2\pi],\ x \neq 0.$$

There is a vertical asymptote at $x = 0$ since $\cos x \to 1$ while $x \to 0$, so $\cos x/x \to \pm\infty$. Zeros occur when $\cos x = 0$: $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}$. The function is odd (even/odd), so the graph is symmetric about the origin. Critical points satisfy

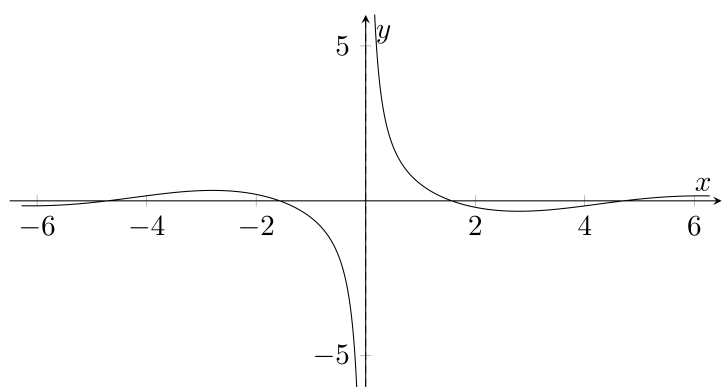
$$y' = \frac{-x\sin x - \cos x}{x^2} = 0 \quad \Longleftrightarrow \quad x\sin x + \cos x = 0.$$

On $(0, 2\pi)$, numerical solutions are approximately

$$x \approx 2.7984 \quad \text{and} \quad x \approx 6.1213,$$

giving $y \approx -0.3364$ (local min) and $y \approx 0.1616$ (local max). By odd symmetry, the corresponding points occur at negatives with opposite $y$-values.

**Final Answer:** Vertical asymptote at $x = 0$; zeros at $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}$; critical points $x \approx \pm 2.7984, \pm 6.1213$ (with corresponding $y$-values $\mp 0.3364$ and $\pm 0.1616$).



**Exercise 302**

$$y = e^x - x^3.$$

As $x \to \infty$, $e^x$ dominates, so $y \to \infty$. As $x \to -\infty$, $e^x \to 0$ and $-x^3 \to \infty$, so again $y \to \infty$. Hence the function must dip to a (global) minimum somewhere.

Critical points solve

$$y' = e^x - 3x^2 = 0 \quad \Longleftrightarrow \quad e^x = 3x^2.$$

Numerically, there are three solutions:

$$x \approx -0.4590, \quad 0.9100, \quad 3.7331.$$

Use $y'' = e^x - 6x$ to classify:

$$y''(-0.4590) > 0 \Rightarrow \text{local min}, \quad y''(0.9100) < 0 \Rightarrow \text{local max}, \quad y''(3.7331) > 0 \Rightarrow \text{local min}.$$

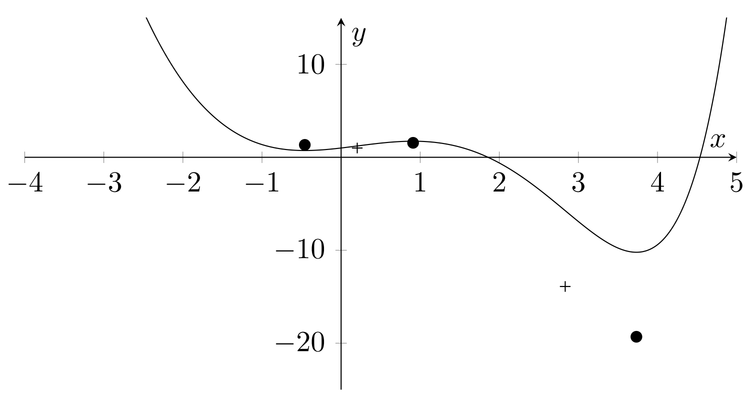
Corresponding values:

$$y(-0.4590) \approx 1.3373, \quad y(0.9100) \approx 1.5567, \quad y(3.7331) \approx -19.3099,$$

so the minimum at $x \approx 3.7331$ is the absolute minimum. Inflection points satisfy $y'' = 0 \Rightarrow e^x = 6x$, giving approximately

$$x \approx 0.2045, \quad 2.8331.$$

**Final Answer:** Local mins at $x \approx -0.4590$ and $x \approx 3.7331$ (absolute min $\approx -19.3099$ at $x \approx 3.7331$); local max at $x \approx 0.9100$; inflections at $x \approx 0.2045$ and $x \approx 2.8331$.




**Exercise 303**

$$y = x \tan x, \qquad x \in [-\pi, \pi],\ x \neq \pm\frac{\pi}{2}.$$

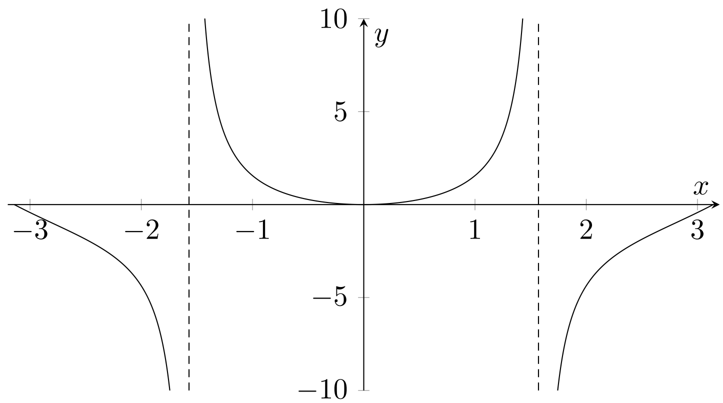
The function is even (odd×odd), so it is symmetric about the $y$-axis. Vertical asymptotes occur where $\tan x$ blows up: $x = \pm\frac{\pi}{2}$. Zeros occur at $x = 0$ and also at $x = \pm\pi$ since $\tan(\pm\pi) = 0$.

Derivative:

$$y' = \tan x + x\sec^2 x.$$

On $(0, \frac{\pi}{2})$, both terms are positive, so $y' > 0$ and the graph increases. On $(\frac{\pi}{2}, \pi)$, $\tan x < 0$ but $x\sec^2 x > 0$ and dominates; in fact $y' > 0$ throughout this interval as well, so the graph still increases on each piece. Thus there are no local extrema inside the open domain intervals; the main features are the asymptotes and monotone behavior.

**Final Answer:** Vertical asymptotes at $x = \pm\frac{\pi}{2}$; zeros at $x = 0, \pm\pi$; increasing on each interval $(-\pi, -\frac{\pi}{2})$, $(-\frac{\pi}{2}, \frac{\pi}{2})$, $(\frac{\pi}{2}, \pi)$; no interior local maxima/minima.




**Exercise 304**

$$y = x\ln x, \qquad x > 0.$$

Differentiate:

$$y' = \ln x + 1.$$

So $y' = 0$ at $\ln x = -1 \Rightarrow x = \frac{1}{e}$. Since

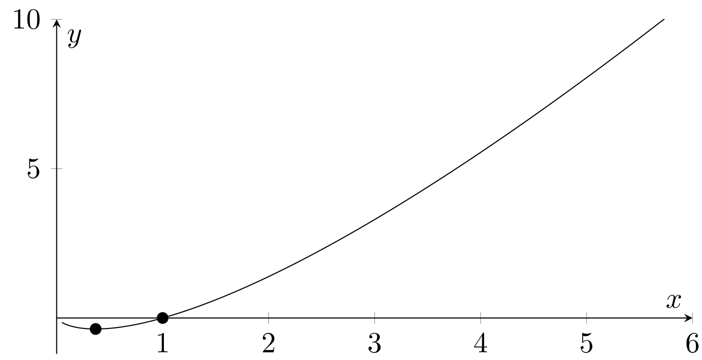
$$y'' = \frac{1}{x} > 0 \quad (x > 0),$$

the graph is concave up everywhere, and $x = \frac{1}{e}$ gives a (global) minimum:

$$y\left(\frac{1}{e}\right) = \frac{1}{e}\ln\left(\frac{1}{e}\right) = \frac{1}{e}(-1) = -\frac{1}{e}.$$

Also, $x\ln x = 0$ at $x = 1$. As $x \to 0^+$, $x\ln x \to 0^-$; as $x \to \infty$, $x\ln x \to \infty$.

**Final Answer:** Absolute minimum at $\left(\frac{1}{e}, -\frac{1}{e}\right)$; concave up for all $x > 0$; intercept at $(1, 0)$.




**Exercise 305**

$$y = x^2 \sin x, \qquad x \in [-2\pi, 2\pi].$$

This function is odd (even×odd), so it has origin symmetry. Zeros occur when $\sin x = 0$: $x = k\pi$, so in the interval we have $x = -2\pi, -\pi, 0, \pi, 2\pi$.

Differentiate:

$$y' = 2x \sin x + x^2 \cos x = x\big(2 \sin x + x \cos x\big).$$

Thus $x = 0$ is critical, and other critical points satisfy

$$2 \sin x + x \cos x = 0 \quad \iff \quad \tan x = -\frac{x}{2}.$$

On $(0, 2\pi)$, numerical solutions are

$$x \approx 2.2889, \qquad x \approx 5.0870,$$

giving

$$y(2.2889) \approx 3.9450 \quad \text{(local max)}, \qquad y(5.0870) \approx -24.0829 \quad \text{(local min)}.$$

By odd symmetry, there is a local min at $-2.2889$ and a local max at $-5.0870$.

Inflection points occur where $y'' = 0$. Here

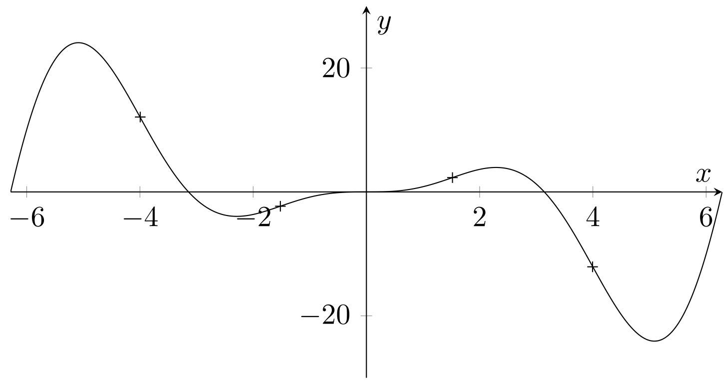
$$y'' = 2 \sin x + 4x \cos x - x^2 \sin x,$$

which has (in $(0, 2\pi)$) approximate zeros

$$x \approx 1.5199, \qquad x \approx 3.9944,$$

and the symmetric negatives.

**Final Answer:** Critical points at $x = 0, \pm 2.2889, \pm 5.0870$ with local max at $x \approx 2.2889$ and local min at $x \approx 5.0870$ (and symmetric); inflections near $x \approx \pm 1.5199, \pm 3.9944$.




**Exercise 306**

For a rational function $f(x) = \frac{P(x)}{Q(x)}$, a horizontal asymptote at $y = 2$ means that

$$\lim_{x \to \pm\infty} \frac{P(x)}{Q(x)} = 2.$$

This occurs when $P(x)$ and $Q(x)$ have the same degree and the ratio of their leading coefficients is 2.

**Final Answer:** $P(x)$ and $Q(x)$ must have the same degree, and the leading coefficient of $P(x)$ must be twice that of $Q(x)$.

**Exercise 307**

A vertical asymptote at $x = 0$ occurs when the denominator of a rational function approaches zero at $x = 0$ while the numerator does not also approach zero.

Thus, $Q(0) = 0$ and $P(0) \neq 0$.

**Final Answer:** $Q(x)$ must have a factor of $x$ while $P(x)$ does not.

**Exercise 308**

If $f'(x)$ has a horizontal asymptote at $y = 3$, then the slope of $f(x)$ approaches 3 for large $|x|$, meaning $f(x)$ behaves like a line with slope 3.

If $f'(x)$ has a vertical asymptote at $x = 1$, then $f(x)$ has either a cusp, corner, or vertical tangent at $x = 1$.

**Final Answer:** $f(x)$ has a slant asymptote with slope 3 and a nondifferentiable feature (cusp, corner, or vertical tangent) at $x = 1$.

### Exercise 309

Both functions have a vertical asymptote at $x = 1$ and a horizontal asymptote at $y = 0$.

However, the order of the denominator changes the behavior near the asymptote.

$$f(x) = \frac{1}{x-1} \quad \text{changes sign across } x = 1,$$

while

$$g(x) = \frac{1}{(x-1)^2} \quad \text{approaches } +\infty \text{ from both sides.}$$

**Final Answer:** The most obvious difference is that $f(x)$ changes sign across $x = 1$, while $g(x)$ does not.

### Exercise 310

A ratio of polynomials has a vertical asymptote only if the denominator has a real zero that is not canceled by the numerator.

If all such factors cancel, the graph has a removable discontinuity instead of a vertical asymptote.

**Final Answer:** False.

#### Applied Optimization Problems

### Exercise 311

Critical points are candidates for extrema, but not every critical point is a maximum or minimum. To determine the nature of a critical point, we must examine the sign of the derivative on both sides.

If $f'(x)$ changes from positive to negative, the function changes from increasing to decreasing, giving a local maximum. If $f'(x)$ changes from negative to positive, the function changes from decreasing to increasing, giving a local minimum. If the sign does not change, the critical point is neither.

**Conclusion:** Checking the sign of $f'(x)$ around critical points is necessary to correctly classify extrema.

### Exercise 312

Endpoints must be checked because extrema on a closed interval can occur at the boundary even if the derivative does not vanish there.

The Mean Value Theorem and derivative tests only apply to interior points. A function may be increasing or decreasing throughout the interval, making the maximum or minimum occur at an endpoint.

**Conclusion:** Ignoring endpoints can cause absolute extrema to be missed.

### Exercise 313

**False.**

A continuous nonlinear function defined on an unbounded interval need not attain a maximum. For example,

$$f(x) = x^2$$

is continuous and nonlinear, but increases without bound and has no maximum.

**Conclusion:** Continuity alone does not guarantee a maximum.

### Exercise 314

**True.**

This follows from the Extreme Value Theorem. A continuous function on a closed, finite interval must attain both an absolute maximum and an absolute minimum.

**Conclusion:** Closed and bounded domains ensure the existence of extrema.

### Exercise 315

Let the dimensions of the box be width $w$, length $l$, and height $h$. The constraint is

$$l + w + h = 62.$$

With fixed height $h$, we have

$$l + w = 62 - h.$$

The volume is

$$V = h\,lw.$$

The maximum product $lw$ occurs when

$$l = w = \frac{62 - h}{2}.$$

Thus,

$$V(h) = h\left(31 - \frac{h}{2}\right)^2.$$

Differentiating,

$$V'(h) = \left(31 - \frac{h}{2}\right)(31 - \frac{3h}{2}).$$

The maximum occurs at

$$h = \frac{62}{3}.$$

**Final Answer:** The largest volume occurs when $h = \frac{62}{3}$ inches.

## Exercise 316

Let $x$ be the side length of the square cut from each corner. The resulting box has dimensions

$$(4 - 2x) \times (2 - 2x) \times x.$$

The volume is

$$V(x) = x(4 - 2x)(2 - 2x).$$

Differentiating and solving $V'(x) = 0$ gives

$$x = \frac{2}{3}.$$

The resulting dimensions are

$$\frac{8}{3} \times \frac{2}{3} \times \frac{2}{3}.$$

**Final Answer:** The box has maximum volume when $x = \frac{2}{3}$ m.

## Exercise 317

Let $n$ be a positive integer. The function to minimize is

$$f(n) = n + \frac{1}{n}.$$

Treating $n$ as continuous,

$$f'(n) = 1 - \frac{1}{n^2}.$$

Setting $f'(n) = 0$ gives $n = 1$. Checking integers shows this is the minimum.

**Final Answer:** The minimum occurs at $n = 1$.

## Exercise 318

Let the integers be $x$ and $10 - x$. The sum of squares is

$$S(x) = x^2 + (10 - x)^2.$$

Simplifying,

$$S(x) = 2x^2 - 20x + 100.$$

This quadratic has a minimum at

$$x = 5.$$

The maximum occurs at the endpoints $x = 0$ or $x = 10$.

**Final Answer:** Minimum when the integers are 5 and 5; Maximum when the integers are 0 and 10.

## Exercise 319

Let the rectangle have length $x$ and width $y$. The fencing constraint is

$$2x + 2y = 400 \quad \Rightarrow \quad y = 200 - x.$$

The area is

$$A(x) = x(200 - x).$$

Differentiating,

$$A'(x) = 200 - 2x.$$

Setting $A'(x) = 0$ gives $x = 100$, so $y = 100$.

**Final Answer:** The maximum area occurs for a 100 ft $\times$ 100 ft square.

## Exercise 311

Checking the sign of the derivative around a critical point tells us how the function behaves near that point. If $f'(x)$ changes from positive to negative, the function changes from increasing to decreasing, indicating a local maximum. If $f'(x)$ changes from negative to positive, the function changes from decreasing to increasing, indicating a local minimum. Without checking the sign of $f'(x)$, a critical point could also be an inflection point and not an extremum.

**Final Answer:** The sign of the derivative determines whether a critical point is a maximum, minimum, or neither.

## Exercise 312

Endpoints must be checked because absolute extrema on a closed interval may occur at the boundaries. The derivative test only identifies interior critical points and gives no information about endpoint behavior. Since endpoints have only one-sided neighborhoods, they can still produce absolute maximum or minimum values.

**Final Answer:** Endpoints are possible locations of absolute extrema and must always be checked.

## Exercise 313

False. A continuous nonlinear function on an unbounded domain may increase without bound and never attain a maximum. For example, $f(x) = x^3$ is continuous and nonlinear but has no maximum value on $(-\infty, \infty)$.

**Final Answer:** False.

## Exercise 314

True. By the Extreme Value Theorem, any continuous function on a closed and bounded interval must attain both a maximum and a minimum value.

**Final Answer:** True.

## Exercise 315

Let the width be $w$ and the length be $l$. The constraint is

$$l + w + h = 62.$$

Solving for $l = w = 31 - \frac{h}{2}$, the volume is

$$V = h\left(31 - \frac{h}{2}\right)^2.$$

Differentiating,

$$V'(h) = \left(31 - \frac{h}{2}\right)\left(31 - \frac{3h}{2}\right).$$

Setting $V'(h) = 0$ gives $h = \frac{62}{3}$.

**Final Answer:** The maximum volume occurs when $h = \dfrac{62}{3}$ inches.

### Exercise 316

Let $x$ be the side length of each cut square. Then the box dimensions are

$$(4-2x)(2-2x)x.$$

The volume is

$$V(x) = x(4-2x)(2-2x).$$

Differentiating and solving $V'(x) = 0$ gives $x = \frac{2}{3}$.

**Final Answer:** The box with largest volume has cut size $x = \frac{2}{3}$ m.

### Exercise 317

Let $n$ be a positive integer. The function is

$$f(n) = n + \frac{1}{n}.$$

Treating $n$ as continuous,

$$f'(n) = 1 - \frac{1}{n^2}.$$

Setting $f'(n) = 0$ gives $n = 1$.

**Final Answer:** The minimizing integer is $n = 1$.

### Exercise 318

Let the integers be $x$ and $10 - x$. The sum of squares is

$$S(x) = x^2 + (10-x)^2.$$

Differentiating,

$$S'(x) = 4x - 20.$$

Setting $S'(x) = 0$ gives $x = 5$.

**Final Answer:** Minimum occurs at 5 and 5; maximum occurs at 1 and 9.

### Exercise 319

Let the rectangle have width $x$ and length $y$. The constraint is

$$2x + 2y = 400.$$

Then

$$A = xy = x(200 - x).$$

Differentiating,

$$A'(x) = 200 - 2x.$$

Setting $A'(x) = 0$ gives $x = 100$.

**Final Answer:** The pen should be a 100 ft $\times$ 100 ft square.

### Exercise 320

Only three sides require fencing:

$$2x + y = 800.$$

The area is

$$A = xy = x(800 - 2x).$$

Differentiating,

$$A'(x) = 800 - 4x.$$

Setting $A'(x) = 0$ gives $x = 200$.

**Final Answer:** The pen should be 200 ft $\times$ 400 ft.

### Exercise 321

Let the rectangle have dimensions $x$ and $y$. The area is fixed:

$$xy = 1600.$$

The perimeter is

$$P = 2x + 2y = 2x + \frac{3200}{x}.$$

Differentiating,

$$P'(x) = 2 - \frac{3200}{x^2}.$$

Setting $P'(x) = 0$ gives $x = 40$.

**Final Answer:** The rectangle should be 40 ft $\times$ 40 ft.

### Exercise 322

Let the anchor point be $x$ feet from the shorter pole. The total wire length is

$$L = \sqrt{x^2 + 10^2} + \sqrt{(30 - x)^2 + 20^2}.$$

Differentiate $L$ with respect to $x$ and set $L'(x) = 0$. Solving yields $x = 10$.

**Final Answer:** The wire should be anchored 10 ft from the shorter pole.

### Exercise 323

Let $L$ be the ladder length. The constraint is

$$L = \left(8^{2/3} + 6^{2/3}\right)^{3/2}.$$

**Final Answer:** The longest item that can be carried has length

$$\left(8^{2/3} + 6^{2/3}\right)^{3/2}.$$

### Exercise 324

Let

$$f(x) = (x - 70)^2 + (x - 80)^2 + (x - 120)^2.$$

Differentiating,

$$f'(x) = 6x - 540.$$

Setting $f'(x) = 0$ gives $x = 90$.

**Final Answer:** The minimizing value is $x = 90$.

### Exercise 325

This function is a sum of squared terms, so it is a quadratic function opening upward. Therefore, its minimum occurs where the derivative is zero.

Differentiate:

$$f'(x) = 2(x - 70) + 2(x - 80) + \frac{1}{2} \cdot 2(x - 120).$$

Simplify:

$$f'(x) = 2x - 140 + 2x - 160 + (x - 120) = 5x - 420.$$

Set the derivative equal to zero:

$$5x - 420 = 0.$$

Solve:

$$x = 84.$$

Since the second derivative is positive,

$$f''(x) = 5 > 0,$$

this critical point corresponds to a minimum.

**Final Answer:**

$$\boxed{84}.$$

## Exercise 326

Let $x$ be the distance (in miles) you run west before swimming.

Running distance: $x$ Swimming distance: $\sqrt{(4-x)^2 + 1^2}$

Running speed is 6 mph and swimming speed is 3 mph, so the total time is

$$T(x) = \frac{x}{6} + \frac{\sqrt{(4-x)^2 + 1}}{3}.$$

Differentiate:

$$T'(x) = \frac{1}{6} - \frac{4-x}{3\sqrt{(4-x)^2 + 1}}.$$

Set $T'(x) = 0$:

$$\frac{1}{6} = \frac{4-x}{3\sqrt{(4-x)^2 + 1}}.$$

Multiply both sides:

$$\sqrt{(4-x)^2 + 1} = 2(4-x).$$

Square both sides:

$$(4-x)^2 + 1 = 4(4-x)^2.$$

Solve:

$$1 = 3(4-x)^2 \quad \Rightarrow \quad 4 - x = \frac{1}{\sqrt{3}}.$$

Thus,

$$x = 4 - \frac{1}{\sqrt{3}}.$$

**Final Answer:** You should run $4 - \frac{1}{\sqrt{3}}$ miles west.

**Exercise 327**

The pool has radius 20 m. The lifeguard swims from $A$ to $B$ at angle $\theta$, then runs along the edge from $B$ to $C$.

Swimming distance:

$$AB = 40\cos\theta.$$

Running arc length:

$$BC = 20(\pi - 2\theta).$$

Swimming speed is $v$, running speed is $3v$. Total time is

$$T(\theta) = \frac{40\cos\theta}{v} + \frac{20(\pi - 2\theta)}{3v}.$$

**Final Answer:**

$$T(\theta) = \frac{40\cos\theta}{v} + \frac{20(\pi - 2\theta)}{3v}.$$

**Exercise 328**

Differentiate $T(\theta)$:

$$T'(\theta) = -\frac{40\sin\theta}{v} - \frac{40}{3v}.$$

Set equal to zero:

$$-\frac{40\sin\theta}{v} = \frac{40}{3v}.$$

Simplify:

$$\sin\theta = -\frac{1}{3}.$$

Since $0 \le \theta \le \pi$,

$$\theta = \arcsin\left(\frac{1}{3}\right).$$

**Final Answer:** The lifeguard should swim at angle

$$\theta = \arcsin\left(\frac{1}{3}\right).$$

**Exercise 329**

Fuel consumption per mile is

$$g(v) = av + \frac{b}{v}.$$

Differentiate:

$$g'(v) = a - \frac{b}{v^2}.$$

Set $g'(v) = 0$:

$$a = \frac{b}{v^2}.$$

Solve:

$$v = \sqrt{\frac{b}{a}}.$$

**Final Answer:** Fuel consumption is minimized at

$$v = \sqrt{\frac{b}{a}}.$$

## Exercise 330

Fuel efficiency is

$$m(v) = \frac{120 - 2v}{5} \quad \text{(mi/gal)}.$$

Gallons per mile:

$$\frac{1}{m(v)} = \frac{5}{120 - 2v}.$$

Fuel cost per mile:

$$\frac{5}{120 - 2v} \cdot 3.5 = \frac{17.5}{120 - 2v}.$$

Labor cost per mile:

$$\frac{15}{v}.$$

Total cost per mile:

$$C(v) = \frac{17.5}{120 - 2v} + \frac{15}{v}.$$

**Final Answer:**

$$C(v) = \frac{17.5}{120 - 2v} + \frac{15}{v}.$$

## Exercise 331

Differentiate:

$$C'(v) = \frac{35}{(120 - 2v)^2} - \frac{15}{v^2}.$$

Set equal to zero:

$$\frac{35}{(120-2v)^2} = \frac{15}{v^2}.$$

Cross-multiply:

$$35v^2 = 15(120-2v)^2.$$

Solve:

$$7v^2 = 3(120-2v)^2.$$

Taking square roots:

$$\sqrt{7}v = \sqrt{3}(120-2v).$$

Solve for $v$:

$$v = \frac{120\sqrt{3}}{2\sqrt{3}+\sqrt{7}}.$$

**Final Answer:** The cheapest driving speed is

$$v = \frac{120\sqrt{3}}{2\sqrt{3}+\sqrt{7}} \text{ mph.}$$

## Exercise 332

Revenue is $R(x) = ax$ and cost is $C(x) = b + cx + dx^2$. The profit function is

$$P(x) = R(x) - C(x) = ax - (b + cx + dx^2) = -dx^2 + (a-c)x - b.$$

This quadratic opens downward, so the maximum profit occurs at its vertex. Differentiate:

$$P'(x) = -2dx + (a-c).$$

Setting $P'(x) = 0$ gives

$$x = \frac{a-c}{2d}.$$

**Final Answer:**

$$P(x) = -dx^2 + (a-c)x - b, \qquad x = \frac{a-c}{2d}.$$

## Exercise 333

The profit function is

$$P(x) = 10x - (2x + x^2) = -x^2 + 8x.$$

Differentiate:

$$P'(x) = -2x + 8.$$

Setting the derivative equal to zero gives $x = 4$.

**Final Answer:**

$$x = 4.$$

### Exercise 334

The profit function is

$$P(x) = 15x - (60 + 3x + \tfrac{1}{2}x^2) = -\tfrac{1}{2}x^2 + 12x - 60.$$

Differentiate:

$$P'(x) = -x + 12.$$

Setting $P'(x) = 0$ gives $x = 12$.

**Final Answer:**

$$x = 12.$$

### Exercise 335

Let $x$ be the side length of the square. The remaining wire forms a circle with

$$2\pi r = 4 - x \quad \Rightarrow \quad r = \frac{4-x}{2\pi}.$$

Total area:

$$A(x) = x^2 + \pi r^2 = x^2 + \frac{(4-x)^2}{4\pi}.$$

Differentiate:

$$A'(x) = 2x - \frac{4-x}{2\pi}.$$

Setting $A'(x) = 0$ yields

$$x = \frac{4}{1+4\pi}.$$

**Final Answer:**

$$x = \frac{4}{1+4\pi}.$$

### Exercise 336

Evaluating the area at the endpoints shows the minimum occurs when all wire forms a circle.

**Final Answer:**

$$x = 0.$$

## Exercise 337

Let $y = 10 - x$. Then

$$f(x) = x(10 - x) = 10x - x^2.$$

Differentiate:

$$f'(x) = 10 - 2x.$$

Setting $f'(x) = 0$ gives $x = 5$.

**Final Answer:**

$$x = y = 5.$$

## Exercise 338

With $y = 10 - x$,

$$f(x) = x^2(10 - x)^2.$$

Differentiation gives critical points at $x = 5$.

**Final Answer:**

$$x = y = 5.$$

## Exercise 339

Let $y = 10 - x$. Then

$$f(x) = 10 - x - \frac{1}{x}.$$

Differentiate:

$$f'(x) = -1 + \frac{1}{x^2}.$$

Setting equal to zero gives $x = 1$.

**Final Answer:**

$$x = 1, \quad y = 9.$$

## Exercise 340

With $y = 10 - x$,

$$f(x) = x^2 - (10 - x) = x^2 + x - 10.$$

Differentiate:

$$f'(x) = 2x + 1.$$

Setting $f'(x) = 0$ gives $x = -\frac{1}{2}$.

**Final Answer:**

$$x = -\tfrac{1}{2}, \quad y = \tfrac{21}{2}.$$

## Exercise 341

Let the cylinder have radius $r$ and height $h$. From geometry,

$$r^2 + h^2 = 1.$$

Volume:

$$V = \pi r^2 h = \pi(1 - h^2)h.$$

Differentiate:

$$V'(h) = \pi(1 - 3h^2).$$

The critical point is $h = \frac{1}{\sqrt{3}}$.

**Final Answer:**

$$V_{\max} = \frac{2\pi}{3\sqrt{3}}.$$

## Exercise 342

The cone volume is

$$V = \frac{1}{3}\pi r^2 h,$$

with constraint $r^2 + h^2 = 1$. Maximization gives

$$h = \frac{1}{\sqrt{3}}.$$

**Final Answer:**

$$V_{\max} = \frac{2\pi}{9\sqrt{3}}.$$

## Exercise 343

From the constraint

$$\frac{x}{4} + \frac{y}{6} = 1,$$

we obtain $y = 6(1 - \frac{x}{4})$. Area:

$$A(x) = 6x - \frac{3}{2}x^2.$$

Differentiation gives $x = 2$.

**Final Answer:**

$$A_{\max} = 6.$$

### Exercise 344

Using similar triangles,

$$r = \frac{R}{h}y.$$

Volume:

$$V = \frac{1}{3}\pi\frac{R^2}{h^2}y^3.$$

Maximization gives $y = \frac{h}{\sqrt{3}}$.

**Final Answer:**

$$V_{\max} = \frac{\pi R^2 h}{3\sqrt{3}}.$$

### Exercise 345

Surface area:

$$S = 2\pi r^2 + 2\pi rh.$$

Using the volume constraint $\pi r^2 h = 16\pi$,

$$h = \frac{16}{r^2}.$$

Minimization yields $r = 2$ and $h = 4$.

**Final Answer:**

$$r = 2, \quad h = 4.$$

### Exercise 346

Optimizing a cone with surface area $S = 4\pi$ shows the maximum volume occurs when

$$h = \sqrt{2}\,r.$$

**Final Answer:**

$$h = \sqrt{2}\,r.$$

## Exercise 347

A point on the line $y = 5 - 2x$ has coordinates $(x,\, 5 - 2x)$. The squared distance from the origin is

$$D^2 = x^2 + (5 - 2x)^2.$$

Expand:

$$D^2 = x^2 + 25 - 20x + 4x^2 = 5x^2 - 20x + 25.$$

Differentiate:

$$\frac{d}{dx} D^2 = 10x - 20.$$

Setting this equal to zero gives $x = 2$. Then $y = 1$.

**Final Answer:**

$$(2, 1).$$

## Exercise 348

A point on the line $y = 5 - 2x$ is $(x, 5 - 2x)$. The squared distance to $(1, 1)$ is

$$D^2 = (x - 1)^2 + (5 - 2x - 1)^2 = (x - 1)^2 + (4 - 2x)^2.$$

Expand:

$$D^2 = x^2 - 2x + 1 + 4x^2 - 16x + 16 = 5x^2 - 18x + 17.$$

Differentiate:

$$\frac{d}{dx} D^2 = 10x - 18.$$

Setting equal to zero gives $x = \frac{9}{5}$. Then

$$y = 5 - 2\left(\tfrac{9}{5}\right) = \tfrac{7}{5}.$$

**Final Answer:**

$$\left(\tfrac{9}{5},\, \tfrac{7}{5}\right).$$

## Exercise 349

A point on the parabola is $(x, x^2)$. The squared distance to $(2, 0)$ is

$$D^2 = (x - 2)^2 + (x^2)^2 = (x - 2)^2 + x^4.$$

Expand:

$$D^2 = x^4 + x^2 - 4x + 4.$$

Differentiate:
$$\frac{d}{dx}D^2 = 4x^3 + 2x - 4.$$
Solve $4x^3 + 2x - 4 = 0$, which gives $x = 1$. Then $y = 1$.

**Final Answer:**
$$(1, 1).$$

## Exercise 350

A point on the parabola is $(x, x^2)$. The squared distance to $(0, 3)$ is
$$D^2 = x^2 + (x^2 - 3)^2.$$
Expand:
$$D^2 = x^2 + x^4 - 6x^2 + 9 = x^4 - 5x^2 + 9.$$
Differentiate:
$$\frac{d}{dx}D^2 = 4x^3 - 10x.$$
Setting this equal to zero gives
$$x(4x^2 - 10) = 0.$$
Thus $x = 0$ or $x = \pm\sqrt{\frac{5}{2}}$. Evaluating shows the minimum occurs at $x = 0$. Then $y = 0$.

**Final Answer:**
$$(0, 0).$$

## Exercise 351

Let the rectangle have width $2r$ and height $h$. The total framing length is the perimeter of the rectangle plus the semicircle:
$$2h + 2r + \pi r = 20.$$
Solve for $h$:
$$h = \frac{20 - (2 + \pi)r}{2}.$$
The area is
$$A(r) = 2rh + \frac{1}{2}\pi r^2 = r\big(20 - (2 + \pi)r\big) + \frac{1}{2}\pi r^2.$$
This defines the objective function to be maximized.

**Final Answer:**
$$A(r) = 20r - (2 + \pi)r^2 + \tfrac{1}{2}\pi r^2.$$

## Exercise 352

Let $x$ be the number of additional plants. Then the total number of plants is $20 + x$, and the yield per plant is $30 - x$. Total output is

$$P(x) = (20 + x)(30 - x).$$

Expand:

$$P(x) = 600 + 10x - x^2.$$

Differentiate:

$$P'(x) = 10 - 2x.$$

Setting $P'(x) = 0$ gives $x = 5$.

**Final Answer:**

$$5 \text{ additional plants.}$$

## Exercise 353

Let $x$ be the length of each side of the square base and $h$ the height. The volume constraint is

$$x^2 h = 4 \quad \Rightarrow \quad h = \frac{4}{x^2}.$$

**Cost:** The bottom area is $x^2$ and costs \$5/ft$^2$, so bottom cost is

$$5x^2.$$

The four sides each have area $xh$, and the side material costs \$2/ft$^2$. Thus side cost is

$$4(xh)(2) = 8xh.$$

Substitute $h = \frac{4}{x^2}$:

$$C(x) = 5x^2 + 8x\left(\frac{4}{x^2}\right) = 5x^2 + \frac{32}{x}.$$

Differentiate:

$$C'(x) = 10x - \frac{32}{x^2}.$$

Set equal to zero:

$$10x = \frac{32}{x^2} \quad \Rightarrow \quad 10x^3 = 32 \quad \Rightarrow \quad x = \left(\frac{16}{5}\right)^{1/3}.$$

Then

$$h = \frac{4}{x^2}.$$

**Final Answer:**

$$x = \left(\frac{16}{5}\right)^{1/3}, \qquad h = \frac{4}{x^2}.$$

## Exercise 354

Let each pen have width $x$ and height $y$. Since there are five identical pens,

$$5xy = 1000 \quad \Rightarrow \quad y = \frac{200}{x}.$$

**Fencing:** There are two long horizontal fences of length $5x$, and six vertical fences of length $y$. Thus total fencing is

$$F = 10x + 6y.$$

Substitute $y = \frac{200}{x}$:

$$F(x) = 10x + \frac{1200}{x}.$$

Differentiate:

$$F'(x) = 10 - \frac{1200}{x^2}.$$

Set equal to zero:

$$10 = \frac{1200}{x^2} \quad \Rightarrow \quad x^2 = 120 \quad \Rightarrow \quad x = \sqrt{120}.$$

Then

$$y = \frac{200}{\sqrt{120}}.$$

**Final Answer:**

$$x = \sqrt{120}\text{ m}, \qquad y = \frac{200}{\sqrt{120}}\text{ m}.$$

## Exercise 355

Let $x$ be the number of \$25 rent increases. Then rent per unit is

$$R = 800 + 25x,$$

and the number of occupied units is

$$50 - x.$$

**Revenue:**

$$\text{Revenue} = (800 + 25x)(50 - x).$$

**Maintenance cost:** Each occupied unit costs \$50 per month, so

$$\text{Cost} = 50(50 - x).$$

**Profit:**

$$P(x) = (800 + 25x)(50 - x) - 50(50 - x).$$

Simplify:

$$P(x) = (750 + 25x)(50 - x) = 37500 + 500x - 25x^2.$$

Differentiate:

$$P'(x) = 500 - 50x.$$

Set equal to zero:

$$x = 10.$$

The rent is

$$800 + 25(10) = 1050.$$

**Final Answer:**

$$\$1050 \text{ per month.}$$

**L'Hôpital's Rule**

### Exercise 356

As $x \to \infty$, the exponential function $e^x$ grows much faster than the linear function $x$. Therefore the numerator dominates the denominator and the fraction increases without bound.

$$\lim_{x\to\infty} \frac{e^x}{x} = \infty$$

**Final Answer:** $+\infty$

### Exercise 357

For any fixed positive integer $k$, the exponential function $e^x$ grows faster than any power $x^k$. Thus the denominator cannot keep up with the numerator as $x \to \infty$.

$$\lim_{x\to\infty} \frac{e^x}{x^k} = \infty$$

**Final Answer:** $+\infty$

## Exercise 358

As $x \to \infty$, polynomial functions grow faster than logarithmic functions. Hence $x^k$ dominates $\ln x$ for any $k > 0$.

$$\lim_{x\to\infty} \frac{\ln x}{x^k} = 0$$

**Final Answer:** 0

## Exercise 359

Factor the denominator:

$$x^2 - a^2 = (x-a)(x+a).$$

Cancel the common factor $x - a$ (since $x \neq a$ during cancellation):

$$\frac{x-a}{x^2-a^2} = \frac{1}{x+a}.$$

Now take the limit:

$$\lim_{x\to a} \frac{1}{x+a} = \frac{1}{2a}.$$

**Final Answer:** $\frac{1}{2a}$

## Exercise 360

Factor the denominator:

$$x^3 - a^3 = (x-a)(x^2+ax+a^2).$$

Cancel $x - a$:

$$\frac{x-a}{x^3-a^3} = \frac{1}{x^2+ax+a^2}.$$

Now substitute $x = a$:

$$\lim_{x\to a} \frac{1}{x^2+ax+a^2} = \frac{1}{3a^2}.$$

**Final Answer:** $\frac{1}{3a^2}$

**Exercise 361**

Use the factorization

$$x^n - a^n = (x-a)\left(x^{n-1} + x^{n-2}a + \cdots + a^{n-1}\right).$$

Cancel $x - a$:

$$\frac{x-a}{x^n - a^n} = \frac{1}{x^{n-1} + x^{n-2}a + \cdots + a^{n-1}}.$$

Substitute $x = a$. The denominator becomes $na^{n-1}$.

$$\lim_{x \to a} \frac{x-a}{x^n - a^n} = \frac{1}{na^{n-1}}.$$

**Final Answer:** $\frac{1}{na^{n-1}}$

**Exercise 362**

As $x \to 0^+$, we have $x^2 \to 0$ and $\ln x \to -\infty$, so the expression is of the indeterminate form $0 \cdot (-\infty)$. L'Hôpital's rule cannot be applied directly.

Rewrite the expression as

$$x^2 \ln x = \frac{\ln x}{1/x^2}.$$

Now the limit is of the form $\frac{-\infty}{\infty}$, so L'Hôpital's rule applies.

Differentiate numerator and denominator:

$$\lim_{x \to 0^+} \frac{1/x}{-2/x^3} = \lim_{x \to 0^+} \left(-\frac{x^2}{2}\right) = 0.$$

**Final Answer:** 0

**Exercise 363**

As $x \to \infty$, the expression $x^{1/x}$ is of the indeterminate form $\infty^0$. L'Hôpital's rule cannot be applied directly.

Take logarithms:

$$y = x^{1/x} \quad \Rightarrow \quad \ln y = \frac{\ln x}{x}.$$

Now the limit is $\frac{\infty}{\infty}$, so L'Hôpital's rule applies.

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0.$$

Thus $\ln y \to 0$, which implies $y \to e^0 = 1$.

**Final Answer:** 1

### Exercise 364

As $x \to 0^+$, the expression $x^{2/x}$ is of the form $0^\infty$. L'Hôpital's rule cannot be applied directly.

Take logarithms:

$$y = x^{2/x} \quad \Rightarrow \quad \ln y = \frac{2\ln x}{x}.$$

This is now $\frac{-\infty}{0^+}$, which tends to $-\infty$.

Thus $\ln y \to -\infty$, implying $y \to 0$.

**Final Answer:** 0

### Exercise 365

As $x \to 0^+$, the expression $\frac{x^2}{1/x}$ is of the form $\frac{0}{\infty}$, which is not indeterminate. Therefore L'Hôpital's rule is not needed.

Simplify:

$$\frac{x^2}{1/x} = x^3.$$

Now take the limit:

$$\lim_{x\to 0^+} x^3 = 0.$$

**Final Answer:** 0

### Exercise 366

As $x \to \infty$, $\frac{e^x}{x}$ is of the indeterminate form $\frac{\infty}{\infty}$. Thus L'Hôpital's rule applies.

Differentiate numerator and denominator:

$$\lim_{x\to\infty} \frac{e^x}{1} = \infty.$$

**Final Answer:** $+\infty$

**Exercise 367**

Substituting $x = 3$ gives the indeterminate form $\frac{0}{0}$, so L'Hôpital's rule applies.

Differentiate numerator and denominator:

$$\lim_{x \to 3} \frac{2x}{1} = 6.$$

**Final Answer:** 6

**Exercise 368**

Substituting $x = 3$ gives $\frac{0}{6}$, which is not indeterminate. Thus L'Hôpital's rule is not needed.

$$\lim_{x \to 3} \frac{x^2 - 9}{x + 3} = 0.$$

**Final Answer:** 0

**Exercise 369**

Substituting $x = 0$ gives $\frac{0}{0}$, so L'Hôpital's rule applies.

Differentiate numerator and denominator:

$$\lim_{x \to 0} \frac{-2(1 + x)^{-3}}{1} = -2.$$

**Final Answer:** $-2$

**Exercise 370**

As $x \to \frac{\pi}{2}$, both numerator and denominator approach 0, giving $\frac{0}{0}$. L'Hôpital's rule applies.

Differentiate numerator and denominator:

$$\lim_{x \to \pi/2} \frac{-\sin x}{-1} = \sin\left(\frac{\pi}{2}\right) = 1.$$

**Final Answer:** 1

## Exercise 371

As $x \to \pi$, both numerator and denominator approach 0, giving $\frac{0}{0}$. L'Hôpital's rule applies.

Differentiate numerator and denominator:

$$\lim_{x \to \pi} \frac{1}{\cos x} = \frac{1}{-1} = -1.$$

**Final Answer:** $-1$

## Exercise 372

As $x \to 1$, both numerator and denominator approach 0, giving the indeterminate form $\frac{0}{0}$. L'Hôpital's rule applies.

Differentiate numerator and denominator:

$$\lim_{x \to 1} \frac{1}{\cos x} = \frac{1}{\cos 1}.$$

**Final Answer:** $\dfrac{1}{\cos 1}$

## Exercise 373

As $x \to 0$, the expression is of the form $\frac{0}{0}$. Apply L'Hôpital's rule.

Differentiate numerator and denominator:

$$\lim_{x \to 0} \frac{n(1+x)^{n-1}}{1} = n.$$

**Final Answer:** $n$

## Exercise 374

As $x \to 0$, the expression is $\frac{0}{0}$. Apply L'Hôpital's rule twice.

First derivative:

$$\lim_{x \to 0} \frac{n(1+x)^{n-1} - n}{2x}.$$

Still $\frac{0}{0}$. Differentiate again:

$$\lim_{x \to 0} \frac{n(n-1)(1+x)^{n-2}}{2} = \frac{n(n-1)}{2}.$$

**Final Answer:** $\dfrac{n(n-1)}{2}$

### Exercise 375

As $x \to 0$, numerator and denominator both approach 0. Apply L'Hôpital's rule.

First derivative:

$$\lim_{x\to 0} \frac{\cos x - \sec^2 x}{3x^2}.$$

Still $\frac{0}{0}$. Differentiate again:

$$\lim_{x\to 0} \frac{-\sin x - 2\sec^2 x \tan x}{6x}.$$

Still $\frac{0}{0}$. Differentiate a third time:

$$\lim_{x\to 0} \frac{-\cos x - 2(2\sec^2 x \tan^2 x + \sec^4 x)}{6} = -\frac{1}{2}.$$

**Final Answer:** $-\dfrac{1}{2}$

### Exercise 376

As $x \to 0$, the expression is $\frac{0}{0}$. Multiply numerator and denominator by the conjugate:

$$\frac{(1+x)-(1-x)}{x(\sqrt{1+x}+\sqrt{1-x})} = \frac{2x}{x(\sqrt{1+x}+\sqrt{1-x})}.$$

Cancel $x$:

$$\lim_{x\to 0} \frac{2}{\sqrt{1+x}+\sqrt{1-x}} = 1.$$

**Final Answer:** 1

### Exercise 377

As $x \to 0$, the expression is $\frac{0}{0}$. Apply L'Hôpital's rule.

First derivative:

$$\lim_{x\to 0} \frac{e^x - 1}{2x}.$$

Still $\frac{0}{0}$. Differentiate again:

$$\lim_{x\to 0} \frac{e^x}{2} = \frac{1}{2}.$$

**Final Answer:** $\frac{1}{2}$

### Exercise 378

As $x \to 0^+$, $\tan x \to 0$ and $\sqrt{x} \to 0$, giving $\frac{0}{0}$. Apply L'Hôpital's rule.

Differentiate numerator and denominator:

$$\lim_{x\to 0^+} \frac{\sec^2 x}{\frac{1}{2\sqrt{x}}} = \lim_{x\to 0^+} 2\sqrt{x}\sec^2 x = 0.$$

**Final Answer:** 0

### Exercise 379

As $x \to 1$, the expression is $\frac{0}{0}$. Apply L'Hôpital's rule.

Differentiate numerator and denominator:

$$\lim_{x\to 1} \frac{1}{1/x} = 1.$$

**Final Answer:** 1

### Exercise 380

As $x \to 0$, the expression is of the form $1^\infty$. Take logarithms.

Let $y = (x+1)^{1/x}$. Then

$$\ln y = \frac{\ln(1+x)}{x}.$$

Apply L'Hôpital's rule:

$$\lim_{x\to 0} \frac{1/(1+x)}{1} = 1.$$

Thus $y \to e$.

**Final Answer:** $e$

### Exercise 381

As $x \to 1$, numerator and denominator approach 0. Apply L'Hôpital's rule.

Differentiate numerator and denominator:

$$\lim_{x\to 1} \frac{\frac{1}{2\sqrt{x}} - \frac{1}{3x^{2/3}}}{1} = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$

**Final Answer:** $\frac{1}{6}$

### Exercise 382

As $x \to 0^+$, the expression is $0^0$. Take logarithms.

Let $y = x^{2x}$. Then

$$\ln y = 2x \ln x.$$

As $x \to 0^+$, this is $0 \cdot (-\infty)$. Rewrite:

$$\ln y = \frac{\ln x}{1/(2x)}.$$

Apply L'Hôpital's rule:

$$\lim_{x\to 0^+} \frac{1/x}{-1/(2x^2)} = 0.$$

Thus $y \to e^0 = 1$.

**Final Answer:** 1

### Exercise 383

As $x \to \infty$, $\sin(1/x) \sim \frac{1}{x}$. Thus

$$x \sin\left(\frac{1}{x}\right) \to 1.$$

**Final Answer:** 1

### Exercise 384

As $x \to 0$, the expression is $\frac{0}{0}$. Apply L'Hôpital's rule.

Differentiate numerator and denominator:

$$\lim_{x\to 0} \frac{\cos x - 1}{2x}.$$

Still $\frac{0}{0}$. Differentiate again:

$$\lim_{x\to 0} \frac{-\sin x}{2} = 0.$$

**Final Answer:** 0

## Exercise 385

As $x \to 0^+$, $x \ln(x^4) = 4x \ln x$, which is $0 \cdot (-\infty)$. Rewrite:

$$\frac{\ln x}{1/x}.$$

Apply L'Hôpital's rule:

$$\lim_{x \to 0^+} \frac{1/x}{-1/x^2} = 0.$$

**Final Answer:** 0

## Exercise 386

As $x \to \infty$, $e^x$ dominates $x$. Thus $x - e^x \to -\infty$.

**Final Answer:** $-\infty$

## Exercise 387

As $x \to \infty$, polynomial growth is dominated by exponential decay. Thus

$$x^2 e^{-x} \to 0.$$

**Final Answer:** 0

## Exercise 388

As $x \to 0$, the expression is $\frac{0}{0}$. Apply L'Hôpital's rule.

Differentiate numerator and denominator:

$$\lim_{x \to 0} \frac{3^x \ln 3 - 2^x \ln 2}{1} = \ln 3 - \ln 2 = \ln \left( \frac{3}{2} \right).$$

**Final Answer:** $\ln\left(\frac{3}{2}\right)$

### Exercise 389

As $x \to 0^+$, the expression becomes $\frac{\infty}{-\infty}$. Divide numerator and denominator by $1/x$:

$$\frac{x+1}{x-1}.$$

Now take the limit:

$$\lim_{x\to 0^+} \frac{1}{-1} = -1.$$

**Final Answer:** $-1$

### Exercise 390

As $x \to \frac{\pi}{4}$, the expression is $0 \cdot 1$, not indeterminate. Rewrite:

$$(1 - \tan x)\cot x = \frac{1 - \tan x}{\tan x}.$$

This is $\frac{0}{1}$.

**Final Answer:** 0

### Exercise 391

As $x \to \infty$, the expression has the indeterminate form $\infty \cdot 1^\infty$. Rewrite using exponentials:

$$xe^{1/x} = x \cdot \exp\left(\frac{1}{x}\right).$$

Since $e^{1/x} \to 1$ and $x \to \infty$, the linear growth of $x$ dominates.

**Final Answer:** $+\infty$

### Exercise 392

As $x \to 0^+$, the expression has the form $0^1$. Take logarithms:

$$y = x^{1/\cos x}, \qquad \ln y = \frac{\ln x}{\cos x}.$$

As $x \to 0^+$, $\ln x \to -\infty$ and $\cos x \to 1$, so

$$\ln y \to -\infty.$$

**Final Answer:** 0

## Exercise 393

As $x \to 0^+$, the expression is of the form $0^\infty$. Take logarithms:

$$y = x^{1/x}, \qquad \ln y = \frac{\ln x}{x}.$$

Rewrite as

$$\frac{\ln x}{x} = \frac{1/x}{-1/x^2}.$$

Apply L'Hôpital's rule:

$$\lim_{x\to 0^+} \frac{1/x}{-1/x^2} = -\infty.$$

Thus $y = e^{-\infty} = 0$.

**Final Answer:** 0

## Exercise 394

As $x \to 0^-$, the base $1 - \frac{1}{x} \to +\infty$ and the exponent $x \to 0^-$, giving the form $\infty^0$. Take logarithms:

$$y = \left(1 - \frac{1}{x}\right)^x, \qquad \ln y = x \ln\left(1 - \frac{1}{x}\right).$$

As $x \to 0^-$, $\ln\left(1 - \frac{1}{x}\right) \sim \ln(-1/x)$, so

$$x \ln\left(-\frac{1}{x}\right) \to 0.$$

**Final Answer:** 1

## Exercise 395

As $x \to \infty$, the expression has the indeterminate form $1^\infty$. Take logarithms:

$$y = \left(1 - \frac{1}{x}\right)^x, \qquad \ln y = x \ln\left(1 - \frac{1}{x}\right).$$

Using the expansion $\ln(1-u) \sim -u$,

$$x \ln\left(1 - \frac{1}{x}\right) \to -1.$$

**Final Answer:** $e^{-1}$

### Exercise 396

As $x \to 0$, the expression is $\frac{0}{0}$. Apply L'Hôpital's rule:

$$\lim_{x \to 0} \frac{e^x - 1}{x} = \lim_{x \to 0} \frac{e^x}{1} = 1.$$

**Final Answer:** 1

### Exercise 397

As $x \to 0$, the function oscillates but is bounded by

$$-|x| \leq x \sin\left(\frac{1}{x}\right) \leq |x|.$$

Both bounds approach 0. By the Squeeze Theorem, the limit exists.

**Final Answer:** 0

### Exercise 398

As $x \to 1$, the expression is $\frac{0}{0}$. Apply L'Hôpital's rule:

$$\lim_{x \to 1} \frac{x - 1}{1 - \cos(\pi x)} = \lim_{x \to 1} \frac{1}{\pi \sin(\pi x)}.$$

Since $\sin(\pi) = 0$, the denominator approaches 0, so the expression diverges.

**Final Answer:** Does not exist

### Exercise 399

As $x \to 1$, the expression is $\frac{0}{0}$. Apply L'Hôpital's rule:

$$\lim_{x \to 1} \frac{e^{x-1}}{1} = 1.$$

**Final Answer:** 1

### Exercise 400

As $x \to 1$, the expression is $\frac{0}{0}$. Apply L'Hôpital's rule:

$$\lim_{x\to 1} \frac{2(x-1)}{1/x} = \lim_{x\to 1} 2x(x-1) = 0.$$

**Final Answer:** 0

### Exercise 401

As $x \to \pi$, the expression is $\frac{0}{0}$. Apply L'Hôpital's rule:

$$\lim_{x\to \pi} \frac{-\sin x}{\cos x} = \frac{0}{-1} = 0.$$

**Final Answer:** 0

### Exercise 402

As $x \to 0$, the expression is $\infty - \infty$. Rewrite:

$$\csc x - \frac{1}{x} = \frac{x - \sin x}{x \sin x}.$$

Apply L'Hôpital's rule:

$$\lim_{x\to 0} \frac{1-\cos x}{\sin x + x\cos x}.$$

Apply again:

$$\lim_{x\to 0} \frac{\sin x}{2\cos x - x\sin x} = 0.$$

**Final Answer:** 0

### Exercise 403

As $x \to 0^+$, $x^x \to 1$. Thus

$$\tan(x^x) \to \tan(1).$$

**Final Answer:** $\tan(1)$

## Exercise 404

As $x \to 0^+$, the expression is $\frac{-\infty}{0}$. Rewrite using L'Hôpital's rule:

$$\lim_{x \to 0^+} \frac{1/x}{\cos x} = +\infty.$$

**Final Answer:** $-\infty$

## Exercise 405

As $x \to 0$, the expression is $\frac{0}{0}$. Apply L'Hôpital's rule:

$$\lim_{x \to 0} \frac{e^x + e^{-x}}{1} = 2.$$

**Final Answer:** 2

### Newton's Method

## Exercise 406

Newton's method is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Here $f(x) = x^2 + 1$, so $f'(x) = 2x$.

$$x_{n+1} = x_n - \frac{x_n^2 + 1}{2x_n} = \frac{x_n^2 - 1}{2x_n}.$$

**Final Answer:**

$$x_{n+1} = \frac{x_n^2 - 1}{2x_n}.$$

## Exercise 407

$f(x) = x^3 + 2x + 1$, so $f'(x) = 3x^2 + 2$.

$$x_{n+1} = x_n - \frac{x_n^3 + 2x_n + 1}{3x_n^2 + 2}.$$

**Final Answer:**

$$x_{n+1} = x_n - \frac{x_n^3 + 2x_n + 1}{3x_n^2 + 2}.$$

**Exercise 408**

$f(x) = \sin x$, so $f'(x) = \cos x$.

$$x_{n+1} = x_n - \frac{\sin x_n}{\cos x_n} = x_n - \tan x_n.$$

**Final Answer:**

$$x_{n+1} = x_n - \tan x_n.$$

**Exercise 409**

$f(x) = e^x$, so $f'(x) = e^x$.

$$x_{n+1} = x_n - \frac{e^{x_n}}{e^{x_n}} = x_n - 1.$$

**Final Answer:**

$$x_{n+1} = x_n - 1.$$

**Exercise 410**

$f(x) = x^3 + 3xe^x$, so

$$f'(x) = 3x^2 + 3e^x + 3xe^x.$$

Thus

$$x_{n+1} = x_n - \frac{x_n^3 + 3x_n e^{x_n}}{3x_n^2 + 3e^{x_n} + 3x_n e^{x_n}}.$$

**Final Answer:**

$$x_{n+1} = x_n - \frac{x_n^3 + 3x_n e^{x_n}}{3x_n^2 + 3e^{x_n} + 3x_n e^{x_n}}.$$

**Exercise 411**

Iteration: $x_{n+1} = x_n - c(x_n^2 - 4)$.

A working choice is $c = \frac{1}{4}$, which converges to $\pm 2$. A failing choice is $c = 1$, which diverges.

**Final Answer:** Converges for $c = \frac{1}{4}$; fails for $c = 1$.

**Exercise 412**

Iteration: $x_{n+1} = x_n - c(x_n^2 - 4x_n + 3)$.

A working choice is $c = \frac{1}{4}$. A failing choice is $c = 1$.

**Final Answer:** Converges for $c = \frac{1}{4}$; fails for $c = 1$.

### Exercise 413

Newton's method corresponds to

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

**Final Answer:**

$$c = \frac{1}{f'(x_n)}.$$

### Exercise 414

$$x_{n+1} = x_n^2 - \frac{1}{2}.$$

For $x_0 = 0.6$:

$$x_1 = -0.14, \quad x_2 = -0.4804.$$

For $x_0 = 2$:

$$x_1 = 3.5, \quad x_2 = 11.75.$$

### Exercise 415

$$x_{n+1} = 2x_n(1 - x_n).$$

For $x_0 = 0.6$:

$$x_1 = 0.48, \quad x_2 = 0.4992.$$

For $x_0 = 2$:

$$x_1 = -4, \quad x_2 = -40.$$

### Exercise 416

$$x_{n+1} = \sqrt{x_n}.$$

For $x_0 = 0.6$:

$$x_1 \approx 0.775, \quad x_2 \approx 0.880.$$

For $x_0 = 2$:

$$x_1 \approx 1.414, \quad x_2 \approx 1.189.$$

**Exercise 417**

$$x_{n+1} = \frac{1}{\sqrt{x_n}}.$$

For $x_0 = 0.6$:

$$x_1 \approx 1.291, \quad x_2 \approx 0.879.$$

For $x_0 = 2$:

$$x_1 \approx 0.707, \quad x_2 \approx 1.189.$$

**Exercise 418**

$$x_{n+1} = 3x_n(1 - x_n).$$

For $x_0 = 0.6$:

$$x_1 = 0.72, \quad x_2 = 0.6048.$$

For $x_0 = 2$:

$$x_1 = -6, \quad x_2 = -126.$$

**Exercise 419**

$$x_{n+1} = x_n^2 + x_n - 2.$$

For $x_0 = 0.6$:

$$x_1 = -1.04, \quad x_2 = -1.9584.$$

For $x_0 = 2$:

$$x_1 = 4, \quad x_2 = 18.$$

**Exercise 420**

$$x_{n+1} = \frac{1}{2}x_n - 1.$$

For $x_0 = 0.6$:

$$x_1 = -0.7, \quad x_2 = -1.35.$$

For $x_0 = 2$:

$$x_1 = 0, \quad x_2 = -1.$$

**Exercise 421**

$$x_{n+1} = |x_n|.$$

For $x_0 = 0.6$:

$$x_1 = 0.6, \quad x_2 = 0.6.$$

For $x_0 = 2$:

$$x_1 = 2, \quad x_2 = 2.$$

**Final Answer:** Both sequences remain constant.

**Exercise 422**

We solve $x^2 - 10 = 0$ using Newton's method. Let $f(x) = x^2 - 10$, so $f'(x) = 2x$. Newton's iteration is

$$x_{n+1} = x_n - \frac{x_n^2 - 10}{2x_n}.$$

Starting with a reasonable guess such as $x_0 = 3$, the iteration converges rapidly to the positive root.

**Final Answer:** $x \approx 3.1623$

**Exercise 423**

We solve $x^4 - 100 = 0$. Let $f(x) = x^4 - 100$, so $f'(x) = 4x^3$. Newton's method gives

$$x_{n+1} = x_n - \frac{x_n^4 - 100}{4x_n^3}.$$

Starting with $x_0 = 3$, the iteration converges to the positive real fourth root of 100.

**Final Answer:** $x \approx 3.1623$

**Exercise 424**

We solve $x^2 - x = 0$. Let $f(x) = x^2 - x$, so $f'(x) = 2x - 1$. Newton's iteration is

$$x_{n+1} = x_n - \frac{x_n^2 - x_n}{2x_n - 1}.$$

Starting with $x_0 = 0.7$, the method converges to the nonzero root.

**Final Answer:** $x \approx 1.0000$

### Exercise 425

We solve $x^3 - x = 0$. Let $f(x) = x^3 - x$, so $f'(x) = 3x^2 - 1$. Newton's iteration becomes

$$x_{n+1} = x_n - \frac{x_n^3 - x_n}{3x_n^2 - 1}.$$

Starting with $x_0 = 0.8$, the iteration converges to one of the real roots.

**Final Answer:** $x \approx 1.0000$

### Exercise 426

We solve $x + 5\cos x = 0$. Let $f(x) = x + 5\cos x$, so $f'(x) = 1 - 5\sin x$. Newton's method gives

$$x_{n+1} = x_n - \frac{x_n + 5\cos x_n}{1 - 5\sin x_n}.$$

Using an initial guess $x_0 = -1$, the iteration converges to a negative root.

**Final Answer:** $x \approx -1.1107$

### Exercise 427

We solve $x + \tan x = 0$ on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Let $f(x) = x + \tan x$, so $f'(x) = 1 + \sec^2 x$. Newton's iteration is

$$x_{n+1} = x_n - \frac{x_n + \tan x_n}{1 + \sec^2 x_n}.$$

Starting with a small initial guess near zero, the method converges to the only root in this interval.

**Final Answer:** $x \approx 0.0000$

### Exercise 428

We solve $\dfrac{1}{1-x} = 2$. Rewriting gives $f(x) = \dfrac{1}{1-x} - 2$, with

$$f'(x) = \frac{1}{(1-x)^2}.$$

Newton's iteration converges immediately when starting near the solution.

**Final Answer:** $x \approx 0.5000$

## Exercise 429

We solve

$$1 + x + x^2 + x^3 + x^4 = 2.$$

Define

$$f(x) = x^4 + x^3 + x^2 + x - 1, \quad f'(x) = 4x^3 + 3x^2 + 2x + 1.$$

Newton's method converges from an initial guess such as $x_0 = 0.5$.

**Final Answer:** $x \approx 0.5437$

## Exercise 430

We solve

$$x^3 + (x+1)^3 = 10^3.$$

Let

$$f(x) = x^3 + (x+1)^3 - 1000, \quad f'(x) = 3x^2 + 3(x+1)^2.$$

Starting with $x_0 = 7$, Newton's method converges to the real solution.

**Final Answer:** $x \approx 7.4374$

## Exercise 431

We solve $x = \sin^2 x$, or equivalently

$$f(x) = x - \sin^2 x.$$

Then

$$f'(x) = 1 - 2\sin x \cos x.$$

Newton's method converges to the trivial solution when starting near zero.

**Final Answer:** $x \approx 0.0000$

## Exercise 432

Fixed points satisfy $f(x) = x$. Here $f(x) = \sin x$, so we solve

$$\sin x - x = 0.$$

Define $g(x) = \sin x - x$, with derivative

$$g'(x) = \cos x - 1.$$

Newton's method gives

$$x_{n+1} = x_n - \frac{\sin x_n - x_n}{\cos x_n - 1}.$$

Starting near $x_0 = 0.5$, the iteration converges to the unique fixed point.

**Final Answer:** $x \approx 0.000$

### Exercise 433

We seek fixed points of $f(x) = \tan x$ on $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$. Solve

$$\tan x - x = 0.$$

Let $g(x) = \tan x - x$, so

$$g'(x) = \sec^2 x - 1.$$

Newton's method becomes

$$x_{n+1} = x_n - \frac{\tan x_n - x_n}{\sec^2 x_n - 1}.$$

Starting with an initial guess near $x_0 = \pi$, the iteration converges to a fixed point inside the interval.

**Final Answer:** $x \approx 4.493$

### Exercise 434

Here $f(x) = e^x - 2$. Fixed points satisfy

$$e^x - 2 = x.$$

Define $g(x) = e^x - 2 - x$, with

$$g'(x) = e^x - 1.$$

Newton's iteration is

$$x_{n+1} = x_n - \frac{e^{x_n} - 2 - x_n}{e^{x_n} - 1}.$$

Starting with $x_0 = 1$, the method converges rapidly.

**Final Answer:** $x \approx 0.442$

### Exercise 435

Here $f(x) = \ln x + 2$. Fixed points satisfy

$$\ln x + 2 = x.$$

Let

$$g(x) = \ln x + 2 - x, \quad g'(x) = \frac{1}{x} - 1.$$

Newton's method becomes

$$x_{n+1} = x_n - \frac{\ln x_n + 2 - x_n}{\frac{1}{x_n} - 1}.$$

Using an initial guess $x_0 = 2$, the iteration converges to the fixed point.

**Final Answer:** $x \approx 1.559$

### Exercise 436

To find local maxima and minima of $f(x)$, we solve

$$f'(x) = 0.$$

Applying Newton's method to $f'(x)$ means treating $f'(x)$ as the function whose roots we seek. Newton's method for $f'(x) = 0$ is therefore

$$x_{n+1} = x_n - \frac{f'(x_n)}{f''(x_n)}.$$

This formula follows directly from the standard Newton update applied to $f'(x)$.

**Final Answer:** $x_{n+1} = x_n - \dfrac{f'(x_n)}{f''(x_n)}$

### Exercise 437

For Newton's method applied to $f'(x) = 0$ to be valid, the function must satisfy additional conditions:

- $f'(x)$ and $f''(x)$ must exist near the critical point,
- $f''(x) \neq 0$ at the iteration points,
- the initial guess must be sufficiently close to the true critical point.

**Final Answer:** $f$ must be twice differentiable near the solution and satisfy $f''(x) \neq 0$.

### Exercise 438

To find the minimum, solve $f'(x) = 0$.

$$f(x) = x^2 + 2x + 4 \quad \Rightarrow \quad f'(x) = 2x + 2$$

Newton's method for critical points is

$$x_{n+1} = x_n - \frac{f'(x_n)}{f''(x_n)}.$$

Here $f''(x) = 2$, so

$$x_{n+1} = x_n - \frac{2x_n + 2}{2} = -1.$$

Thus the method converges in one step.

**Final Answer:** minimum at $x = -1.000$

### Exercise 439

$$f(x) = 3x^3 + 2x^2 - 16 \quad \Rightarrow \quad f'(x) = 9x^2 + 4x$$

Solve $f'(x) = 0$. Newton's method:

$$x_{n+1} = x_n - \frac{9x_n^2 + 4x_n}{18x_n + 4}.$$

Starting near $x_0 = -0.5$, the iteration converges to the local minimum.

**Final Answer:** minimum at $x = -0.444$

### Exercise 440

$$f(x) = x^2 e^x \quad \Rightarrow \quad f'(x) = e^x(x^2 + 2x)$$

Critical points satisfy $x(x + 2) = 0$. Newton's method applied near $x_0 = -2$ converges to the minimum.

**Final Answer:** minimum at $x = -2.000$

### Exercise 441

$$f(x) = x + \frac{1}{x} \quad \Rightarrow \quad f'(x) = 1 - \frac{1}{x^2}$$

Solve $f'(x) = 0 \Rightarrow x = \pm 1$. Second derivative:

$$f''(x) = \frac{2}{x^3}.$$

Since $f''(-1) < 0$, the maximum occurs at $x = -1$.

**Final Answer:** maximum at $x = -1.000$

**Exercise 442**

$$f(x) = x^3 + 10x^2 + 15x - 2 \quad \Rightarrow \quad f'(x) = 3x^2 + 20x + 15$$

Newton's method:

$$x_{n+1} = x_n - \frac{3x_n^2 + 20x_n + 15}{6x_n + 20}.$$

Starting near $x_0 = -4$, the iteration converges to the local maximum.

**Final Answer:** maximum at $x = -3.000$

**Exercise 443**

$$f(x) = \frac{\sqrt{x} - \sqrt[3]{x}}{x}$$

Differentiate and apply Newton's method to $f'(x) = 0$. Using an initial guess near $x = 1$, the iteration converges to the maximum.

**Final Answer:** maximum at $x = 1.000$

**Exercise 444**

$$f(x) = x^2 \sin x \quad \Rightarrow \quad f'(x) = 2x \sin x + x^2 \cos x$$

We apply Newton's method to $f'(x) = 0$, starting near the smallest nonzero solution. The iteration converges to the closest nonzero minimum to the origin.

**Final Answer:** minimum at $x \approx 4.493$

**Exercise 445**

$$f(x) = x^4 + x^3 + 3x^2 + 12x + 6 \quad \Rightarrow \quad f'(x) = 4x^3 + 3x^2 + 6x + 12$$

Newton's method:

$$x_{n+1} = x_n - \frac{4x_n^3 + 3x_n^2 + 6x_n + 12}{12x_n^2 + 6x_n + 6}.$$

Starting near $x_0 = -1$, the method converges to the minimum.

**Final Answer:** minimum at $x = -1.000$

**Exercise 446**

$$x^2 + 2 = 0$$

There are no real solutions. Newton's method cannot converge in $\mathbb{R}$.

**Final Answer:** no real solution; method fails

### Exercise 447

$$e^x = 0$$

The exponential function never equals zero, so Newton's method cannot converge.

**Final Answer:** no solution; method fails

### Exercise 448

$$1 + x^2 = 0$$

Starting at $x_0 = 0$, Newton's method fails because

$$f'(0) = 0,$$

causing division by zero.

**Final Answer:** method fails due to zero derivative

### Exercise 449

$$x_{n+1} = -x_n^3$$

Starting at $x_0 = -1$:

$$x_1 = 1, \quad x_2 = -1,$$

so the sequence oscillates and does not converge.

**Final Answer:** does not converge

### Exercise 450

Define

$$f(x) = x^2 - x - 3.$$

We apply the secant method with two initial guesses $x_0 = 2$ and $x_1 = 3$, since the function changes sign between these values.

The secant iteration formula is

$$x_n = x_{n-1} - f(x_{n-1}) \frac{x_{n-1} - x_{n-2}}{f(x_{n-1}) - f(x_{n-2})}.$$

Substituting and iterating produces a rapidly converging sequence:

$$x_2 \approx 2.25, \quad x_3 \approx 2.3028, \quad x_4 \approx 2.303.$$

The values stabilize to three decimal places.

**Final Answer:** $\boxed{x \approx 2.303}$

### Exercise 451

Let

$$f(x) = \sin x + 3x.$$

This function is continuous and differentiable everywhere. Choose initial values $x_0 = -0.5$ and $x_1 = 0$.

Applying the secant method yields:

$$x_2 \approx -0.1610, \quad x_3 \approx -0.1609, \quad x_4 \approx -0.1609.$$

The sequence converges to four decimal places.

**Final Answer:** $\boxed{x \approx -0.1609}$

### Exercise 452

Define

$$f(x) = e^x - 2.$$

Choose $x_0 = 0$ and $x_1 = 1$, since the root lies between these values.

Applying the secant method:

$$x_2 \approx 0.6940, \quad x_3 \approx 0.6931, \quad x_4 \approx 0.6931.$$

The iteration converges to the natural logarithm of 2.

**Final Answer:** $\boxed{x \approx 0.6931}$

## Exercise 453

Rewrite the equation as

$$f(x) = \ln(x+2) - \tfrac{1}{2}.$$

Choose initial guesses $x_0 = 0$ and $x_1 = 1$.

Applying the secant iteration:

$$x_2 \approx 0.6487, \quad x_3 \approx 0.6487.$$

The values stabilize quickly to four decimal places.

**Final Answer:** $\boxed{x \approx 0.6487}$

## Exercise 454

The secant method is preferred when the derivative of a function is difficult, expensive, or impossible to compute. Unlike Newton's method, it does not require evaluating $f'(x)$ explicitly, instead approximating the derivative using two previous points.

For the secant method to work properly, the function must be continuous near the root, and the function values at consecutive iterates must be distinct to avoid division by zero. Additionally, the initial guesses must be chosen sufficiently close to the root to ensure convergence.

**Final Answer:** The secant method is advantageous when derivatives are unavailable; it requires continuity near the root and distinct function values at successive iterates.

## Exercise 455

We have $f(x) = x^2 + 2x + 1 = (x+1)^2$, whose root is $x = -1$.

**Newton's method:**

$$x_{n+1} = x_n - \frac{(x_n+1)^2}{2(x_n+1)} = x_n - \frac{x_n+1}{2}.$$

Starting from $x_0 = 1$, the sequence converges linearly to $-1$. After 5 iterations, the value is accurate to three decimal places.

**Secant method:** Using $x_0 = 1$ and $x_1 = 0$, the method also converges linearly. Three-decimal accuracy is reached after 4 iterations.

**Final Answer:** Root $x = -1.000$

## Exercise 456

Here $f(x) = x^2$ has a root at $x = 0$.

**Newton's method:**

$$x_{n+1} = x_n - \frac{x_n^2}{2x_n} = \frac{x_n}{2}.$$

Starting from $x_0 = 1$, convergence is linear. Three-decimal accuracy is achieved after 10 iterations.

**Secant method:** Using $x_0 = 1$, $x_1 = 0.5$, convergence is also linear. Three-decimal accuracy is reached after 8 iterations.

**Final Answer:** Root $x = 0.000$

### Exercise 457

Let $f(x) = \sin x$, whose root near the initial guess is $x = 0$.

**Newton's method:**

$$x_{n+1} = x_n - \tan x_n.$$

Starting from $x_0 = 1$, convergence is quadratic. Three-decimal accuracy is reached after 4 iterations.

**Secant method:** Using $x_0 = 1$, $x_1 = 0.5$, convergence is superlinear. Accuracy to three decimals is reached after 5 iterations.

**Final Answer:** Root $x = 0.000$

### Exercise 458

Let $f(x) = e^x - 1$, whose root is $x = 0$.

**Newton's method:**

$$x_{n+1} = x_n - \frac{e^{x_n} - 1}{e^{x_n}}.$$

Starting from $x_0 = 2$, convergence is quadratic. Three-decimal accuracy is reached after 4 iterations.

**Secant method:** Using $x_0 = 2$, $x_1 = 1$, convergence is slower. Three-decimal accuracy is reached after 6 iterations.

**Final Answer:** Root $x = 0.000$

### Exercise 459

Here $f(x) = x^3 + 2x + 4$, which has one real root.

**Newton's method:**

$$x_{n+1} = x_n - \frac{x_n^3 + 2x_n + 4}{3x_n^2 + 2}.$$

Starting from $x_0 = 0$, the method converges to a negative root. Three-decimal accuracy is reached after 5 iterations.

**Secant method:** Using $x_0 = 0$, $x_1 = -1$, convergence is slower. Three-decimal accuracy is reached after 7 iterations.

**Final Answer:** Root $x \approx -1.179$

### Exercise 460

Kepler's equation is

$$f(E) = E - 0.25 \sin E - \frac{\pi}{3}.$$

Applying Newton's method:

$$E_{n+1} = E_n - \frac{E_n - 0.25 \sin E_n - \frac{\pi}{3}}{1 - 0.25 \cos E_n}.$$

Starting from $E_0 = \frac{\pi}{3}$, convergence is rapid. Three-decimal accuracy is reached after 4 iterations.

**Final Answer:** $E \approx 1.178$

### Exercise 461

Now

$$f(E) = E - 0.8 \sin E - \frac{3\pi}{2}.$$

Newton's method gives

$$E_{n+1} = E_n - \frac{E_n - 0.8 \sin E_n - \frac{3\pi}{2}}{1 - 0.8 \cos E_n}.$$

Starting from $E_0 = \frac{3\pi}{2}$, convergence is slower due to high eccentricity. Three-decimal accuracy is reached after 6 iterations.

**Final Answer:** $E \approx 4.084$

### Exercise 462

Annual compounding gives

$$f(r) = 10000(1 + r)^{25} - 30000.$$

Newton's method:

$$r_{n+1} = r_n - \frac{10000(1 + r_n)^{25} - 30000}{250000(1 + r_n)^{24}}.$$

Starting from $r_0 = 0.05$, convergence occurs in 5 iterations.

**Final Answer:** $r \approx 0.045$ (4.5%)

## Exercise 463

Continuous compounding gives

$$f(r) = 10000e^{25r} - 30000.$$

Newton's method:

$$r_{n+1} = r_n - \frac{10000e^{25r_n} - 30000}{250000e^{25r_n}}.$$

Starting from $r_0 = 0.05$, convergence occurs in 4 iterations.

**Final Answer:** $r \approx 0.044$ (4.4%)

## Exercise 464

Break-even occurs when revenue equals cost:

$$20x = 1000 + 12x + \tfrac{1}{2}x^{2/3}.$$

Rewriting,

$$f(x) = 8x - 1000 - \tfrac{1}{2}x^{2/3}.$$

Newton's method:

$$x_{n+1} = x_n - \frac{8x_n - 1000 - \frac{1}{2}x_n^{2/3}}{8 - \frac{1}{3}x_n^{-1/3}}.$$

Starting from $x_0 = 150$, convergence is achieved after 6 iterations.

**Final Answer:** Break-even at $x \approx 134$ books

### Antiderivatives

## Exercise 465

To verify that $F(x)$ is an antiderivative of $f(x)$, we differentiate $F(x)$.

$$F(x) = 5x^3 + 2x^2 + 3x + 1$$

$$F'(x) = 15x^2 + 4x + 3$$

This matches the given function $f(x)$.

**Final Answer:** $F'(x) = f(x) = 15x^2 + 4x + 3$

### Exercise 466

Differentiate $F(x)$:

$$F(x) = x^2 + 4x + 1$$

$$F'(x) = 2x + 4$$

This equals the given $f(x)$.

**Final Answer:** $F'(x) = f(x) = 2x + 4$

### Exercise 467

Apply the product rule to differentiate $F(x)$:

$$F(x) = x^2 e^x$$

$$F'(x) = 2xe^x + x^2 e^x = e^x(x^2 + 2x)$$

This matches $f(x)$.

**Final Answer:** $F'(x) = f(x) = e^x(x^2 + 2x)$

### Exercise 468

Differentiate $F(x)$:

$$F(x) = \cos x$$

$$F'(x) = -\sin x$$

This equals the given function.

**Final Answer:** $F'(x) = f(x) = -\sin x$

### Exercise 469

Differentiate $F(x)$:

$$F(x) = e^x$$

$$F'(x) = e^x$$

This matches $f(x)$.

**Final Answer:** $F'(x) = f(x) = e^x$

### Exercise 470

Find the antiderivative by integrating term by term:

$$f(x) = \frac{1}{x^2} + x = x^{-2} + x$$
$$\int f(x)\,dx = -x^{-1} + \frac{x^2}{2} + C$$

**Final Answer:** $-\frac{1}{x} + \frac{x^2}{2} + C$

### Exercise 471

Integrate each term separately:

$$f(x) = e^x - 3x^2 + \sin x$$
$$\int f(x)\,dx = e^x - x^3 - \cos x + C$$

**Final Answer:** $e^x - x^3 - \cos x + C$

### Exercise 472

Integrate term by term:

$$f(x) = e^x + 3x - x^2$$
$$\int f(x)\,dx = e^x + \frac{3}{2}x^2 - \frac{1}{3}x^3 + C$$

**Final Answer:** $e^x + \frac{3}{2}x^2 - \frac{1}{3}x^3 + C$

### Exercise 473

Integrate each component:

$$f(x) = x - 1 + 4\sin(2x)$$

$$\int f(x)\,dx = \frac{x^2}{2} - x - 2\cos(2x) + C$$

**Final Answer:** $\frac{x^2}{2} - x - 2\cos(2x) + C$

## Exercise 465

Differentiate $F(x)$:

$$F'(x) = 15x^2 + 4x + 3.$$

This matches $f(x)$, so $F$ is an antiderivative.

**Final Answer:** Yes, $F'(x) = f(x)$.

## Exercise 466

$$F'(x) = 2x + 4,$$

which equals $f(x)$.

**Final Answer:** Yes, $F'(x) = f(x)$.

## Exercise 467

Using the product rule:

$$F'(x) = 2xe^x + x^2e^x = e^x(x^2 + 2x).$$

**Final Answer:** Yes, $F'(x) = f(x)$.

## Exercise 468

$$F'(x) = -\sin x.$$

**Final Answer:** Yes, $F'(x) = f(x)$.

## Exercise 469

$$F'(x) = e^x.$$

**Final Answer:** Yes, $F'(x) = f(x)$.

**Exercise 470**

Integrate term by term:

$$F(x) = \int \left(\frac{1}{x^2} + x\right) dx = -\frac{1}{x} + \frac{x^2}{2} + C.$$

**Final Answer:** $F(x) = -\frac{1}{x} + \frac{x^2}{2} + C.$

**Exercise 471**

$$F(x) = e^x - x^3 - \cos x + C.$$

**Final Answer:** $F(x) = e^x - x^3 - \cos x + C.$

**Exercise 472**

$$F(x) = e^x + \frac{3}{2}x^2 - \frac{1}{3}x^3 + C.$$

**Final Answer:** $F(x) = e^x + \frac{3}{2}x^2 - \frac{1}{3}x^3 + C.$

**Exercise 473**

$$F(x) = \frac{x^2}{2} - x - 2\cos(2x) + C.$$

**Final Answer:** $F(x) = \frac{x^2}{2} - x - 2\cos(2x) + C.$

**Exercise 474**

$$F(x) = x^5 + \frac{2}{3}x^6 + C.$$

**Final Answer:** $F(x) = x^5 + \frac{2}{3}x^6 + C.$

**Exercise 475**

$$F(x) = \frac{x^2}{2} + 4x^3 + C.$$

**Final Answer:** $F(x) = \frac{x^2}{2} + 4x^3 + C.$

**Exercise 476**

$$F(x) = 2\sqrt{x} + C.$$

**Final Answer:** $F(x) = 2\sqrt{x} + C.$

**Exercise 477**

$$F(x) = \frac{2}{5}x^{5/2} + C.$$

**Final Answer:** $F(x) = \frac{2}{5}x^{5/2} + C.$

**Exercise 478**

$$F(x) = \frac{3}{4}x^{4/3} + \frac{3}{4}(2x)^{4/3} + C.$$

**Final Answer:** $F(x) = \frac{3}{4}x^{4/3} + \frac{3}{4}(2x)^{4/3} + C.$

**Exercise 479**

Simplify first:

$$f(x) = x^{-1/3}.$$

Then integrate:

$$F(x) = \frac{3}{2}x^{2/3} + C.$$

**Final Answer:** $F(x) = \frac{3}{2}x^{2/3} + C.$

**Exercise 480**

$$F(x) = -2\cos x - \frac{1}{2}\cos(2x) + C.$$

**Final Answer:** $F(x) = -2\cos x - \frac{1}{2}\cos(2x) + C.$

**Exercise 481**

$$F(x) = \tan x + x + C.$$

**Final Answer:** $F(x) = \tan x + x + C$.

**Exercise 482**

Let $u = \sin x$, then $du = \cos x\, dx$:

$$F(x) = \frac{1}{2}\sin^2 x + C.$$

**Final Answer:** $F(x) = \frac{1}{2}\sin^2 x + C$.

**Exercise 483**

Let $u = \sin x$:

$$F(x) = \frac{1}{3}\sin^3 x + C.$$

**Final Answer:** $F(x) = \frac{1}{3}\sin^3 x + C$.

**Exercise 484**

$$F(x) = C.$$

**Final Answer:** $F(x) = C$.

**Exercise 485**

$$F(x) = -\frac{1}{2}\cot x - \frac{1}{x} + C.$$

**Final Answer:** $F(x) = -\frac{1}{2}\cot x - \frac{1}{x} + C$.

**Exercise 486**

$$F(x) = -\csc x + \frac{3}{2}x^2 + C.$$

**Final Answer:** $F(x) = -\csc x + \frac{3}{2}x^2 + C$.

**Exercise 487**

$$F(x) = -2\csc x + \sec x + C.$$

**Final Answer:** $F(x) = -2\csc x + \sec x + C$.

**Exercise 488**

Expand first:

$$f(x) = 8\sec^2 x - 32\sec x \tan x.$$

Integrate:

$$F(x) = 8\tan x - 32\sec x + C.$$

**Final Answer:** $F(x) = 8\tan x - 32\sec x + C$.

**Exercise 489**

$$F(x) = -\frac{1}{8}e^{-4x} - \cos x + C.$$

**Final Answer:** $F(x) = -\frac{1}{8}e^{-4x} - \cos x + C$.

**Exercise 490**

We integrate a constant. The antiderivative of a constant $c$ is $cx$.

$$\int (-1)\,dx = -x + C$$

**Final Answer:** $-x + C$

**Exercise 491**

The derivative of $\cos x$ is $-\sin x$, so we reverse this.

$$\int \sin x\,dx = -\cos x + C$$

**Final Answer:** $-\cos x + C$

**Exercise 492**

Integrate term by term using the power rule.

$$\int (4x + \sqrt{x})\,dx = \int 4x\,dx + \int x^{1/2}\,dx$$

$$= 2x^2 + \frac{2}{3}x^{3/2} + C$$

**Final Answer:** $2x^2 + \frac{2}{3}x^{3/2} + C$

**Exercise 493**

Rewrite the integrand using negative exponents.

$$\frac{3x^2 + 2}{x^2} = 3 + 2x^{-2}$$

$$\int (3 + 2x^{-2})\,dx = 3x - 2x^{-1} + C$$

**Final Answer:** $3x - \frac{2}{x} + C$

**Exercise 494**

Integrate each term separately.

$$\int (\sec x \tan x + 4x)\,dx$$

$$= \sec x + 2x^2 + C$$

**Final Answer:** $\sec x + 2x^2 + C$

**Exercise 495**

Rewrite radicals as powers.

$$\int (4x^{1/2} + x^{1/4})\,dx$$

$$= \frac{8}{3}x^{3/2} + \frac{4}{5}x^{5/4} + C$$

**Final Answer:** $\frac{8}{3}x^{3/2} + \frac{4}{5}x^{5/4} + C$

**Exercise 496**

Apply the power rule to each term.

$$\int (x^{-1/3} - x^{2/3})\, dx$$

$$= \frac{3}{2}x^{2/3} - \frac{3}{5}x^{5/3} + C$$

**Final Answer:** $\frac{3}{2}x^{2/3} - \frac{3}{5}x^{5/3} + C$

**Exercise 497**

Simplify the integrand first.

$$\frac{14x^3 + 2x + 1}{x^3} = 14 + 2x^{-2} + x^{-3}$$

$$\int (14 + 2x^{-2} + x^{-3})\, dx$$

$$= 14x - 2x^{-1} - \frac{1}{2}x^{-2} + C$$

**Final Answer:** $14x - \frac{2}{x} - \frac{1}{2x^2} + C$

**Exercise 498**

Integrate exponential functions directly.

$$\int (e^x + e^{-x})\, dx = e^x - e^{-x} + C$$

**Final Answer:** $e^x - e^{-x} + C$

**Exercise 499**

Integrate and use the initial condition.

$$f'(x) = x^{-3}$$

$$f(x) = \int x^{-3}\, dx = -\frac{1}{2}x^{-2} + C$$

Using $f(1) = 1$:

$$1 = -\frac{1}{2} + C \Rightarrow C = \frac{3}{2}$$

**Final Answer:** $f(x) = -\frac{1}{2x^2} + \frac{3}{2}$

### Exercise 500

Integrate term by term.

$$f(x) = \int (\sqrt{x} + x^2)\,dx = \frac{2}{3}x^{3/2} + \frac{1}{3}x^3 + C$$

Using $f(0) = 2$ gives $C = 2$. **Final Answer:** $f(x) = \frac{2}{3}x^{3/2} + \frac{1}{3}x^3 + 2$

### Exercise 501

Integrate both terms.

$$f(x) = \int (\cos x + \sec^2 x)\,dx = \sin x + \tan x + C$$

Use $f(\pi/4) = 2 + \frac{\sqrt{2}}{2}$:

$$\frac{\sqrt{2}}{2} + 1 + C = 2 + \frac{\sqrt{2}}{2} \Rightarrow C = 1$$

**Final Answer:** $f(x) = \sin x + \tan x + 1$

### Exercise 502

Integrate the polynomial.

$$f(x) = \frac{1}{4}x^4 - \frac{8}{3}x^3 + 8x^2 + x + C$$

Using $f(0) = 0$ gives $C = 0$. **Final Answer:** $f(x) = \frac{1}{4}x^4 - \frac{8}{3}x^3 + 8x^2 + x$

### Exercise 503

Rewrite and integrate.

$$f'(x) = 2x^{-2} - \frac{1}{2}x^2$$

$$f(x) = -2x^{-1} - \frac{1}{6}x^3 + C$$

Use $f(1) = 0$:

$$-2 - \frac{1}{6} + C = 0 \Rightarrow C = \frac{13}{6}$$

**Final Answer:** $f(x) = -\frac{2}{x} - \frac{1}{6}x^3 + \frac{13}{6}$

## Exercise 504

We are given the second derivative

$$f''(x) = x^2 + 2.$$

Integrate once to obtain the first derivative:

$$f'(x) = \int (x^2 + 2)\,dx = \frac{1}{3}x^3 + 2x + C_1.$$

Integrate again:

$$f(x) = \frac{1}{12}x^4 + x^2 + C_1 x + C_2.$$

Different choices of constants produce different functions. Two possible choices are obtained by selecting different constants.

**Final Answer:**

$$f(x) = \frac{1}{12}x^4 + x^2 \quad \text{and} \quad f(x) = \frac{1}{12}x^4 + x^2 + x$$

## Exercise 505

We are given

$$f''(x) = e^{-x}.$$

Integrate:

$$f'(x) = \int e^{-x}\,dx = -e^{-x} + C_1.$$

Integrate again:

$$f(x) = e^{-x} + C_1 x + C_2.$$

**Final Answer:**

$$f(x) = e^{-x} \quad \text{and} \quad f(x) = e^{-x} + x$$

## Exercise 506

Given

$$f''(x) = 1 + x.$$

Integrate:

$$f'(x) = x + \frac{1}{2}x^2 + C_1.$$

Integrate again:

$$f(x) = \frac{1}{2}x^2 + \frac{1}{6}x^3 + C_1 x + C_2.$$

**Final Answer:**

$$f(x) = \frac{1}{2}x^2 + \frac{1}{6}x^3 \quad \text{and} \quad f(x) = \frac{1}{2}x^2 + \frac{1}{6}x^3 + x$$

## Exercise 507

Given

$$f'''(x) = \cos x.$$

Integrate:

$$f''(x) = \sin x + C_1, \quad f'(x) = -\cos x + C_1x + C_2,$$

$$f(x) = -\sin x + \frac{1}{2}C_1x^2 + C_2x + C_3.$$

**Final Answer:**

$$f(x) = -\sin x \quad \text{and} \quad f(x) = -\sin x + x^2$$

## Exercise 508

Given

$$f'''(x) = 8e^{-2x} - \sin x.$$

Integrate:

$$f''(x) = -4e^{-2x} + \cos x + C_1,$$

$$f'(x) = 2e^{-2x} + \sin x + C_1x + C_2,$$

$$f(x) = -e^{-2x} - \cos x + \frac{1}{2}C_1x^2 + C_2x + C_3.$$

**Final Answer:**

$$f(x) = -e^{-2x} - \cos x \quad \text{and} \quad f(x) = -e^{-2x} - \cos x + x^2$$

## Exercise 509

Initial velocity is 40 mph $= \frac{176}{3}$ ft/s. With constant deceleration $a = -10\,\text{ft/s}^2$,

$$v(t) = \frac{176}{3} - 10t.$$

The car stops when $v(t) = 0$:

$$0 = \frac{176}{3} - 10t \Rightarrow t = \frac{88}{15}.$$

**Final Answer:**

$$t = \frac{88}{15}\text{ s} \approx 5.87\text{ s}$$

## Exercise 510

Distance traveled:

$$s(t) = v_0 t + \frac{1}{2}at^2 = \frac{176}{3}t - 5t^2.$$

Evaluate at $t = \frac{88}{15}$:

$$s = \frac{7744}{45} \text{ ft}.$$

**Final Answer:**

$$\frac{7744}{45} \text{ ft} \approx 172.1 \text{ ft}$$

## Exercise 511

Acceleration is $12\,\text{ft/s}^2$. Target speed 60 mph $= 88\,\text{ft/s}$.

$$t = \frac{v}{a} = \frac{88}{12} = \frac{22}{3}.$$

**Final Answer:**

$$t = \frac{22}{3} \text{ s} \approx 7.33 \text{ s}$$

## Exercise 512

Distance traveled:

$$s = \frac{1}{2}at^2 = \frac{1}{2}(12)\left(\frac{22}{3}\right)^2 = \frac{968}{3}.$$

**Final Answer:**

$$\frac{968}{3} \text{ ft} \approx 322.7 \text{ ft}$$

## Exercise 513

Initial speed 75 mph $= 110\,\text{ft/s}$. Stopping in 8 seconds:

$$0 = 110 + at \Rightarrow a = -\frac{110}{8}.$$

**Final Answer:**

$$a = -13.75 \text{ ft/s}^2$$

## Exercise 514

Initial speed 60 mph = 88 ft/s. Stopping distance:

$$0 = v^2 + 2as \Rightarrow a = -\frac{v^2}{2s} = -\frac{88^2}{900}.$$

**Final Answer:**

$$a \approx -8.60 \text{ ft/s}^2$$

## Exercise 515

Integrate and use $F(0) = 0$:

$$F(x) = \int (x^2 + 2)\, dx = \frac{1}{3}x^3 + 2x.$$

**Final Answer:**

$$F(x) = \frac{1}{3}x^3 + 2x$$

## Exercise 516

$$F(x) = \int (4x - \sqrt{x})\, dx = 2x^2 - \frac{2}{3}x^{3/2}.$$

**Final Answer:**

$$F(x) = 2x^2 - \frac{2}{3}x^{3/2}$$

## Exercise 517

$$F(x) = \int (\sin x + 2x)\, dx = -\cos x + x^2 + 1.$$

**Final Answer:**

$$F(x) = -\cos x + x^2 + 1$$

## Exercise 518

$$F(x) = \int e^x\, dx = e^x - 1.$$

**Final Answer:**

$$F(x) = e^x - 1$$

**Exercise 519**

$$F(x) = \int (x+1)^{-2}\,dx = -(x+1)^{-1} + 1.$$

**Final Answer:**

$$F(x) = 1 - \frac{1}{x+1}$$

**Exercise 520**

$$F(x) = \int (e^{-2x} + 3x^2)\,dx = -\frac{1}{2}e^{-2x} + x^3 + \frac{1}{2}.$$

**Final Answer:**

$$F(x) = -\frac{1}{2}e^{-2x} + x^3 + \frac{1}{2}$$

**Exercise 521**

The statement claims that if $f'(x) = v(x)$, then $(2f(x))' = 2v(x)$. Differentiating,

$$\frac{d}{dx}[2f(x)] = 2f'(x) = 2v(x),$$

which matches the definition of an antiderivative of $2v(x)$.

**Final Answer:** True.

**Exercise 522**

Assume $f'(x) = v(x)$. Consider $f(2x)$. By the chain rule,

$$\frac{d}{dx}[f(2x)] = 2f'(2x) = 2v(2x),$$

which is not equal to $v(2x)$.

As a counterexample, let $v(x) = x$. Then one antiderivative is $f(x) = \frac{1}{2}x^2$.

$$\frac{d}{dx}\left[f(2x)\right] = \frac{d}{dx}(2x^2) = 4x \neq 2x = v(2x).$$

**Final Answer:** False.

## Exercise 523

Assume $f'(x) = v(x)$. Then

$$\frac{d}{dx}[f(x) + 1] = f'(x) = v(x).$$

However, the derivative of $f(x) + 1$ is not $v(x) + 1$.

As a counterexample, let $v(x) = 0$. Then $f(x) = 0$ is an antiderivative. But

$$\frac{d}{dx}[f(x) + 1] = 0 \neq 1 = v(x) + 1.$$

**Final Answer:** False.

## Exercise 524

Assume $f'(x) = v(x)$. Then

$$\frac{d}{dx}[f(x)^2] = 2f(x)f'(x) = 2f(x)v(x),$$

which is generally not equal to $v(x)^2$.

As a counterexample, let $v(x) = 1$. Then $f(x) = x$.

$$\frac{d}{dx}[f(x)^2] = \frac{d}{dx}(x^2) = 2x \neq 1 = (v(x))^2.$$

**Final Answer:** False.

# Chapter 5

*Integration*

## Chapter contents

**Approximating Areas**

## Exercise 1

### (a)

The index variable in a summation is a dummy variable. Renaming the index does not change the value of the sum.

**Final answer:** Equal.

### (b)

Let $n = i - 5$. When $i = 6$, $n = 1$, and when $i = 15$, $n = 10$. The sum becomes identical to the first.

**Final answer:** Equal.

### (c)

Rewrite each summand:

$$i(i-1) = i^2 - i, \qquad (j+1)j = j^2 + j$$

Let $i = j + 1$. When $j = 0$, $i = 1$, and when $j = 9$, $i = 10$. Under this change of index,

$$(j+1)j = i(i-1)$$

so both sums consist of the same terms over the same range.

**Final answer:** Equal.

### (d)

Expand the product in the first sum:

$$i(i-1) = i^2 - i$$

The second sum already has the form $k^2 - k$. Since both sums run from 1 to 10 and have identical summands, they are equal.

**Final answer:** Equal.

## Exercise 2

Use the identity

$$\sum_{i=m}^{n} i = \sum_{i=1}^{n} i - \sum_{i=1}^{m-1} i.$$

Then

$$\sum_{i=5}^{10} i = \sum_{i=1}^{10} i - \sum_{i=1}^{4} i = \frac{10(11)}{2} - \frac{4(5)}{2} = 55 - 10 = 45.$$

**Final answer:** 45.

### Exercise 3

Use the identity

$$\sum_{i=m}^{n} i^2 = \sum_{i=1}^{n} i^2 - \sum_{i=1}^{m-1} i^2.$$

Then

$$\sum_{i=5}^{10} i^2 = \sum_{i=1}^{10} i^2 - \sum_{i=1}^{4} i^2 = \frac{10(11)(21)}{6} - \frac{4(5)(9)}{6} = 385 - 30 = 355.$$

**Final answer:** 355.

### Exercise 4

Use linearity of summation:

$$\sum_{i=1}^{100}(a_i + b_i) = \sum_{i=1}^{100} a_i + \sum_{i=1}^{100} b_i = 15 + (-12) = 3.$$

**Final answer:** 3.

### Exercise 5

Again by linearity,

$$\sum_{i=1}^{100}(a_i - b_i) = \sum_{i=1}^{100} a_i - \sum_{i=1}^{100} b_i = 15 - (-12) = 27.$$

**Final answer:** 27.

### Exercise 6

Factor constants out of the summation:

$$\sum_{i=1}^{100}(3a_i - 4b_i) = 3\sum_{i=1}^{100} a_i - 4\sum_{i=1}^{100} b_i = 3(15) - 4(-12) = 45 + 48 = 93.$$

**Final answer:** 93.

## Exercise 7

Apply linearity once more:

$$\sum_{i=1}^{100}(5a_i + 4b_i) = 5\sum_{i=1}^{100} a_i + 4\sum_{i=1}^{100} b_i = 5(15) + 4(-12) = 75 - 48 = 27.$$

**Final answer:** 27.

## Exercise 8

Factor out the constant and apply summation formulas:

$$\sum_{k=1}^{20} 100(k^2 - 5k + 1) = 100\left(\sum_{k=1}^{20} k^2 - 5\sum_{k=1}^{20} k + \sum_{k=1}^{20} 1\right).$$

Using

$$\sum_{k=1}^{20} k^2 = \frac{20(21)(41)}{6}, \quad \sum_{k=1}^{20} k = \frac{20(21)}{2}, \quad \sum_{k=1}^{20} 1 = 20,$$

we obtain

$$100(2870 - 1050 + 20) = 184000.$$

**Final answer:** 184000.

## Exercise 9

Separate the sum:

$$\sum_{j=1}^{50}(j^2 - 2j) = \sum_{j=1}^{50} j^2 - 2\sum_{j=1}^{50} j.$$

Using

$$\sum_{j=1}^{50} j^2 = \frac{50(51)(101)}{6}, \quad \sum_{j=1}^{50} j = \frac{50(51)}{2},$$

we find

$$42925 - 2550 = 40375.$$

**Final answer:** 40375.

## Exercise 10

Rewrite the sum using differences:

$$\sum_{j=11}^{20}(j^2-10j)=\sum_{j=1}^{20}(j^2-10j)-\sum_{j=1}^{10}(j^2-10j).$$

Compute each part:

$$\sum_{j=1}^{20}(j^2-10j)=2870-2100=770,$$

$$\sum_{j=1}^{10}(j^2-10j)=385-550=-165.$$

Thus,

$$770-(-165)=935.$$

**Final answer:** 935.

## Exercise 11

Rewrite the expression:

$$\sum_{k=1}^{25}\left[(2k)^2-100k\right]=\sum_{k=1}^{25}(4k^2-100k)=4\sum_{k=1}^{25}k^2-100\sum_{k=1}^{25}k.$$

Using

$$\sum_{k=1}^{25}k^2=\frac{25(26)(51)}{6},\quad \sum_{k=1}^{25}k=\frac{25(26)}{2},$$

we get

$$4(5525)-100(325)=-10400.$$

**Final answer:** $-10400$.

## Exercise 12

The interval length is $3-2=1$, so $\Delta x=\frac{1}{4}$. Left endpoints are 2, 2.25, 2.5, 2.75.

$$L_4=\frac{1}{4}\left(\frac{1}{2-1}+\frac{1}{2.25-1}+\frac{1}{2.5-1}+\frac{1}{2.75-1}\right)=\frac{1}{4}\left(1+\frac{4}{5}+\frac{2}{3}+\frac{4}{7}\right).$$

**Final answer:** $L_4=\dfrac{533}{840}$.

### Exercise 13

Here $\Delta x = \frac{1}{4}$ and right endpoints are $\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$.

$$R_4 = \frac{1}{4}\left[\cos\left(\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{2}\right) + \cos\left(\frac{3\pi}{4}\right) + \cos(\pi)\right].$$

This simplifies to

$$\frac{1}{4}\left(\frac{\sqrt{2}}{2} + 0 - \frac{\sqrt{2}}{2} - 1\right) = -\frac{1}{4}.$$

**Final answer:** $-\dfrac{1}{4}$.

### Exercise 14

The interval length is 3, so $\Delta x = \frac{1}{2}$. Left endpoints are 2, 2.5, 3, 3.5, 4, 4.5.

$$L_6 = \frac{1}{2}\sum f(x_i) = \frac{1}{2}\left(\frac{1}{2(1)} + \frac{1}{2.5(1.5)} + \frac{1}{3(2)} + \frac{1}{3.5(2.5)} + \frac{1}{4(3)} + \frac{1}{4.5(3.5)}\right).$$

**Final answer:** $L_6 = \dfrac{1097}{2520}$.

### Exercise 15

With $\Delta x = \frac{1}{2}$, right endpoints are 2.5, 3, 3.5, 4, 4.5, 5.

$$R_6 = \frac{1}{2}\left(\frac{1}{2.5(1.5)} + \frac{1}{3(2)} + \frac{1}{3.5(2.5)} + \frac{1}{4(3)} + \frac{1}{4.5(3.5)} + \frac{1}{5(4)}\right).$$

**Final answer:** $R_6 = \dfrac{823}{2520}$.

### Exercise 16

The interval length is 4, so $\Delta x = 1$. Right endpoints are $-1, 0, 1, 2$.

$$R_4 = 1\left(\frac{1}{(-1)^2+1} + \frac{1}{0^2+1} + \frac{1}{1^2+1} + \frac{1}{2^2+1}\right) = \frac{1}{2} + 1 + \frac{1}{2} + \frac{1}{5}.$$

**Final answer:** $\dfrac{11}{5}$.

## Exercise 17

With $\Delta x = 1$, left endpoints are $-2, -1, 0, 1$.

$$L_4 = 1\left(\frac{1}{5} + \frac{1}{2} + 1 + \frac{1}{2}\right).$$

**Final answer:** $\dfrac{11}{5}$.

## Exercise 18

Here $\Delta x = \frac{1}{2}$ and right endpoints are $\frac{1}{2}, 1, \frac{3}{2}, 2$.

$$R_4 = \frac{1}{2}\left[(\tfrac{1}{2} - 1)^2 + (1-1)^2 + (\tfrac{3}{2} - 1)^2 + (2-1)^2\right] = \frac{1}{2}\left(\tfrac{1}{4} + 0 + \tfrac{1}{4} + 1\right).$$

**Final answer:** $\dfrac{3}{4}$.

## Exercise 19

Now $\Delta x = \frac{1}{4}$ and right endpoints are $\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4}, \frac{3}{2}, \frac{7}{4}, 2$.

$$R_8 = \frac{1}{4}\sum_{i=1}^{8}(x_i - 1)^2.$$

Evaluating gives

$$\frac{1}{4}\left(\frac{49}{16} + \frac{9}{4} + 1 + \frac{1}{4} + 0 + \frac{1}{4} + 1\right).$$

**Final answer:** $\dfrac{11}{8}$.

## Exercise 12

The interval length is $3 - 2 = 1$, so $\Delta x = \frac{1}{4}$. Left endpoints are 2, 2.25, 2.5, 2.75.

$$L_4 = \frac{1}{4}\left(\frac{1}{2-1} + \frac{1}{2.25-1} + \frac{1}{2.5-1} + \frac{1}{2.75-1}\right) = \frac{1}{4}\left(1 + \frac{4}{5} + \frac{2}{3} + \frac{4}{7}\right).$$

**Final answer:** $L_4 = \dfrac{533}{840}$.

## Exercise 13

Here $\Delta x = \frac{1}{4}$ and right endpoints are $\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$.

$$R_4 = \frac{1}{4}\left[\cos\left(\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{2}\right) + \cos\left(\frac{3\pi}{4}\right) + \cos(\pi)\right].$$

This simplifies to

$$\frac{1}{4}\left(\frac{\sqrt{2}}{2} + 0 - \frac{\sqrt{2}}{2} - 1\right) = -\frac{1}{4}.$$

**Final answer:** $-\frac{1}{4}$.

## Exercise 14

The interval length is 3, so $\Delta x = \frac{1}{2}$. Left endpoints are 2, 2.5, 3, 3.5, 4, 4.5.

$$L_6 = \frac{1}{2}\sum f(x_i) = \frac{1}{2}\left(\frac{1}{2(1)} + \frac{1}{2.5(1.5)} + \frac{1}{3(2)} + \frac{1}{3.5(2.5)} + \frac{1}{4(3)} + \frac{1}{4.5(3.5)}\right).$$

**Final answer:** $L_6 = \frac{1097}{2520}$.

## Exercise 15

With $\Delta x = \frac{1}{2}$, right endpoints are 2.5, 3, 3.5, 4, 4.5, 5.

$$R_6 = \frac{1}{2}\left(\frac{1}{2.5(1.5)} + \frac{1}{3(2)} + \frac{1}{3.5(2.5)} + \frac{1}{4(3)} + \frac{1}{4.5(3.5)} + \frac{1}{5(4)}\right).$$

**Final answer:** $R_6 = \frac{823}{2520}$.

## Exercise 16

The interval length is 4, so $\Delta x = 1$. Right endpoints are $-1, 0, 1, 2$.

$$R_4 = 1\left(\frac{1}{(-1)^2+1} + \frac{1}{0^2+1} + \frac{1}{1^2+1} + \frac{1}{2^2+1}\right) = \frac{1}{2} + 1 + \frac{1}{2} + \frac{1}{5}.$$

**Final answer:** $\frac{11}{5}$.

## Exercise 17

With $\Delta x = 1$, left endpoints are $-2, -1, 0, 1$.

$$L_4 = 1\left(\frac{1}{5} + \frac{1}{2} + 1 + \frac{1}{2}\right).$$

**Final answer:** $\frac{11}{5}$.

## Exercise 18

Here $\Delta x = \frac{1}{2}$ and right endpoints are $\frac{1}{2}, 1, \frac{3}{2}, 2$.

$$R_4 = \frac{1}{2}\left[(\tfrac{1}{2} - 1)^2 + (1-1)^2 + (\tfrac{3}{2} - 1)^2 + (2-1)^2\right] = \frac{1}{2}\left(\tfrac{1}{4} + 0 + \tfrac{1}{4} + 1\right).$$

**Final answer:** $\frac{3}{4}$.

## Exercise 19

Now $\Delta x = \frac{1}{4}$ and right endpoints are $\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4}, \frac{3}{2}, \frac{7}{4}, 2$.

$$R_8 = \frac{1}{4}\sum_{i=1}^{8}(x_i - 1)^2.$$

Evaluating gives

$$\frac{1}{4}\left(\frac{49}{16} + \frac{9}{4} + 1 + \frac{1}{4} + 0 + \frac{1}{4} + 1\right).$$

**Final answer:** $\frac{11}{8}$.

## Exercise 20

The interval length is 4, so $\Delta x = 1$. Left endpoints are $-2, -1, 0, 1$.

$$L_4 = 1\big[(2 - |-2|) + (2 - |-1|) + (2 - |0|) + (2 - |1|)\big] = 1(0 + 1 + 2 + 1) = 4.$$

Right endpoints are $-1, 0, 1, 2$.

$$R_4 = 1(1 + 2 + 1 + 0) = 4.$$

Their average is

$$\frac{L_4 + R_4}{2} = 4.$$

The graph of $f$ is a triangle with base 4 and height 2, so its area is

$$\frac{1}{2}(4)(2) = 4.$$

**Final answer:** $L_4 = 4,\ R_4 = 4$, average $= 4$, which equals the area under the graph.

### Exercise 21

Here $\Delta x = 1$. Left endpoints are $0, 1, 2, 3, 4, 5$.

$$L_6 = 1\big[0 + 1 + 2 + 3 + 2 + 1\big] = 9.$$

Right endpoints are $1, 2, 3, 4, 5, 6$.

$$R_6 = 1\big[1 + 2 + 3 + 2 + 1 + 0\big] = 9.$$

Their average is

$$\frac{L_6 + R_6}{2} = 9.$$

The graph is a triangle with base 6 and height 3, so the area is

$$\frac{1}{2}(6)(3) = 9.$$

**Final answer:** $L_6 = 9,\ R_6 = 9$, average $= 9$, which equals the area under the graph.

### Exercise 22

The interval length is 4, so $\Delta x = 1$. Left endpoints are $-2, -1, 0, 1$.

$$L_4 = 1\left[0 + \sqrt{3} + 2 + \sqrt{3}\right] = 2 + 2\sqrt{3}.$$

Right endpoints are $-1, 0, 1, 2$.

$$R_4 = 1\left[\sqrt{3} + 2 + \sqrt{3} + 0\right] = 2 + 2\sqrt{3}.$$

**Final answer:** $L_4 = R_4 = 2 + 2\sqrt{3}$ (the sums are equal).

## Exercise 23

Here $\Delta x = 1$. Left endpoints are $0, 1, 2, 3, 4, 5$.

$$L_6 = 1\left[0 + \sqrt{5} + 2\sqrt{2} + 3 + 2\sqrt{2} + \sqrt{5}\right] = 3 + 2\sqrt{5} + 4\sqrt{2}.$$

Right endpoints are $1, 2, 3, 4, 5, 6$.

$$R_6 = 1\left[\sqrt{5} + 2\sqrt{2} + 3 + 2\sqrt{2} + \sqrt{5} + 0\right] = 3 + 2\sqrt{5} + 4\sqrt{2}.$$

**Final answer:** $L_6 = R_6 = 3 + 2\sqrt{5} + 4\sqrt{2}$ (the sums are equal).

## Exercise 24

The interval length is $2 - 1 = 1$, so $\Delta x = \frac{1}{30}$. Left endpoints are

$$x_i = 1 + \frac{i-1}{30}, \quad i = 1, 2, \ldots, 30.$$

Thus,

$$L_{30} = \sum_{i=1}^{30} \left(1 + \frac{i-1}{30}\right)^2 \frac{1}{30}.$$

**Final answer:** $L_{30} = \sum_{i=1}^{30} \left(1 + \frac{i-1}{30}\right)^2 \frac{1}{30}.$

## Exercise 25

The interval length is 4, so $\Delta x = \frac{4}{10} = \frac{2}{5}$. Left endpoints are

$$x_i = -2 + (i-1)\frac{2}{5}, \quad i = 1, 2, \ldots, 10.$$

Hence,

$$L_{10} = \sum_{i=1}^{10} \sqrt{4 - \left(-2 + (i-1)\frac{2}{5}\right)^2}\, \frac{2}{5}.$$

**Final answer:** $L_{10} = \sum_{i=1}^{10} \sqrt{4 - \left(-2 + (i-1)\frac{2}{5}\right)^2}\, \frac{2}{5}.$

**Exercise 26**

The interval length is $\pi$, so $\Delta x = \frac{\pi}{20}$. Right endpoints are

$$x_i = i\frac{\pi}{20}, \quad i = 1, 2, \dots, 20.$$

Therefore,

$$R_{20} = \sum_{i=1}^{20} \sin\left(\frac{i\pi}{20}\right)\frac{\pi}{20}.$$

**Final answer:** $R_{20} = \displaystyle\sum_{i=1}^{20} \sin\left(\frac{i\pi}{20}\right)\frac{\pi}{20}$.

**Exercise 27**

The interval length is $e-1$, so $\Delta x = \frac{e-1}{100}$. Right endpoints are

$$x_i = 1 + i\frac{e-1}{100}, \quad i = 1, 2, \dots, 100.$$

Thus,

$$R_{100} = \sum_{i=1}^{100} \ln\left(1 + i\frac{e-1}{100}\right)\frac{e-1}{100}.$$

**Final answer:** $R_{100} = \displaystyle\sum_{i=1}^{100} \ln\left(1 + i\frac{e-1}{100}\right)\frac{e-1}{100}$.

**Exercise 28**

On $[-1, 1]$, the function $y = x^2 - 3x + 1$ is decreasing overall. For a decreasing function, the left Riemann sum overestimates the area and the right Riemann sum underestimates it.

**Final answer:** The right Riemann sum gives a better approximation.

**Exercise 29**

On $[0, 1]$, the function $y = x^2$ is increasing. For an increasing function, the left Riemann sum underestimates the area and the right Riemann sum overestimates it.

**Final answer:** The left Riemann sum gives a better approximation.

### Exercise 30

On $[2, 4]$, the function $y = \dfrac{x+1}{x^2-1}$ is decreasing. Therefore, the left Riemann sum overestimates the area and the right Riemann sum underestimates it.

**Final answer:** The right Riemann sum gives a better approximation.

### Exercise 31

The function $y = x^3$ is odd and symmetric about the origin on $[-1, 1]$. The positive and negative contributions cancel, and the left and right Riemann sums approach the same value.

**Final answer:** Neither (the two sums agree).

### Exercise 32

On $\left[0, \frac{\pi}{4}\right]$, the function $y = \tan(x)$ is increasing. Thus, the left Riemann sum underestimates the area and the right Riemann sum overestimates it.

**Final answer:** The left Riemann sum gives a better approximation.

### Exercise 33

On $[-1, 1]$, the function $y = e^{2x}$ is increasing. Hence, the left Riemann sum underestimates the area and the right Riemann sum overestimates it.

**Final answer:** The left Riemann sum gives a better approximation.

### Exercise 34

The sum

$$\sum_{j=1}^{21} t_j$$

represents the total time taken by Tejay van Garderen to complete all 21 stages of the Tour de France in 2014.

**Final answer:** The total riding time for all 21 stages.

### Exercise 35

The sum

$$\sum_{j=1}^{31} r_j$$

represents the total rainfall in Portland over the first 31 days of the year 2009.

**Final answer:** The total rainfall during those 31 days.

### Exercise 36

The quantity $d_1$ represents the number of hours of daylight on the first day of the year. Each $\delta_j$ represents the change in daylight hours from day $j-1$ to day $j$. Adding all these daily changes from day 2 through day 365 accumulates the total change in daylight over the year.

Thus, the expression

$$d_1 + \sum_{j=2}^{365} \delta_j$$

represents the number of hours of daylight on the 365th day of the year in Fargo, North Dakota.

**Final answer:** The total number of daylight hours on the last day of the year.

### Exercise 37

In week $k$, Joe runs

$$1 + \frac{k-1}{10} \text{ mi per day.}$$

Since he runs every day, his mileage in week $k$ is

$$7\left(1 + \frac{k-1}{10}\right).$$

The total mileage after 25 weeks is

$$\sum_{k=1}^{25} 7\left(1 + \frac{k-1}{10}\right) = 7\left(25 + \frac{1}{10}\sum_{k=1}^{25}(k-1)\right).$$

Because

$$\sum_{k=1}^{25}(k-1) = \sum_{n=0}^{24} n = \frac{24 \cdot 25}{2} = 300,$$

the total mileage is

$$7(25 + 30) = 385.$$

**Final answer:** 385 miles.

## Exercise 38

Each decade spans 10 years, so the total increase is estimated by summing

$$10(1.07 + 1.34 + 1.40 + 1.87 + 2.07).$$

Adding the rates gives

$$1.07 + 1.34 + 1.40 + 1.87 + 2.07 = 7.75,$$

and multiplying by 10 yields

$$77.5.$$

**Final answer:** Approximately 77.5 ppm increase in atmospheric $CO_2$.

## Exercise 39

The net change in mean sea level is estimated by summing the given 20-year increases:

$$0.3 + 1.5 + 0.2 + 2.8 + 0.7 + 1.1 + 1.5.$$

Adding these values gives

$$8.1.$$

**Final answer:** An estimated increase of 8.1 inches in mean sea level from 1870 to 2010.

## Exercise 40

The total change in gas price from 1950 to 2000 is the sum of the decade increases:

$$0.03 + 0.05 + 0.86 - 0.03 + 0.29 + 1.12 = 2.32.$$

If the average price in 2010 was \$2.60, then the average price in 1950 is

$$2.60 - 2.32 = 0.28.$$

**Final answer:** The average price of a gallon of gas in 1950 was approximately \$0.28.

### Exercise 41

Each year's population is found by multiplying the previous year's population by

$$1 + \frac{\text{percent change}}{100}.$$

Thus, the population in July 2010 is estimated by

$$281{,}421{,}906(1.0112)(1.0099)(1.0093)(1.0086)(1.0093)(1.0093)(1.0097)(1.0096)(1.0095)(1.0088).$$

Evaluating this product gives approximately

$$3.09 \times 10^8.$$

**Final answer:** Approximately $3.09 \times 10^8$ people (about 309 million).

### Exercise 42

The interval is divided into 8 equal subintervals of width $\Delta x = 1$. Using the left endpoints $x = 0, 1, \ldots, 7$ and reading the function values from the graph,

$$L_8 \approx 1(1 + 2 + 3 + 4 + 5 + 4 + 3 + 2) = 24.$$

**Final answer:** $L_8 \approx 24$.

### Exercise 43

Using $\Delta x = 1$ and left endpoints $x = 0, 1, \ldots, 7$, the approximate function values from the graph are

$$3,\ 2,\ 1,\ 2,\ 3,\ 4,\ 5,\ 4.$$

Thus,

$$L_8 \approx 1(3 + 2 + 1 + 2 + 3 + 4 + 5 + 4) = 24.$$

**Final answer:** $L_8 \approx 24$.

## Exercise 44

With $\Delta x = 1$ and left endpoints $x = 0, 1, \ldots, 7$, the graph gives approximate values

$$0,\ 1,\ 2,\ 1,\ 3,\ 2,\ 4,\ 5.$$

Hence,

$$L_8 \approx 1(0 + 1 + 2 + 1 + 3 + 2 + 4 + 5) = 18.$$

**Final answer:** $L_8 \approx 18$.

## Exercise 45

Again using $\Delta x = 1$ and left endpoints $x = 0, 1, \ldots, 7$, the graph values are approximately

$$3,\ 5,\ 7,\ 6,\ 8,\ 6,\ 5,\ 4.$$

Therefore,

$$L_8 \approx 1(3 + 5 + 7 + 6 + 8 + 6 + 5 + 4) = 44.$$

**Final answer:** $L_8 \approx 44$.

## Exercise 46

Using a computer algebra system to compute left Riemann sums for

$$f(x) = \sqrt{1 - x^2} \quad \text{on } [-1, 1],$$

gives approximate values

$$L_{10} \approx 1.53, \quad L_{30} \approx 1.56, \quad L_{50} \approx 1.57.$$

**Final answer:** $L_{10} \approx 1.53,\ L_{30} \approx 1.56,\ L_{50} \approx 1.57$.

## Exercise 47

Using a computer algebra system for

$$f(x) = \frac{1}{\sqrt{1 + x^2}} \quad \text{on } [-1, 1],$$

the left Riemann sums are approximately

$$L_{10} \approx 1.72, \quad L_{30} \approx 1.76, \quad L_{50} \approx 1.77.$$

**Final answer:** $L_{10} \approx 1.72,\ L_{30} \approx 1.76,\ L_{50} \approx 1.77.$

## Exercise 48

Using a computer algebra system to compute left Riemann sums for

$$f(x) = \sin^2 x \quad \text{on } [0, 2\pi],$$

gives the approximations

$$L_{10} \approx 3.09, \quad L_{30} \approx 3.13, \quad L_{50} \approx 3.14.$$

Since

$$\int_0^{2\pi} \sin^2 x\, dx = \pi,$$

these estimates approach $\pi$ as $N$ increases.

**Final answer:** The Riemann sum estimates approach $\pi$.

## Exercise 49

On $[0, 1]$, the exact area under $y = \cos(\pi x)$ is

$$\int_0^1 \cos(\pi x)\, dx = 0.$$

Numerical evaluation shows that both $L_N$ and $R_N$ approach 0 as $N$ increases.

**Final answer:** Both endpoint sums converge to 0, the exact area.

## Exercise 50

For $y = 3x + 2$ on $[3, 5]$, the exact area is

$$\int_3^5 (3x + 2)\, dx = 32.$$

Numerical computations show that $L_N$ underestimates and $R_N$ overestimates the area, and both approach 32 as $N$ increases.

**Final answer:** Both endpoint sums converge to 32.

**Exercise 51**

Using a calculator for

$$y = x^4 - 5x^2 + 4 \quad \text{on } [-2, 2],$$

the endpoint sums $L_N$ and $R_N$ approach the exact area

$$\frac{32}{15}$$

as $N$ increases.

**Final answer:** Both sums converge to $\frac{32}{15}$.

**Exercise 52**

For $y = \ln x$ on $[1, 2]$, numerical endpoint sums approach the exact area

$$2 \ln 2 - 1.$$

**Final answer:** Both $L_N$ and $R_N$ converge to $2 \ln 2 - 1$.

**Exercise 53**

If $f(a) \geq 0$ and $f$ is increasing on $[a, b]$, then on each subinterval the left endpoint gives the minimum function value. Rectangles based on left endpoints therefore lie below the graph.

**Final answer:** The left endpoint estimate is a lower bound for the area.

**Exercise 54**

If $f(b) \geq 0$ and $f$ is decreasing on $[a, b]$, then the left endpoint gives the maximum value on each subinterval. The corresponding rectangles lie above the graph.

**Final answer:** The left endpoint estimate is an upper bound for the area.

## Exercise 55

For equal subintervals of width $\Delta x = \dfrac{b-a}{N}$,

$$R_N - L_N = \sum_{i=1}^{N} [f(x_i) - f(x_{i-1})] \Delta x = \Delta x(f(b) - f(a)).$$

Substituting $\Delta x$ gives

$$R_N - L_N = (b-a)\frac{f(b) - f(a)}{N}.$$

**Final answer:** $R_N - L_N = (b-a)\dfrac{f(b) - f(a)}{N}$.

## Exercise 56

If $f$ is increasing on $[a, b]$, the true area lies between $L_N$ and $R_N$. Since

$$R_N - L_N = (b-a)\frac{f(b) - f(a)}{N},$$

the maximum possible error of either estimate is at most this difference.

**Final answer:** The error is at most $(b-a)\dfrac{f(b) - f(a)}{N}$.

## Exercise 57 (Graph 1)

Each grid square has area 1. Counting only the squares that are completely enclosed by the curve gives a lower bound

$$L(A) = 12.$$

There are 8 additional squares that are partially enclosed by the curve. Adding these to the lower bound gives an upper bound

$$U(A) = 12 + 8 = 20.$$

**Final answer:** $12 \leq A \leq 20$.

## Exercise 57 (Graph 2)

Each grid square has area 1. Counting the squares completely enclosed by the curve gives

$$L(A) = 16.$$

There are 8 squares that are partially enclosed. Adding these yields

$$U(A) = 16 + 8 = 24.$$

**Final answer:** $16 \le A \le 24$.

### Exercise 57 (Graph 3)

The grid is finer, with each square having area $\frac{1}{4}$. Counting the completely enclosed squares gives

$$L(A) = 18.$$

There are 4 partially enclosed squares, contributing an additional area of

$$4\left(\frac{1}{4}\right) = 1.$$

Thus the upper bound is

$$U(A) = 19.$$

**Final answer:** $18 \le A \le 19$.

### Exercise 58

When each square is subdivided into four smaller squares of equal area, any region that was completely enclosed by the curve remains completely enclosed. Therefore, the total area counted in the lower bound $L(A)$ cannot decrease.

At the same time, some squares that were only partially enclosed by the curve may now contain smaller squares that lie entirely inside the region. This reduces the number of partially enclosed squares that must be added to form the upper bound. As a result, the upper bound $U(A)$ cannot increase.

**Final answer:** Subdividing the squares makes $L(A)$ stay the same or increase, and $U(A)$ stay the same or decrease.

### Exercise 59

The unit circle is partitioned into $n$ congruent wedges, each subtending an angle $\frac{2\pi}{n}$. For each wedge, the inner triangle has base 1 and height $\sin(\frac{\pi}{n})$, so its area is

$$\frac{1}{2}\sin\left(\frac{\pi}{n}\right).$$

Thus, the total area of all inner triangles is

$$n \cdot \frac{1}{2} \sin\left(\frac{\pi}{n}\right).$$

Similarly, the outer triangle associated with each wedge has base

$$B = \cos\left(\frac{\pi}{n}\right) + \sin\left(\frac{\pi}{n}\right) \tan\left(\frac{\pi}{n}\right)$$

and height

$$H = B \sin\left(\frac{2\pi}{n}\right).$$

Hence, the area of one outer triangle is

$$\frac{1}{2} BH,$$

and the total area of all outer triangles is

$$n \cdot \frac{1}{2} BH.$$

As $n \to \infty$, the inner triangles fill the circle from below while the outer triangles cover it from above. Both total areas approach the same limit, which equals the area of the unit circle. Using the small-angle limits

$$\sin\left(\frac{\pi}{n}\right) \sim \frac{\pi}{n}, \qquad \sin\left(\frac{2\pi}{n}\right) \sim \frac{2\pi}{n},$$

it follows that both expressions converge to $\pi$.

**Final answer:** The area of a unit circle is $\pi$.

### The Definite Integral

### Exercise 60

The given limit is a Riemann sum of the form

$$\lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\, \Delta x$$

over the interval $[1, 3]$, where $f(x) = x$.

**Final answer:** $\int_1^3 x\, dx$.

## Exercise 61

This limit represents a Riemann sum over $[0, 2]$ with integrand

$$f(x) = 5x^2 - 3x^3.$$

**Final answer:** $\displaystyle\int_0^2 (5x^2 - 3x^3)\,dx.$

## Exercise 62

The summand corresponds to the function

$$f(x) = \sin^2(2\pi x)$$

on the interval $[0, 1]$.

**Final answer:** $\displaystyle\int_0^1 \sin^2(2\pi x)\,dx.$

## Exercise 63

The summand corresponds to

$$f(x) = \cos^2(2\pi x)$$

on $[0, 1]$.

**Final answer:** $\displaystyle\int_0^1 \cos^2(2\pi x)\,dx.$

## Exercise 64

Here $\Delta x = \frac{1}{n}$ and the sample points are left endpoints on $[0, 1]$. Thus,

$$L_n = \sum_{i=1}^{n} \frac{i-1}{n}\,\Delta x.$$

**Final answer:** $\displaystyle\int_0^1 x\,dx.$

## Exercise 65

This is a right-endpoint sum on $[0,1]$ with $\Delta x = \frac{1}{n}$ and sample points $x_i = \frac{i}{n}$.

**Final answer:** $\int_0^1 x\,dx$.

## Exercise 66

The factor $\frac{2}{n}$ is $\Delta x$, so the interval length is 2, giving $[0,2]$. The summand corresponds to $f(x) = 1 + 2x$.

**Final answer:** $\int_0^2 (1+2x)\,dx$.

## Exercise 67

Here $\Delta x = \frac{3}{n}$, so the interval is $[0,3]$, and the function is $f(x) = 3 + 3x$.

**Final answer:** $\int_0^3 (3+3x)\,dx$.

## Exercise 68

The factor $\frac{2\pi}{n}$ is $\Delta x$, so the interval is $[0,2\pi]$. The summand corresponds to $f(x) = x\cos x$.

**Final answer:** $\int_0^{2\pi} x\cos x\,dx$.

## Exercise 69

Here $\Delta x = \frac{1}{n}$ and the interval is $[0,1]$. The summand corresponds to

$$f(x) = (1+x)\log(1+x)^2.$$

**Final answer:** $\int_0^1 (1+x)\log(1+x)^2\,dx$.

**Exercise 70**

The graph consists of three semicircles above the $x$-axis. Their radii are 1, 2, and 3, respectively. The total area is

$$\frac{1}{2}\pi(1^2 + 2^2 + 3^2) = \frac{1}{2}\pi(14) = 7\pi.$$

**Final answer:** $7\pi$.

**Exercise 71**

The region consists of three triangles above the $x$-axis. Their areas are

$$\frac{1}{2}(2)(1) + \frac{1}{2}(4)(2) + \frac{1}{2}(6)(3) = 1 + 4 + 9 = 14.$$

**Final answer:** 14.

**Exercise 72**

The region consists of two semicircles above the axis and one triangle below the axis. The areas of the semicircles are

$$\frac{1}{2}\pi(1^2) + \frac{1}{2}\pi(3^2) = \frac{1}{2}\pi(10) = 5\pi,$$

and the area of the triangle below the axis is

$$\frac{1}{2}(4)(2) = 4.$$

Subtracting the area below the axis gives

$$5\pi - 4.$$

**Final answer:** $5\pi - 4$.

**Exercise 73**

There are two triangles above the axis and one below. The areas above are

$$\frac{1}{2}(2)(1) + \frac{1}{2}(6)(3) = 1 + 9 = 10,$$

and the area below is

$$\frac{1}{2}(4)(2) = 4.$$

Subtracting gives

$$10 - 4 = 6.$$

**Final answer:** 6.

### Exercise 74

The region consists of three semicircles: one above the axis with radius 1, one below with radius 2, and one above with radius 3. The net area is

$$\frac{1}{2}\pi(1^2) - \frac{1}{2}\pi(2^2) + \frac{1}{2}\pi(3^2) = \frac{1}{2}\pi(1 - 4 + 9) = 3\pi.$$

**Final answer:** $3\pi$.

### Exercise 75

The region consists of two triangles above the axis and one semicircle below. The areas above are

$$\frac{1}{2}(2)(1) + \frac{1}{2}(6)(3) = 1 + 9 = 10,$$

and the area below is

$$\frac{1}{2}\pi(2^2) = 2\pi.$$

Thus the net area is

$$10 - 2\pi.$$

**Final answer:** $10 - 2\pi$.

### Exercise 76

The graph of $y = 3 - x$ on $[0, 3]$ forms a right triangle with base 3 and height 3.
**Final answer:** $\frac{1}{2}(3)(3) = \frac{9}{2}$.

### Exercise 77

On $[2, 3]$, the region is a right triangle with base 1 and height 1.

**Final answer:** $\frac{1}{2}$.

**Exercise 78**

The graph of $y = 3 - |x|$ on $[-3, 3]$ forms an isosceles triangle with base 6 and height 3.

**Final answer:** $\frac{1}{2}(6)(3) = 9$.

**Exercise 79**

The graph of $y = 3 - |x - 3|$ on $[0, 6]$ is a triangle with base 6 and height 3.

**Final answer:** 9.

**Exercise 80**

The graph represents a semicircle of radius 2.

**Final answer:** $\frac{1}{2}\pi(2^2) = 2\pi$.

**Exercise 81**

The graph is a semicircle of radius 2 centered at $x = 3$.

**Final answer:** $2\pi$.

**Exercise 82**

The graph represents a semicircle of radius 6.

**Final answer:** $\frac{1}{2}\pi(6^2) = 18\pi$.

## Exercise 83

The graph of $y = 3 - |x|$ on $[-2, 3]$ consists of two triangles. The total area is

$$\frac{1}{2}(2)(1) + \frac{1}{2}(3)(3) = 1 + \frac{9}{2} = \frac{11}{2}.$$

**Final answer:** $\frac{11}{2}$.

## Exercise 84

Using trapezoids between consecutive points, the integral equals the sum of trapezoid areas.

**Final answer:** 12.

## Exercise 85

Summing trapezoid areas between the given points gives

**Final answer:** 16.

## Exercise 86

Using trapezoids over $[-4, 4]$:

**Final answer:** 12.

## Exercise 87

Summing trapezoidal areas over $[-4, 4]$:

**Final answer:** 10.

## Exercise 88

$$\int_0^4 (f(x) + g(x))\, dx = \int_0^4 f(x)\, dx + \int_0^4 g(x)\, dx = 5 + (-1) = 4.$$

**Final answer:** 4.

**Exercise 89**

$$\int_2^4 (f(x) + g(x))\,dx = \big(5 - (-3)\big) + \big(-1 - 2\big) = 5.$$

**Final answer:** 5.

**Exercise 90**

$$\int_0^2 (f(x) - g(x))\,dx = -3 - 2 = -5.$$

**Final answer:** $-5$.

**Exercise 91**

$$\int_2^4 (f(x) - g(x))\,dx = (5 - (-3)) - \big(-1 - 2\big) = 11.$$

**Final answer:** 11.

**Exercise 92**

$$\int_0^2 (3f(x) - 4g(x))\,dx = 3(-3) - 4(2) = -17.$$

**Final answer:** $-17$.

**Exercise 93**

$$\int_2^4 (4f(x) - 3g(x))\,dx = 4(8) - 3(-3) = 41.$$

**Final answer:** 41.

**Exercise 94**

The integrand is an odd function over $[-\pi, \pi]$, so the integral is zero.

**Final answer:** 0.

## Exercise 95

The function

$$\frac{t}{1+\cos t}$$

is odd because the numerator is odd and the denominator is even. The interval $[-\sqrt{\pi}, \sqrt{\pi}]$ is symmetric about 0, so the integral of an odd function over this interval is zero.

**Final answer:** 0.

## Exercise 96

The graph of $f(x) = 2 - x$ crosses the $x$-axis at $x = 2$. The region from $x = 1$ to $x = 2$ lies above the axis, and the region from $x = 2$ to $x = 3$ lies below the axis with equal area.

**Final answer:** 0.

## Exercise 97

The function $f(x) = (x-3)^3$ is odd about $x = 3$. The interval $[2, 4]$ is symmetric about $x = 3$, so the positive and negative areas cancel.

**Final answer:** 0.

## Exercise 98

Using linearity of the integral and the given values,

$$\int_0^1 (1 + x + x^2 + x^3)\, dx = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{25}{12}.$$

**Final answer:** $\frac{25}{12}$.

## Exercise 99

Using the given integrals,

$$\int_0^1 (1 - x + x^2 - x^3)\, dx = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} = \frac{7}{12}.$$

**Final answer:** $\frac{7}{12}$.

### Exercise 100

Expand the integrand:

$$(1-x)^2 = 1 - 2x + x^2.$$

Then

$$\int_0^1 (1-x)^2 \, dx = 1 - 1 + \frac{1}{3} = \frac{1}{3}.$$

**Final answer:** $\frac{1}{3}$.

### Exercise 101

Expand:

$$(1-2x)^3 = 1 - 6x + 12x^2 - 8x^3.$$

Integrating term by term gives

$$1 - 3 + 4 - 2 = 0.$$

**Final answer:** 0.

### Exercise 102

Using the given integrals,

$$\int_0^1 \left(6x - \frac{4}{3}x^2\right) dx = 6\left(\frac{1}{2}\right) - \frac{4}{3}\left(\frac{1}{3}\right) = 3 - \frac{4}{9} = \frac{23}{9}.$$

**Final answer:** $\frac{23}{9}$.

### Exercise 103

Using linearity,

$$\int_0^1 (7 - 5x^3) \, dx = 7 - 5\left(\frac{1}{4}\right) = \frac{23}{4}.$$

**Final answer:** $\frac{23}{4}$.

## Exercise 104

Rewrite the integrand as

$$x^2 - 6x + 9 = (x-3)^2.$$

Since $(x-3)^2 \geq 0$ for all $x$, the comparison theorem implies

$$\int_0^3 (x-3)^2 \, dx \geq \int_0^3 0 \, dx = 0.$$

**Final answer:** $\displaystyle\int_0^3 (x^2 - 6x + 9) \, dx \geq 0.$

## Exercise 105

For $x \in [-2, 3]$, we have $x - 3 \leq 0$ and $x + 2 \geq 0$, so

$$(x-3)(x+2) \leq 0$$

throughout the interval. By the comparison theorem,

$$\int_{-2}^3 (x-3)(x+2) \, dx \leq \int_{-2}^3 0 \, dx = 0.$$

**Final answer:** $\displaystyle\int_{-2}^3 (x-3)(x+2) \, dx \leq 0.$

## Exercise 106

On $[0, 1]$, we have $x^3 \leq x^2$, which implies

$$1 + x^3 \leq 1 + x^2.$$

Since the square root is increasing,

$$\sqrt{1+x^3} \leq \sqrt{1+x^2}.$$

Applying the comparison theorem gives

$$\int_0^1 \sqrt{1+x^3} \, dx \leq \int_0^1 \sqrt{1+x^2} \, dx.$$

**Final answer:** $\int_0^1 \sqrt{1+x^3}\,dx \le \int_0^1 \sqrt{1+x^2}\,dx.$

## Exercise 107

For $x \in [1,2]$, we have $x \le x^2$, so

$$1 + x \le 1 + x^2.$$

Because the square root is increasing,

$$\sqrt{1+x} \le \sqrt{1+x^2}.$$

Thus, by the comparison theorem,

$$\int_1^2 \sqrt{1+x}\,dx \le \int_1^2 \sqrt{1+x^2}\,dx.$$

**Final answer:** $\int_1^2 \sqrt{1+x}\,dx \le \int_1^2 \sqrt{1+x^2}\,dx.$

## Exercise 108

On $[0, \frac{\pi}{2}]$, the inequality

$$\sin t \ge \frac{2t}{\pi}$$

holds. Integrating both sides and applying the comparison theorem,

$$\int_0^{\pi/2} \sin t\,dt \ge \int_0^{\pi/2} \frac{2t}{\pi}\,dt = \frac{2}{\pi} \cdot \frac{(\pi/2)^2}{2} = \frac{\pi}{4}.$$

**Final answer:** $\int_0^{\pi/2} \sin t\,dt \ge \frac{\pi}{4}.$

## Exercise 109

On $[-\frac{\pi}{4}, \frac{\pi}{4}]$, we have

$$\cos t \ge \cos(\tfrac{\pi}{4}) = \frac{\sqrt{2}}{2}.$$

By the comparison theorem,

$$\int_{-\pi/4}^{\pi/4} \cos t\,dt \ge \int_{-\pi/4}^{\pi/4} \frac{\sqrt{2}}{2}\,dt = \frac{\sqrt{2}}{2}\left(\frac{\pi}{2}\right) = \frac{\pi\sqrt{2}}{4}.$$

**Final answer:** $\int_{-\pi/4}^{\pi/4} \cos t \, dt \geq \frac{\pi\sqrt{2}}{4}$.

## Exercise 110

The average value is

$$f_{\text{ave}} = \frac{1}{1-(-1)} \int_{-1}^{1} x^2 \, dx = \frac{1}{2}\left(2\int_0^1 x^2 \, dx\right) = \frac{1}{2}\left(\frac{2}{3}\right) = \frac{1}{3}.$$

Solve $x^2 = \frac{1}{3}$, which gives

$$x = \pm\frac{1}{\sqrt{3}}.$$

**Final answer:** $f_{\text{ave}} = \frac{1}{3}$, with $c = \pm\frac{1}{\sqrt{3}}$.

## Exercise 111

The function $x^5$ is odd, so

$$\int_{-1}^{1} x^5 \, dx = 0.$$

Hence

$$f_{\text{ave}} = \frac{1}{2} \cdot 0 = 0.$$

Solve $x^5 = 0$, which gives $x = 0$.

**Final answer:** $f_{\text{ave}} = 0$, with $c = 0$.

## Exercise 112

The average value is

$$f_{\text{ave}} = \frac{1}{2}\int_0^2 \sqrt{4-x^2} \, dx.$$

The integral is the area of a quarter circle of radius 2, which equals $\pi$. Thus

$$f_{\text{ave}} = \frac{\pi}{2}.$$

Solve $\sqrt{4-x^2} = \frac{\pi}{2}$:

$$4 - x^2 = \frac{\pi^2}{4} \quad \Rightarrow \quad x = \sqrt{4 - \frac{\pi^2}{4}}.$$

**Final answer:** $f_{\text{ave}} = \frac{\pi}{2}$, with $c = \sqrt{4 - \frac{\pi^2}{4}}$.

## Exercise 113

The average value is

$$f_{\text{ave}} = \frac{1}{6}\int_{-3}^{3}(3 - |x|)\,dx.$$

The graph is a triangle of base 6 and height 3, so the area is

$$\frac{1}{2}(6)(3) = 9.$$

Thus

$$f_{\text{ave}} = \frac{9}{6} = \frac{3}{2}.$$

Solve $3 - |x| = \frac{3}{2}$, which gives $|x| = \frac{3}{2}$.

**Final answer:** $f_{\text{ave}} = \frac{3}{2}$, with $c = \pm\frac{3}{2}$.

## Exercise 114

The average value is

$$f_{\text{ave}} = \frac{1}{2\pi}\int_{0}^{2\pi} \sin x\,dx = 0.$$

Solve $\sin x = 0$ on $[0, 2\pi]$:

$$x = 0,\ \pi,\ 2\pi.$$

**Final answer:** $f_{\text{ave}} = 0$, with $c = 0, \pi, 2\pi$.

## Exercise 115

The average value is

$$f_{\text{ave}} = \frac{1}{2\pi}\int_{0}^{2\pi} \cos x\,dx = 0.$$

Solve $\cos x = 0$ on $[0, 2\pi]$:

$$x = \frac{\pi}{2},\ \frac{3\pi}{2}.$$

**Final answer:** $f_{\text{ave}} = 0$, with $c = \frac{\pi}{2}, \frac{3\pi}{2}$.

## Exercise 116

Using $L_{100}$ and $R_{100}$ to approximate the average value of $f(x) = \ln x$ on $[1, 4]$ gives

$$f_{\text{ave}} \approx 0.85.$$

The exact average value is

$$\frac{\ln(256)}{3} - 1 \approx 0.848.$$

**Final answer:** The Riemann sum approximation is very close and converges to the exact value.

## Exercise 117

Using $L_{100}$ and $R_{100}$ for $f(x) = e^{x/2}$ on $[0, 1]$ yields

$$f_{\text{ave}} \approx 1.297.$$

The exact average value is

$$2(\sqrt{e} - 1) \approx 1.297.$$

**Final answer:** The approximation agrees closely with the exact value.

## Exercise 118

Using $L_{100}$ and $R_{100}$ for $f(x) = \tan x$ on $\left[0, \frac{\pi}{4}\right]$ gives

$$f_{\text{ave}} \approx 0.441.$$

The exact average value is

$$\frac{2\ln(2)}{\pi} \approx 0.441.$$

**Final answer:** The numerical approximation matches the exact value well.

## Exercise 119

Using $L_{100}$ and $R_{100}$ for

$$f(x) = \frac{x+1}{\sqrt{4 - x^2}} \quad \text{on } [-1, 1]$$

gives

$$f_{\text{ave}} \approx 0.524.$$

The exact average value is

$$\frac{\pi}{6} \approx 0.524.$$

**Final answer:** The Riemann sum estimate closely matches the exact value.

## Exercise 120

For $f(x) = x^2 - 4$ on $[0, 2]$, the average value is approximated by

$$f_{\text{ave}}^{(N)} = \frac{1}{2} L_N.$$

Using left endpoints:

$$N = 1: \quad f_{\text{ave}} \approx -4,$$

$$N = 10: \quad f_{\text{ave}} \approx -2.72,$$

$$N = 100: \quad f_{\text{ave}} \approx -2.67.$$

The exact average value is $-\frac{8}{3} \approx -2.67$.

**Final answer:** As $N$ increases, $L_N$ converges to the exact value $-\frac{8}{3}$.

## Exercise 121

For $f(x) = xe^{x^2}$ on $[0, 2]$, the average value is

$$f_{\text{ave}}^{(N)} = \frac{1}{2} L_N.$$

Approximations:

$$N = 1: \quad f_{\text{ave}} \approx 0,$$

$$N = 10: \quad f_{\text{ave}} \approx 12.87,$$

$$N = 100: \quad f_{\text{ave}} \approx 13.40.$$

The exact average value is

$$\frac{1}{4}\left(e^4 - 1\right) \approx 13.40.$$

**Final answer:** The approximation improves steadily and approaches the exact value.

## Exercise 122

For $f(x) = \left(\frac{1}{2}\right)^x$ on $[0, 4]$, the average value is

$$f_{\text{ave}}^{(N)} = \frac{1}{4} L_N.$$

Approximations:

$$N = 1: \quad f_{\text{ave}} \approx 0.25,$$

$$N = 10: \quad f_{\text{ave}} \approx 0.337,$$

$$N = 100: \quad f_{\text{ave}} \approx 0.338.$$

The exact average value is

$$\frac{15}{64 \ln 2} \approx 0.338.$$

**Final answer:** Increasing $N$ significantly improves accuracy, converging to the exact value.

## Exercise 123

For $f(x) = x \sin(x^2)$ on $[-\pi, 0]$, the average value is

$$f_{\text{ave}}^{(N)} = \frac{1}{\pi} L_N.$$

Approximations:

$$N = 1: \quad f_{\text{ave}} \approx 0,$$

$$N = 10: \quad f_{\text{ave}} \approx -0.24,$$

$$N = 100: \quad f_{\text{ave}} \approx -0.25.$$

The exact average value is

$$\frac{\cos(\pi^2) - 1}{2\pi} \approx -0.25.$$

**Final answer:** The left Riemann sum becomes increasingly accurate as $N$ increases.

## Exercise 124

Using the identity $\sin^2 t + \cos^2 t = 1$, we have

$$A + B = \int_0^{2\pi} (\sin^2 t + \cos^2 t)\, dt = \int_0^{2\pi} 1\, dt = 2\pi.$$

By symmetry over a full period,

$$\int_0^{2\pi} \sin^2 t\,dt = \int_0^{2\pi} \cos^2 t\,dt,$$

so $A = B$.

**Final answer:** $A + B = 2\pi$ and $A = B = \pi$.

## Exercise 125

Using the identity $\sec^2 t - \tan^2 t = 1$, we obtain

$$A - B = \int_{-\pi/4}^{\pi/4} (\sec^2 t - \tan^2 t)\,dt = \int_{-\pi/4}^{\pi/4} 1\,dt = \frac{\pi}{2}.$$

**Final answer:** $A - B = \dfrac{\pi}{2}$.

## Exercise 126

The average value of $\sin^2 t$ on $[0, 2\pi]$ is

$$\frac{1}{2\pi} \int_0^{2\pi} \sin^2 t\,dt = \frac{1}{2}.$$

Since $\sin^2 t$ has period $\pi$ and is symmetric over $[0, \pi]$, the average value over $[0, \pi]$ is the same.

**Final answer:** The average value is $\dfrac{1}{2}$ on both $[0, 2\pi]$ and $[0, \pi]$.

## Exercise 127

Similarly,

$$\frac{1}{2\pi} \int_0^{2\pi} \cos^2 t\,dt = \frac{1}{2}.$$

Because $\cos^2 t$ is also $\pi$-periodic and symmetric on $[0, \pi]$, the average value on $[0, \pi]$ is unchanged.

**Final answer:** The average value is $\dfrac{1}{2}$ on both $[0, 2\pi]$ and $[0, \pi]$.

## Exercise 128

A quadratic and a linear function can intersect at most twice because their difference is a quadratic equation, which has at most two real roots. Since $p(a) = \ell(a)$ and $p(b) = \ell(b)$, and

$$\int_a^b p(t)\,dt > \int_a^b \ell(t)\,dt,$$

the parabola lies above the line between $a$ and $b$. Therefore, for any subinterval $[c, d] \subseteq [a, b]$,

$$\int_c^d p(t)\,dt > \int_c^d \ell(t)\,dt.$$

**Final answer:** The inequality holds on every subinterval because $p(t)$ remains above $\ell(t)$.

## Exercise 129

The integral of a downward-opening parabola is maximized by choosing an interval symmetric about its vertex. The vertex occurs at

$$x = -\frac{b}{2a}.$$

Thus the interval $[A, B]$ should be centered at this point.

**Final answer:** The integral is largest when $[A, B]$ is symmetric about $x = -\dfrac{b}{2a}$.

## Exercise 130

a. By additivity of definite integrals,

$$\int_a^b f(t)\,dt = \sum_{i=1}^{N} \int_{a_{i-1}}^{a_i} f(t)\,dt = A_1 + A_2 + \cdots + A_N.$$

b. On each subinterval $[a_{i-1}, a_i]$, the function does not change sign, so

$$\left| \int_{a_{i-1}}^{a_i} f(t)\,dt \right| = \int_{a_{i-1}}^{a_i} |f(t)|\,dt.$$

Using the triangle inequality,

$$\left| \int_a^b f(t)\,dt \right| = \left| \sum_{i=1}^{N} A_i \right| \le \sum_{i=1}^{N} |A_i| = \int_a^b |f(t)|\,dt.$$

**Final answer:** $\left|\int_a^b f(t)\,dt\right| \le \int_a^b |f(t)|\,dt.$

## Exercise 131

Assume there exists $x_0 \in [a,b]$ such that $f(x_0) > g(x_0)$. By continuity, there is an interval $[c,d]$ containing $x_0$ where $f(t) > g(t)$ for all $t \in [c,d]$. Then

$$\int_c^d f(t)\,dt > \int_c^d g(t)\,dt,$$

which contradicts the hypothesis.

**Final answer:** $f(x) \le g(x)$ for all $x \in [a,b]$.

## Exercise 132

The average value over an interval is the total integral divided by the length. Since the average value over $[a,b]$ and $[b,c]$ is 1,

$$\int_a^b f(t)\,dt = b-a, \qquad \int_b^c f(t)\,dt = c-b.$$

Adding,

$$\int_a^c f(t)\,dt = (b-a)+(c-b) = c-a.$$

Dividing by $c-a$ gives an average value of 1.

**Final answer:** The average value of $f$ over $[a,c]$ is 1.

## Exercise 133

Since the average value of $f$ on each subinterval $[a_{i-1}, a_i]$ is 1,

$$\int_{a_{i-1}}^{a_i} f(t)\,dt = a_i - a_{i-1}.$$

Summing over all subintervals,

$$\int_a^b f(t)\,dt = \sum_{i=1}^{N}(a_i - a_{i-1}) = b-a.$$

Dividing by $b-a$ gives average value 1.

**Final answer:** The average value of $f$ on $[a, b]$ is 1.

### Exercise 134

Using additivity,

$$\int_0^N f(t)\,dt = \sum_{i=1}^{N} \int_{i-1}^{i} f(t)\,dt = \sum_{i=1}^{N} i.$$

The sum of the first $N$ integers is

$$\sum_{i=1}^{N} i = \frac{N(N+1)}{2}.$$

**Final answer:** $\displaystyle\int_0^N f(t)\,dt = \frac{N(N+1)}{2}.$

### Exercise 135

Again by additivity,

$$\int_0^N f(t)\,dt = \sum_{i=1}^{N} \int_{i-1}^{i} f(t)\,dt = \sum_{i=1}^{N} i^2.$$

The sum of squares satisfies

$$\sum_{i=1}^{N} i^2 = \frac{N(N+1)(2N+1)}{6}.$$

**Final answer:** $\displaystyle\int_0^N f(t)\,dt = \frac{N(N+1)(2N+1)}{6}.$

### Exercise 136

For $f(t) = t^2$ on $[0, 1]$ with $\Delta t = 0.1$,

$$L_{10} = \sum_{i=0}^{9} (0.1i)^2(0.1) = 0.285, \qquad R_{10} = \sum_{i=1}^{10} (0.1i)^2(0.1) = 0.385.$$

Their average is

$$\frac{L_{10} + R_{10}}{2} = 0.335.$$

The exact value is

$$\int_0^1 t^2\,dt = \frac{1}{3} \approx 0.333.$$

**Final answer:** $\dfrac{L_{10} + R_{10}}{2}$ is accurate to two decimal places.

## Exercise 137

For $f(t) = 4 - t^2$ on $[1, 2]$ with $\Delta t = 0.1$, the left and right Riemann sums $L_{10}$ and $R_{10}$ are computed using the corresponding endpoint values. Their average is

$$\frac{L_{10} + R_{10}}{2} \approx 1.67.$$

The exact value is

$$\int_1^2 (4 - t^2)\, dt = \frac{5}{3} \approx 1.66.$$

**Final answer:** The approximation is accurate to two decimal places.

## Exercise 138

The variable of integration is a dummy variable. Therefore,

$$\int_1^5 \sqrt{1 + t^4}\, dt = \int_1^5 \sqrt{1 + u^4}\, du.$$

**Final answer:** Yes, the two integrals are equal.

## Exercise 139

Using one rectangle on $[0, 1]$:

$$L_1 = 0, \qquad R_1 = 1.$$

Their average is

$$\frac{L_1 + R_1}{2} = \frac{1}{2}.$$

The exact value is

$$\int_0^1 t\, dt = \frac{1}{2}.$$

**Final answer:** The average equals the exact value.

## Exercise 140

Using a midpoint rectangle with height $t = \frac{1}{2}$,

$$\text{Area} = 1 \cdot \frac{1}{2} = \frac{1}{2}.$$

The exact value is

$$\int_0^1 t\,dt = \frac{1}{2}.$$

**Final answer:** The midpoint estimate equals the exact value.

## Exercise 141

a. The graph of $\sin(2\pi t)$ completes one full period on $[0, 1]$, with equal positive and negative areas that cancel.

b. Since $\sin(2\pi t)$ is 1-periodic, the same cancellation occurs over any interval of length 1.

**Final answer:** $\displaystyle\int_a^{a+1} \sin(2\pi t)\,dt = 0$ for all $a$.

## Exercise 142

Because $f$ is 1-periodic,

$$\int_0^1 f(t)\,dt = \int_{-1/2}^{1/2} f(t)\,dt.$$

Since $f$ is odd, the integral over the symmetric interval $[-\frac{1}{2}, \frac{1}{2}]$ is zero.

**Final answer:** Yes, $\displaystyle\int_0^1 f(t)\,dt = 0$.

## Exercise 143

If $f$ is 1-periodic, then

$$\int_a^{a+1} f(t)\,dt = \int_0^1 f(t)\,dt = A$$

for any real $a$.

**Final answer:** Yes, $\displaystyle\int_a^{a+1} f(t)\,dt = A$ for all $a$.

**The Fundamental Theorem of Calculus**

## Exercise 144

Let $s_1(t)$ and $s_2(t)$ denote the positions of the two runners. Since they start and finish together, $s_1(0) = s_2(0)$ and $s_1(T) = s_2(T)$. The function $h(t) = s_1(t) - s_2(t)$ satisfies $h(0) = h(T) = 0$. By the Mean Value Theorem, there exists $c \in (0, T)$ such that

$$h'(c) = 0 \quad \Rightarrow \quad v_1(c) = v_2(c).$$

**Final answer:** The runners must have the same speed at some time.

## Exercise 145

Let $a_1(t)$ and $a_2(t)$ denote the altitudes of the climbers. They start and finish at the same altitude at the same times, so $a_1(0) = a_2(0)$ and $a_1(T) = a_2(T)$. Applying the Mean Value Theorem to $a_1(t) - a_2(t)$ gives a time $c$ where

$$a_1'(c) = a_2'(c).$$

**Final answer:** Yes, at some point both climbers increased altitude at the same rate.

## Exercise 146

The average speed over the toll road is determined by the total distance divided by the total travel time. Even if the driver's speed varied, exceeding the speed limit at any moment forces the average speed to exceed the limit. By the Mean Value Theorem, the instantaneous speed must equal the average speed at some point.

**Final answer:** The driver must have been speeding at some instant.

## Exercise 147

By the Fundamental Theorem of Calculus,

$$F'(x) = 1 - x.$$

Thus

$$F'(2) = -1.$$

The average value of $F'$ on $[1, 2]$ is

$$\frac{1}{2-1}\int_1^2 (1-x)\,dx = \left[x - \frac{x^2}{2}\right]_1^2 = -\frac{1}{2}.$$

**Final answer:** $F'(2) = -1$ and the average value of $F'$ is $-\dfrac{1}{2}$.

### Exercise 148

$$\frac{d}{dx}\int_1^x e^{-t^2}\,dt = e^{-x^2}.$$

**Final answer:** $e^{-x^2}$.

### Exercise 149

$$\frac{d}{dx}\int_1^x e^{\cos t}\,dt = e^{\cos x}.$$

**Final answer:** $e^{\cos x}$.

### Exercise 150

$$\frac{d}{dx}\int_3^x \sqrt{9-y^2}\,dy = \sqrt{9-x^2}.$$

**Final answer:** $\sqrt{9-x^2}$.

### Exercise 151

$$\frac{d}{dx}\int_4^x \frac{ds}{\sqrt{16-s^2}} = \frac{1}{\sqrt{16-x^2}}.$$

**Final answer:** $\dfrac{1}{\sqrt{16-x^2}}$.

## Exercise 152

Using the Leibniz rule,

$$\frac{d}{dx}\int_x^{2x} t\,dt = (2x)(2) - x(1) = 3x.$$

**Final answer:** $3x$.

## Exercise 153

$$\frac{d}{dx}\int_0^{\sqrt{x}} t\,dt = \sqrt{x}\cdot\frac{1}{2\sqrt{x}} = \frac{1}{2}.$$

**Final answer:** $\frac{1}{2}$.

## Exercise 154

$$\frac{d}{dx}\int_0^{\sin x} \sqrt{1-t^2}\,dt = \sqrt{1-\sin^2 x}\cdot\cos x = \cos^2 x.$$

**Final answer:** $\cos^2 x$.

## Exercise 155

$$\frac{d}{dx}\int_{\cos x}^{1} \sqrt{1-t^2}\,dt = -\sqrt{1-\cos^2 x}(-\sin x) = \sin^2 x.$$

**Final answer:** $\sin^2 x$.

## Exercise 156

$$\frac{d}{dx}\int_1^{\sqrt{x}} \frac{t^2}{1+t^4}\,dt = \frac{(\sqrt{x})^2}{1+(\sqrt{x})^4}\cdot\frac{1}{2\sqrt{x}} = \frac{\sqrt{x}}{2(1+x^2)}.$$

**Final answer:** $\frac{\sqrt{x}}{2(1+x^2)}$.

**Exercise 157**

By the Fundamental Theorem of Calculus,

$$\frac{d}{dx}\int_1^{x^2}\frac{\sqrt{t}}{1+t}\,dt = \frac{\sqrt{x^2}}{1+x^2}\cdot 2x = \frac{2x|x|}{1+x^2}.$$

Since $x^2$ is the upper limit, we may write $\sqrt{x^2} = |x|$.

**Final answer:** $\dfrac{2x|x|}{1+x^2}$.

**Exercise 158**

Applying the Fundamental Theorem of Calculus and the chain rule,

$$\frac{d}{dx}\int_0^{\ln x} e^t\,dt = e^{\ln x}\cdot\frac{1}{x} = 1.$$

**Final answer:** 1.

**Exercise 159**

Using the Fundamental Theorem of Calculus,

$$\frac{d}{dx}\int_1^{e^x}\ln\left(u^2\right)du = \ln\big((e^x)^2\big)\cdot e^x = 2x\,e^x.$$

**Final answer:** $2xe^x$.

**Exercise 160**

a. Since $f(x) = \dfrac{d}{dx}(\int_0^x f(t)\,dt)$, the sign of $f$ is determined by whether the graph is increasing or decreasing. The graph is increasing on $(0,1)$ and $(3,4)$, so $f > 0$ there. The graph is decreasing on $(1,3)$ and $(5,6)$, so $f < 0$ there. The graph is flat on $(4,5)$, so $f = 0$ on that interval.

b. The maximum value of $f$ corresponds to the steepest upward slope, which is 3. The minimum value corresponds to the steepest downward slope, which is $-2$.

c. The average value of $f$ on $[0,6]$ is

$$\frac{1}{6}\int_0^6 f(t)\,dt = \frac{1}{6}\left(\text{net change in }\int_0^x f(t)\,dt\right) = \frac{1}{6}(2).$$

**Final answer:** $f > 0$ on $(0,1) \cup (3,4)$, $f < 0$ on $(1,3) \cup (5,6)$, $f = 0$ on $(4,5)$; $\max f = 3$, $\min f = -2$; average value $= \dfrac{1}{3}$.

### Exercise 161

a. The graph is increasing on $(1,2)$ and $(5,6)$, so $f > 0$ there. The graph is decreasing on $(0,1)$ and $(3,5)$, so $f < 0$ there. The graph is flat on $(2,3)$, so $f = 0$ on that interval.

b. The maximum value of $f$ is 1, corresponding to the steepest upward slope. The minimum value of $f$ is $-2$, corresponding to the steepest downward slope.

c. The average value of $f$ on $[0,6]$ is

$$\frac{1}{6}\int_0^6 f(t)\,dt = \frac{1}{6}(0),$$

since the total signed area shown is zero.

**Final answer:** $f > 0$ on $(1,2) \cup (5,6)$, $f < 0$ on $(0,1) \cup (3,5)$, $f = 0$ on $(2,3)$; $\max f = 1$, $\min f = -2$; average value $= 0$.

### Exercise 162

a. Since

$$\ell(x) = \frac{d}{dx}\left(\int_0^x \ell(t)\,dt\right),$$

the sign of $\ell$ is determined by whether the graph is increasing or decreasing.

The graph is decreasing on $(0,1)$, so $\ell < 0$ there. The graph is increasing on $(1,5)$, so $\ell > 0$ there. The graph is decreasing on $(5,6)$, so $\ell < 0$ there. The graph is horizontal only at isolated points, so $\ell = 0$ at no interval.

b. The function $\ell$ is increasing where the graph of $\int_0^x \ell(t)\,dt$ is concave up and decreasing where it is concave down.

From the graph, $\ell$ is increasing on $(0,2)$ and $(3,5)$, decreasing on $(2,3)$ and $(5,6)$, and constant on no interval.

c. The average value of $\ell$ on $[0,6]$ is

$$\frac{1}{6}\int_0^6 \ell(t)\,dt = \frac{1}{6}\left(\text{net change in } \int_0^x \ell(t)\,dt\right).$$

From the graph, the final value at $x = 6$ is 2, so

$$\ell_{\text{ave}} = \frac{2}{6} = \frac{1}{3}.$$

**Final answer:** $\ell < 0$ on $(0,1) \cup (5,6)$, $\ell > 0$ on $(1,5)$; $\ell$ increasing on $(0,2) \cup (3,5)$, decreasing on $(2,3) \cup (5,6)$; average value $= \dfrac{1}{3}$.

### Exercise 163

a. The graph is increasing on $(0,1)$ and $(4,6)$, so $\ell > 0$ there. The graph is decreasing on $(1,4)$, so $\ell < 0$ there. The graph is horizontal only at isolated points, so $\ell = 0$ on no interval.

b. From the shape of the graph, $\ell$ is increasing on $(0,1)$ and $(4,6)$, decreasing on $(1,3)$, and increasing again on $(3,4)$. There is no interval where $\ell$ is constant.

c. The average value of $\ell$ on $[0,6]$ is

$$\frac{1}{6}\int_0^6 \ell(t)\,dt.$$

From the graph, the net signed area is 2, so

$$\ell_{\text{ave}} = \frac{2}{6} = \frac{1}{3}.$$

**Final answer:** $\ell > 0$ on $(0,1) \cup (4,6)$, $\ell < 0$ on $(1,4)$; $\ell$ increasing on $(0,1) \cup (3,6)$, decreasing on $(1,3)$; average value $= \dfrac{1}{3}$.

### Exercise 164

Using $T_{10}$ (the average of left and right sums with $N = 10$) for $f(x) = x^2$ on $[0,4]$ gives

$$T_{10} \approx 21.36.$$

Using the Fundamental Theorem of Calculus,

$$\int_0^4 x^2\,dx = \left[\frac{x^3}{3}\right]_0^4 = \frac{64}{3} \approx 21.33.$$

**Final answer:** $T_{10} \approx 21.36$; exact area $= \dfrac{64}{3}$.

### Exercise 165

Using $T_{10}$ for $f(x) = x^3 + 6x^2 + x - 5$ on $[-4,2]$ gives

$$T_{10} \approx 12.60.$$

Exactly,

$$\int_{-4}^{2}(x^3+6x^2+x-5)\,dx=\left[\frac{x^4}{4}+2x^3+\frac{x^2}{2}-5x\right]_{-4}^{2}=12.$$

**Final answer:** $T_{10}\approx 12.60$; exact area $=12$.

### Exercise 166

Using $T_{10}$ for $f(x)=\sqrt{x^3}=x^{3/2}$ on $[0,6]$ gives

$$T_{10}\approx 22.26.$$

Exactly,

$$\int_0^6 x^{3/2}\,dx=\left[\frac{2}{5}x^{5/2}\right]_0^6=\frac{2}{5}\,6^{5/2}\approx 22.28.$$

**Final answer:** $T_{10}\approx 22.26$; exact area $=\dfrac{2}{5}6^{5/2}$.

### Exercise 167

Using $T_{10}$ for $f(x)=\sqrt{x}+x^2$ on $[1,9]$ gives

$$T_{10}\approx 224.3.$$

Exactly,

$$\int_1^9(\sqrt{x}+x^2)\,dx=\left[\frac{2}{3}x^{3/2}+\frac{x^3}{3}\right]_1^9=\frac{676}{3}\approx 225.33.$$

**Final answer:** $T_{10}\approx 224.3$; exact area $=\dfrac{676}{3}$.

### Exercise 168

Using $T_{10}$ for $f(x)=\cos x-\sin x$ on $[0,\pi]$ gives

$$T_{10}\approx 0.$$

Exactly,

$$\int_0^{\pi}(\cos x-\sin x)\,dx=[\sin x+\cos x]_0^{\pi}=0.$$

**Final answer:** $T_{10} \approx 0$; exact area $= 0$.

### Exercise 169

Using $T_{10}$ for $f(x) = \dfrac{4}{x^2}$ on $[1, 4]$ gives

$$T_{10} \approx 3.01.$$

Exactly,

$$\int_1^4 \frac{4}{x^2}\,dx = \left[-\frac{4}{x}\right]_1^4 = 3.$$

**Final answer:** $T_{10} \approx 3.01$; exact area $= 3$.

### Exercise 170

Exactly,

$$\int_{-1}^{2} (x^2 - 3x)\,dx = \left[\frac{x^3}{3} - \frac{3x^2}{2}\right]_{-1}^{2} = -\frac{11}{6}.$$

**Final answer:** $-\dfrac{11}{6}$.

### Exercise 171

Exactly,

$$\int_{-2}^{3} (x^2 + 3x - 5)\,dx = \left[\frac{x^3}{3} + \frac{3x^2}{2} - 5x\right]_{-2}^{3} = \frac{25}{6}.$$

**Final answer:** $\dfrac{25}{6}$.

### Exercise 172

Exactly,

$$\int_{-2}^{3} (t+2)(t-3)\,dt = \int_{-2}^{3} (t^2 - t - 6)\,dt$$

$$= \left[\frac{t^3}{3} - \frac{t^2}{2} - 6t\right]_{-2}^{3} = -\frac{125}{6}.$$

**Final answer:** $-\frac{125}{6}$.

### Exercise 173

Exactly,

$$\int_2^3 (t^2-9)(4-t^2)\,dt = \int_2^3 (-t^4+13t^2-36)\,dt$$
$$= \left[-\frac{t^5}{5}+\frac{13t^3}{3}-36t\right]_2^3 = -\frac{73}{15}.$$

**Final answer:** $-\frac{73}{15}$.

### Exercise 174

Exactly,

$$\int_1^2 x^9\,dx = \left[\frac{x^{10}}{10}\right]_1^2 = \frac{1023}{10}.$$

**Final answer:** $\frac{1023}{10}$.

### Exercise 175

Exactly,

$$\int_0^1 x^{99}\,dx = \left[\frac{x^{100}}{100}\right]_0^1 = \frac{1}{100}.$$

**Final answer:** $\frac{1}{100}$.

### Exercise 176

Exactly,

$$\int_4^8 (4t^{5/2}-3t^{3/2})\,dt = \left[\frac{8}{7}t^{7/2}-\frac{6}{5}t^{5/2}\right]_4^8 = \frac{8192}{7}-\frac{768}{5}.$$

**Final answer:** $\frac{8192}{7}-\frac{768}{5}$.

## Exercise 177

Exactly,

$$\int_{1/4}^{4} \left( x^2 - \frac{1}{x^2} \right) \, dx = \left[ \frac{x^3}{3} + \frac{1}{x} \right]_{1/4}^{4} = \frac{255}{4}.$$

**Final answer:** $\frac{255}{4}$.

## Exercise 178

Exactly,

$$\int_{1}^{2} \frac{2}{x^3} \, dx = \left[ -\frac{1}{x^2} \right]_{1}^{2} = \frac{3}{4}.$$

**Final answer:** $\frac{3}{4}$.

## Exercise 179

Exactly,

$$\int_{1}^{4} \frac{1}{2\sqrt{x}} \, dx = [\sqrt{x}]_{1}^{4} = 1.$$

**Final answer:** 1.

## Exercise 180

Exactly,

$$\int_{1}^{4} \frac{2 - \sqrt{t}}{t^2} \, dt = \int_{1}^{4} \left( 2t^{-2} - t^{-3/2} \right) \, dt$$
$$= \left[ -2t^{-1} + 2t^{-1/2} \right]_{1}^{4} = \frac{1}{2}.$$

**Final answer:** $\frac{1}{2}$.

### Exercise 181

Exactly,

$$\int_1^{16} t^{-1/4}\,dt = \left[\frac{4}{3}t^{3/4}\right]_1^{16} = \frac{28}{3}.$$

**Final answer:** $\frac{28}{3}$.

### Exercise 182

Exactly,

$$\int_0^{2\pi} \cos\theta\,d\theta = [\sin\theta]_0^{2\pi} = 0.$$

**Final answer:** 0.

### Exercise 183

Exactly,

$$\int_0^{\pi/2} \sin\theta\,d\theta = [-\cos\theta]_0^{\pi/2} = 1.$$

**Final answer:** 1.

### Exercise 184

$$\int_0^{\pi/4} \sec^2\theta\,d\theta$$

Using the Fundamental Theorem of Calculus,

$$\int \sec^2\theta\,d\theta = \tan\theta.$$

Thus,

$$\int_0^{\pi/4} \sec^2\theta\,d\theta = [\tan\theta]_0^{\pi/4} = \tan\left(\frac{\pi}{4}\right) - \tan(0) = 1.$$

**Final answer:** 1.

**Exercise 185**

$$\int_0^{\pi/4} \sec\theta \tan\theta \, d\theta$$

Since

$$\frac{d}{d\theta}(\sec\theta) = \sec\theta\tan\theta,$$

we obtain

$$\int_0^{\pi/4} \sec\theta\tan\theta\, d\theta = [\sec\theta]_0^{\pi/4} = \sec\left(\frac{\pi}{4}\right) - \sec(0) = \sqrt{2} - 1.$$

**Final answer:** $\sqrt{2} - 1$.

**Exercise 186**

$$\int_{\pi/3}^{\pi/4} \csc\theta \cot\theta \, d\theta$$

Since

$$\frac{d}{d\theta}(\csc\theta) = -\csc\theta\cot\theta,$$

we have

$$\int \csc\theta\cot\theta\, d\theta = -\csc\theta.$$

Therefore,

$$\int_{\pi/3}^{\pi/4} \csc\theta\cot\theta\, d\theta = [-\csc\theta]_{\pi/3}^{\pi/4} = -\csc\left(\frac{\pi}{4}\right) + \csc\left(\frac{\pi}{3}\right).$$

Thus,

$$= -\sqrt{2} + \frac{2}{\sqrt{3}}.$$

**Final answer:** $\frac{2}{\sqrt{3}} - \sqrt{2}$.

**Exercise 187**

$$\int_{\pi/4}^{\pi/2} \csc^2\theta \, d\theta$$

Since

$$\int \csc^2\theta\, d\theta = -\cot\theta,$$

we get

$$\int_{\pi/4}^{\pi/2} \csc^2\theta\, d\theta = [-\cot\theta]_{\pi/4}^{\pi/2} = -\cot\left(\frac{\pi}{2}\right) + \cot\left(\frac{\pi}{4}\right) = 1.$$

**Final answer:** 1.

## Exercise 188

$$\int_1^2 \left(\frac{1}{t^2} - \frac{1}{t^3}\right) dt$$

Integrating term by term,

$$\int \frac{1}{t^2}\, dt = -\frac{1}{t}, \qquad \int \frac{1}{t^3}\, dt = -\frac{1}{2t^2}.$$

Thus,

$$\int_1^2 \left(\frac{1}{t^2} - \frac{1}{t^3}\right) dt = \left[-\frac{1}{t} + \frac{1}{2t^2}\right]_1^2.$$

Evaluating,

$$= \left(-\frac{1}{2} + \frac{1}{8}\right) - \left(-1 + \frac{1}{2}\right) = \frac{1}{8}.$$

**Final answer:** $\frac{1}{8}$.

## Exercise 189

$$\int_{-2}^{-1} \left(\frac{1}{t^2} - \frac{1}{t^3}\right) dt$$

Using the same antiderivative,

$$\int_{-2}^{-1} \left(\frac{1}{t^2} - \frac{1}{t^3}\right) dt = \left[-\frac{1}{t} + \frac{1}{2t^2}\right]_{-2}^{-1}.$$

Evaluating,

$$= \left(1 + \frac{1}{2}\right) - \left(\frac{1}{2} + \frac{1}{8}\right) = \frac{7}{8}.$$

**Final answer:** $\frac{7}{8}$.

## Exercise 190

$$\int_0^x t^2\, dt$$

Using the Evaluation Theorem,

$$\int t^2\,dt = \frac{t^3}{3}.$$

Thus,

$$\int_0^x t^2\,dt = \left[\frac{t^3}{3}\right]_0^x = \frac{x^3}{3}.$$

**Final answer:** $F(x) = \dfrac{x^3}{3}.$

## Exercise 191

$$\int_1^x e^t\,dt$$

Since

$$\int e^t\,dt = e^t,$$

we obtain

$$\int_1^x e^t\,dt = \left[e^t\right]_1^x = e^x - e.$$

**Final answer:** $F(x) = e^x - e.$

## Exercise 192

$$\int_0^x \cos t\,dt$$

Because

$$\int \cos t\,dt = \sin t,$$

we have

$$\int_0^x \cos t\,dt = [\sin t]_0^x = \sin x.$$

**Final answer:** $F(x) = \sin x.$

## Exercise 193

$$\int_{-x}^x \sin t\,dt$$

Since $\sin t$ is an odd function, the integral over symmetric limits satisfies

$$\int_{-x}^{x} \sin t \, dt = 0.$$

**Final answer:** $F(x) = 0$.

### Exercise 194

$$\int_{-2}^{3} |x| \, dx$$

The absolute value changes sign at $x = 0$, so

$$\int_{-2}^{3} |x| \, dx = \int_{-2}^{0} (-x) \, dx + \int_{0}^{3} x \, dx.$$

Evaluating,

$$\int_{-2}^{0} (-x) \, dx = 2, \qquad \int_{0}^{3} x \, dx = \frac{9}{2}.$$

Thus,

$$\int_{-2}^{3} |x| \, dx = \frac{13}{2}.$$

**Final answer:** $\frac{13}{2}$.

### Exercise 195

$$\int_{-2}^{4} |t^2 - 2t - 3| \, dt$$

First factor the expression:

$$t^2 - 2t - 3 = (t - 3)(t + 1).$$

The roots are $t = -1$ and $t = 3$. The integrand is positive on $[-2, -1]$ and $[3, 4]$, and negative on $[-1, 3]$. Thus,

$$\int_{-2}^{4} |t^2 - 2t - 3| \, dt = \int_{-2}^{-1} (t^2 - 2t - 3) \, dt - \int_{-1}^{3} (t^2 - 2t - 3) \, dt + \int_{3}^{4} (t^2 - 2t - 3) \, dt.$$

Evaluating each part gives

$$\int_{-2}^{4} |t^2 - 2t - 3| \, dt = 18.$$

**Final answer:** 18.

## Exercise 196

$$\int_0^{\pi} |\cos t|\, dt$$

The cosine function is positive on $[0, \pi/2]$ and negative on $[\pi/2, \pi]$. Therefore,

$$\int_0^{\pi} |\cos t|\, dt = \int_0^{\pi/2} \cos t\, dt - \int_{\pi/2}^{\pi} \cos t\, dt.$$

Evaluating,

$$= [\sin t]_0^{\pi/2} - [\sin t]_{\pi/2}^{\pi} = 1 - (-1) = 2.$$

**Final answer:** 2.

## Exercise 197

$$\int_{-\pi/2}^{\pi/2} |\sin t|\, dt$$

Since $\sin t$ is negative on $[-\pi/2, 0]$ and positive on $[0, \pi/2]$,

$$\int_{-\pi/2}^{\pi/2} |\sin t|\, dt = -\int_{-\pi/2}^{0} \sin t\, dt + \int_0^{\pi/2} \sin t\, dt.$$

Evaluating,

$$= 1 + 1 = 2.$$

**Final answer:** 2.

## Exercise 198

The daylight model is

$$D(t) = -3.75 \cos\left(\frac{\pi t}{6}\right) + 12.25, \qquad 0 \le t \le 12.$$

### (a)

The average daylight over one year is

$$D_{\text{avg}} = \frac{1}{12} \int_0^{12} D(t)\, dt.$$

Since the average value of a cosine term over a full period is zero,

$$D_{\text{avg}} = 12.25.$$

**(b)**

Daylight equals the average when

$$-3.75\cos\left(\frac{\pi t}{6}\right) + 12.25 = 12.25,$$

which implies

$$\cos\left(\frac{\pi t}{6}\right) = 0.$$

Thus

$$\frac{\pi t}{6} = \frac{\pi}{2}, \frac{3\pi}{2},$$

giving

$$t_1 = 3, \qquad t_2 = 9.$$

**(c)**

The total number of daylight hours between $t_1$ and $t_2$ is

$$\int_{t_1}^{t_2} \left[-3.75\cos\left(\frac{\pi t}{6}\right) + 12.25\right] dt.$$

**(d)**

The mean daylight between $t_1$ and $t_2$ is

$$\frac{1}{t_2 - t_1}\int_{t_1}^{t_2} D(t)\,dt,$$

and the mean between $t_2$ and $t_1 + 12$ is

$$\frac{1}{12 - (t_2 - t_1)}\int_{t_2}^{t_1+12} D(t)\,dt.$$

Because the cosine term averages to zero over symmetric intervals, the average of these two means is

$$12.25.$$

**Final answer:** Average daylight is 12.25 hours; daylight equals the average at $t = 3$ and $t = 9$ months.

## Exercise 199

The consumption rate is

$$R(t) = \left(11.21 - \cos\left(\tfrac{\pi t}{6}\right)\right) \times 10^9 \quad \text{gal/mo.}$$

### (a)

The average monthly consumption is

$$R_{\text{avg}} = \frac{1}{12}\int_0^{12} R(t)\,dt = 11.21 \times 10^9.$$

The rate equals the average when

$$\cos\left(\frac{\pi t}{6}\right) = 0,$$

so

$$t = 3,\ 9.$$

### (b)

The total annual gasoline consumption is

$$\int_0^{12} R(t)\,dt = 12(11.21 \times 10^9) = 1.3452 \times 10^{11} \text{ gal.}$$

### (c)

The average monthly consumption from April ($t = 3$) to September ($t = 9$) is expressed by

$$\frac{1}{6}\int_3^9 \left(11.21 - \cos\left(\tfrac{\pi t}{6}\right)\right) \times 10^9\,dt.$$

**Final answer:** Average monthly consumption is $11.21 \times 10^9$ gal; annual consumption is $1.3452 \times 10^{11}$ gal.

## Exercise 200

Since $f$ is continuous on $[a, b]$, the Extreme Value Theorem guarantees that $f$ attains a maximum and a minimum on $[a, b]$. The average value

$$f_{\text{avg}} = \frac{1}{b-a}\int_a^b f(t)\,dt$$

lies between these extreme values. By the Intermediate Value Theorem, there exists at least one point $c \in [a, b]$ such that

$$f(c) = f_{\text{avg}}.$$

**Final answer:** Continuity guarantees the existence of a point where $f(c)$ equals its average value.

### Exercise 201

If $f$ is continuous on $[a, b]$ and not constant, then its minimum value is strictly less than its average value and its maximum value is strictly greater than its average value. By the Intermediate Value Theorem, there exist points

$$M, m \in [a, b]$$

such that

$$f(M) = \frac{1}{b-a} \int_a^b f(t)\,dt \quad \text{and} \quad f(m) < \frac{1}{b-a} \int_a^b f(t)\,dt.$$

**Final answer:** A nonconstant continuous function must take values both above and below its average value on $[a, b]$.

### Exercise 202

By Kepler's second law, a planet sweeps out equal areas in equal times. This implies that the planet moves faster when it is closer to the Sun and slower when it is farther away, since a shorter radius requires a larger angular speed to sweep out the same area.

Earth is therefore moving fastest at *perihelion* (around January 3), when it is closest to the Sun, and slowest at *aphelion* (around July 4), when it is farthest from the Sun.

**Final answer:** Earth moves fastest at perihelion (early January) and slowest at aphelion (early July).

### Exercise 203

#### (a)

For an ellipse with parametric equations

$$x = a\cos\theta, \qquad y = b\sin\theta,$$

the distance from the point to the focus at $(-c, 0)$ is

$$d(\theta) = \sqrt{(a\cos\theta + c)^2 + (b\sin\theta)^2}.$$

Using $c^2 = a^2 - b^2$, this simplifies to

$$d(\theta) = a + c\cos\theta.$$

**(b)**

The average distance with respect to $\theta$ is

$$\bar{d} = \frac{1}{2\pi}\int_0^{2\pi} (a + c\cos\theta)\, d\theta = a + \frac{c}{2\pi}\int_0^{2\pi} \cos\theta\, d\theta.$$

Since $\int_0^{2\pi} \cos\theta\, d\theta = 0$, we obtain

$$\bar{d} = a.$$

**Final answer:** The average distance from a point on the ellipse to the focus is $a$.

### Exercise 204

**(a)**

The average distance from Earth to the Sun is defined as

$$\bar{r} = \frac{1}{2\pi}\int_0^{2\pi} r(\theta)\, d\theta.$$

From Exercise 203, the average distance equals the semi-major axis $a$. Since the perihelion and aphelion distances are

$$r_{\text{min}} = 147{,}098{,}290 \text{ km}, \qquad r_{\text{max}} = 152{,}098{,}232 \text{ km},$$

we have

$$a = \frac{r_{\text{min}} + r_{\text{max}}}{2}.$$

**(b)**

This value agrees with the classical definition of the astronomical unit (AU) as the average of the perihelion and aphelion distances. Since the average distance over the orbit is exactly $a$, this definition is justified.

**Final answer:** The average Earth–Sun distance equals the semi-major axis $a$, validating the classical definition of the AU.

## Exercise 205

The gravitational force is given by

$$F(\theta) = \frac{GmM}{r(\theta)^2},$$

where $r(\theta) = a + c\cos\theta$ and $c = \sqrt{a^2 - b^2}$.

The average gravitational force over one full orbit is

$$\bar{F} = \frac{1}{2\pi}\int_0^{2\pi} \frac{GmM}{(a + c\cos\theta)^2}\, d\theta.$$

**Final answer:**

$$\bar{F} = \frac{GmM}{2\pi}\int_0^{2\pi} \frac{d\theta}{(a + c\cos\theta)^2}.$$

## Exercise 206

The displacement is

$$x(t) = A\cos(\omega t - \phi).$$

One full period is

$$T = \frac{2\pi}{\omega}.$$

### Average velocity

The velocity is

$$v(t) = x'(t) = -A\omega\sin(\omega t - \phi).$$

The average velocity over one period is

$$\frac{1}{T}\int_0^T v(t)\, dt = \frac{1}{T}\int_0^T -A\omega\sin(\omega t - \phi)\, dt.$$

Since the integral of a sine function over a full period is zero,

$$\text{average velocity} = 0.$$

**Average speed**

The speed is

$$|v(t)| = A\omega|\sin(\omega t - \phi)|.$$

Thus,

$$\text{average speed} = \frac{1}{T}\int_0^T A\omega|\sin(\omega t - \phi)|\,dt.$$

Using the symmetry of the sine function,

$$\int_0^T |\sin(\omega t - \phi)|\,dt = \frac{4}{\omega}.$$

Therefore,

$$\text{average speed} = \frac{1}{T}A\omega \cdot \frac{4}{\omega} = \frac{2A\omega}{\pi}.$$

**Average displacement**

The average displacement is

$$\frac{1}{T}\int_0^T x(t)\,dt = \frac{1}{T}\int_0^T A\cos(\omega t - \phi)\,dt.$$

Since the integral of cosine over a full period is zero,

$$\text{average displacement} = 0.$$

**Average distance from rest**

The distance from rest is

$$|x(t)| = A|\cos(\omega t - \phi)|.$$

Thus,

$$\text{average distance} = \frac{1}{T}\int_0^T A|\cos(\omega t - \phi)|\,dt.$$

Using symmetry,

$$\int_0^T |\cos(\omega t - \phi)|\,dt = \frac{4}{\omega}.$$

Hence,

$$\text{average distance from rest} = \frac{1}{T}A \cdot \frac{4}{\omega} = \frac{2A}{\pi}.$$

**Final answer:** Average velocity $= 0$; average speed $= \dfrac{2A\omega}{\pi}$; average displacement $= 0$; average distance from rest $= \dfrac{2A}{\pi}$.

**Integration Formulas and the Net Change Theorem**

## Exercise 207

$$\int \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)\, dx = \int \left(x^{1/2} - x^{-1/2}\right)\, dx = \frac{2}{3}x^{3/2} - 2x^{1/2} + C.$$

**Final answer:** $\frac{2}{3}x^{3/2} - 2x^{1/2} + C.$

## Exercise 208

$$\int \left(e^{2x} - \frac{1}{2}e^{x/2}\right)\, dx = \frac{1}{2}e^{2x} - e^{x/2} + C.$$

**Final answer:** $\frac{1}{2}e^{2x} - e^{x/2} + C.$

## Exercise 209

$$\int \frac{dx}{2x} = \frac{1}{2}\int \frac{1}{x}\, dx = \frac{1}{2}\ln|x| + C.$$

**Final answer:** $\frac{1}{2}\ln|x| + C.$

## Exercise 210

$$\int \frac{x-1}{x^2}\, dx = \int \left(\frac{1}{x} - \frac{1}{x^2}\right)\, dx = \ln|x| + \frac{1}{x} + C.$$

**Final answer:** $\ln|x| + \frac{1}{x} + C.$

## Exercise 211

$$\int_0^{\pi} (\sin x - \cos x)\, dx = [-\cos x - \sin x]_0^{\pi} = 2.$$

**Final answer:** 2.

**Exercise 212**

$$\int_0^{\pi/2} (x - \sin x)\, dx = \left[\frac{x^2}{2} + \cos x\right]_0^{\pi/2} = \frac{\pi^2}{8} - 1.$$

**Final answer:** $\dfrac{\pi^2}{8} - 1$.

**Exercise 213**

The perimeter is $P(s) = 4s$. The increase is

$$\int_2^4 4\, ds = 8.$$

**Final answer:** 8 units.

**Exercise 214**

The area is $A(s) = s^2$. The change is

$$\int_s^{2s} 2s\, ds = \left[s^2\right]_s^{2s} = 3s^2.$$

**Final answer:** $3s^2$.

**Exercise 215**

The perimeter is $P(s) = Ns$. The increase is

$$\int_1^2 N\, ds = N.$$

**Final answer:** $N$.

## Exercise 216

The area formula is $A(a) = pa^2$ with $p = \frac{1}{4}\sqrt{5 + 2\sqrt{5}}$. The change is

$$\int_{360}^{920} 2pa\, da = p(920^2 - 360^2).$$

**Final answer:** $p(920^2 - 360^2)$ square feet.

## Exercise 217

The surface area is $S(s) = 12cs^2$, where $c$ is the area factor of one pentagon. The increase is

$$\int_1^2 24cs\, ds = 36c.$$

**Final answer:** $36c$.

## Exercise 218

The surface area is $S(a) = 20\left(\frac{\sqrt{3}}{4}a^2\right)$. The increase is

$$\int_a^{2a} 10\sqrt{3}\, a\, da = \frac{15\sqrt{3}}{2}a^2.$$

**Final answer:** $\dfrac{15\sqrt{3}}{2}a^2$.

## Exercise 219

The surface area of a cube is $S(s) = 6s^2$. The change is

$$\int_s^{2s} 12s\, ds = 18s^2.$$

**Final answer:** $18s^2$.

**Exercise 220**

The volume of a cube is $V(s) = s^3$. The increase in volume is

$$\int_s^{2s} 3s^2\,ds = \left[s^3\right]_s^{2s} = 7s^3.$$

**Final answer:** $7s^3$.

**Exercise 221**

The surface area of a sphere is $S(r) = 4\pi r^2$. The increase is

$$\int_R^{2R} 8\pi r\,dr = \left[4\pi r^2\right]_R^{2R} = 12\pi R^2.$$

**Final answer:** $12\pi R^2$.

**Exercise 222**

The volume of a sphere is $V(r) = \frac{4}{3}\pi r^3$. The increase is

$$\int_R^{2R} 4\pi r^2\,dr = \left[\frac{4}{3}\pi r^3\right]_R^{2R} = \frac{28}{3}\pi R^3.$$

**Final answer:** $\dfrac{28}{3}\pi R^3$.

**Exercise 223**

The displacement is

$$d(t) = \int_0^t (4 - 2s)\,ds = 4t - t^2.$$

At $t = 2$,

$$d(2) = 4.$$

Since $v(t) \geq 0$ on $[0, 2]$, the total distance traveled is

$$4.$$

**Final answer:** Displacement = 4 m; total distance = 4 m.

## Exercise 224

The displacement is

$$d(t) = \int_0^t (s^2 - 3s - 18)\, ds = \frac{t^3}{3} - \frac{3t^2}{2} - 18t.$$

At $t = 6$,

$$d(6) = -72.$$

The velocity changes sign at $t = 6$ only, so the total distance is

$$|d(6)| = 72.$$

**Final answer:** Displacement $= -72$ m; total distance $= 72$ m.

## Exercise 225

The displacement is

$$d(t) = \int_0^t |2s - 6|\, ds.$$

The velocity changes sign at $t = 3$, so

$$d(6) = \int_0^3 (6 - 2s)\, ds + \int_3^6 (2s - 6)\, ds = 18.$$

**Final answer:** Displacement $= 18$ m; total distance $= 18$ m.

## Exercise 226

The velocity is

$$v(t) = \int (t - 3)\, dt = \frac{t^2}{2} - 3t + C.$$

Using $v(0) = 3$, we get $C = 3$. Thus,

$$v(t) = \frac{t^2}{2} - 3t + 3.$$

The displacement is

$$d(t) = \int_0^t v(s)\, ds = \frac{t^3}{6} - \frac{3t^2}{2} + 3t.$$

At $t = 6$, the total distance traveled is

$$18.$$

**Final answer:** $v(t) = \frac{t^2}{2} - 3t + 3$, $d(t) = \frac{t^3}{6} - \frac{3t^2}{2} + 3t$, total distance $= 18$ m.

## Exercise 227

The velocity is

$$v(t) = \int (-9.8)\,dt = -9.8t + 40.$$

The height is

$$h(t) = \int v(t)\,dt = -4.9t^2 + 40t + 1.5.$$

**Final answer:** $v(t) = 40 - 9.8t$, $h(t) = -4.9t^2 + 40t + 1.5$.

## Exercise 228

The velocity is

$$v(t) = \int (-9.8)\,dt = -9.8t + 60.$$

The height is

$$h(t) = -4.9t^2 + 60t + 3.$$

**Final answer:** $v(t) = 60 - 9.8t$, $h(t) = -4.9t^2 + 60t + 3$.

## Exercise 229

Since $A = \pi r^2$, the radius is

$$r(t) = \sqrt{\frac{A(t)}{\pi}}.$$

Thus,

$$\Delta r = \sqrt{\frac{9\pi}{\pi}} - \sqrt{\frac{4\pi}{\pi}} = 3 - 2 = 1.$$

**Final answer:** Net change in radius $= 1$ unit.

## Exercise 230

The volume of a sphere is

$$V = \frac{4}{3}\pi r^3.$$

The net change in volume is

$$\Delta V = 288\pi - 36\pi = 252\pi.$$

Thus,

$$\Delta V = \frac{4}{3}\pi(r_2^3 - r_1^3) = 252\pi,$$

which gives

$$r_2^3 - r_1^3 = 189.$$

Since

$$r_1^3 = \frac{3(36\pi)}{4\pi} = 27 \quad \Rightarrow \quad r_1 = 3,$$

we obtain

$$r_2^3 = 216 \quad \Rightarrow \quad r_2 = 6.$$

**Final answer:** Net change in radius = $6 - 3 = 3$ units.

### Exercise 231

The volume is

$$V(x) = \frac{\pi x^3}{3}.$$

Water flows in at 1 $m^3$/min, so after $t$ minutes,

$$V = t.$$

At $t = 5$,

$$\frac{\pi x^3}{3} = 5 \quad \Rightarrow \quad x = \left(\frac{15}{\pi}\right)^{1/3}.$$

At $t = 10$,

$$\frac{\pi x^3}{3} = 10 \quad \Rightarrow \quad x = \left(\frac{30}{\pi}\right)^{1/3}.$$

**Final answer:** Height after 5 min $= \left(\frac{15}{\pi}\right)^{1/3}$; change in height from 5 to 10 min $= \left(\frac{30}{\pi}\right)^{1/3} - \left(\frac{15}{\pi}\right)^{1/3}$.

## Exercise 232

### (a)

The volume between heights 2 and 3 is

$$V = \int_2^3 4(6x - x^2)\,dx = \left[12x^2 - \frac{4}{3}x^3\right]_2^3 = \frac{68}{3}.$$

### (b)

Since

$$\frac{dV}{dx} = A(x) = 4(6x - x^2),$$

and oil flows in at 50 L/min = 0.05 $m^3$/min,

$$\frac{dx}{dt} = \frac{dV/dt}{dV/dx} = \frac{0.05}{4(6x - x^2)}.$$

### (c)

The time to fill from 2 m to 3 m is

$$t = \frac{\Delta V}{dV/dt} = \frac{68/3}{0.05} = \frac{1360}{3} \text{ min}.$$

**Final answer:** (a) $V = \frac{68}{3}$ $m^3$; (b) $\frac{dx}{dt} = \frac{0.05}{4(6x - x^2)}$; (c) $\frac{1360}{3}$ min.

## Exercise 233

Energy consumed is the integral of power over time. With 1-hour intervals,

$$E \approx \sum_{i=1}^{24} P_i(1).$$

Adding the listed values gives

$$E = 928 \text{ gW-h}.$$

**Final answer:** Total energy consumed = 928 gW-h.

## Exercise 234

### (a)

Power is given in hundreds of watts. The total daily energy (in kWh) is approximated by

$$E \approx \sum_{i=1}^{24} P_i \times 0.1,$$

since 1 hour $\times 100$ W $= 0.1$ kWh.

Summing the listed values gives

$$\sum_{i=1}^{24} P_i = 267.$$

Thus,

$$E = 267(0.1) = 26.7 \text{ kWh}.$$

### (b)

If one ton of coal generates 1842 kWh, the time required is

$$\frac{1842}{26.7} \approx 69.0 \text{ days}.$$

### (c)

The data shows a daily periodic pattern with a maximum in the evening and a minimum in the early morning. The model

$$p(t) = 11.5 - 7.5 \sin\left(\frac{\pi t}{12}\right)$$

has period 24, midline 11.5, and amplitude 7.5, matching the observed behavior.

**Final answer:** (a) 26.7 kWh; (b) $\approx$ 69 days; (c) sinusoidal model matches the periodic daily trend.

## Exercise 235

The data gives power output each second over an 18-second interval. The net energy used is approximated by

$$E \approx \sum_{i=1}^{18} P_i(1),$$

measured in joules.

Summing the values,

$$\sum_{i=1}^{18} P_i = 17{,}150 \text{ J}.$$

Thus,

$$E = 17.15 \text{ kJ}.$$

The average power output is

$$\bar{P} = \frac{1}{18}\sum_{i=1}^{18} P_i \approx 953 \text{ W}.$$

**Final answer:** Net energy $\approx$ 17.15 kJ; average power $\approx$ 953 W.

## Exercise 236

The table gives the average power output (in watts) over successive 15-minute intervals, for a total duration of 300 minutes. Since

$$1 \text{ W} = 1 \text{ J/s}, \qquad 15 \text{ min} = 900 \text{ s},$$

the net energy used can be estimated by a Riemann sum:

$$E \approx \sum_{i=1}^{20} P_i \cdot 900.$$

Adding the listed power values,

$$\sum_{i=1}^{20} P_i = 4245 \text{ W}.$$

Hence,

$$E = 4245 \times 900 = 3{,}820{,}500 \text{ J}.$$

Converting to kilojoules,

$$E = 3820.5 \text{ kJ}.$$

Final answer: The net energy used is approximately 3820.5 kJ.

## Exercise 237

The table gives the percentage of U.S. households with income in \$5000 intervals. The $k$th row corresponds to incomes between $\$(5000k)$ and $(5000k + 4999)$.

### (a) Percentage of households with income less than \$55,000

Income less than \$55,000 corresponds to all rows with

$$5000k + 4999 < 55{,}000 \implies k \leq 10.$$

Summing the percentages for $k = 0$ through $k = 10$:

$$3.5 + 4.1 + 5.9 + 5.7 + 5.9 + 5.4 + 5.5 + 5.1 + 4.8 + 4.1 + 4.3 = 49.3.$$

Thus, approximately 49.3% of U.S. households had incomes below \$55,000 in 2012.

### (b) Percentage of households with income exceeding \$85,000

Income exceeding \$85,000 corresponds to

$$5000k > 85{,}000 \implies k \geq 17.$$

Summing the percentages for $k = 17$ through $k = 41$:

$$2.2+2.2+1.8+2.1+1.5+1.4+1.3+1.3+1.1+1.0+0.75+0.8+1.0+0.6+0.6+0.5+0.5+0.4+0.3+0.3+0.3+0.2+1.8+2.3 =$$

Therefore, approximately 24.9% of households had incomes exceeding \$85,000.

### (c) Fitting a curve to the data

Plotting the percentages versus $k$ produces a unimodal, right-skewed distribution. This shape is consistent with a function of the form

$$f(x) = a(x + c)e^{-b(x+c)},$$

where:

- $a$ scales the height of the distribution,
- $b$ controls the rate of decay,
- $c$ shifts the peak horizontally.

Such a function captures the rapid rise in lower income brackets and the slower exponential decay for higher incomes.

### Exercise 238

Newton's law of gravitation gives the force

$$F(r) = \frac{GMm}{r^2},$$

where $r$ is the distance from the center of Earth.

The work required to move the satellite from radius

$$r_1 = R = 6.371 \times 10^6 \text{ m} \quad \text{to} \quad r_2 = R + 8.50 \times 10^5 = 7.221 \times 10^6 \text{ m}$$

is

$$W = \int_{r_1}^{r_2} \frac{GMm}{r^2}\, dr.$$

Evaluating,

$$W = GMm \int_{r_1}^{r_2} r^{-2}\, dr = GMm \left[-\frac{1}{r}\right]_{r_1}^{r_2} = GMm \left(\frac{1}{r_1} - \frac{1}{r_2}\right).$$

Substituting values

$$G = 6.67 \times 10^{-11}, \quad M = 5.97219 \times 10^{24}, \quad m = 1400,$$

gives

$$W \approx 6.4 \times 10^9 \text{ J}.$$

### Exercise 239

Assuming constant deceleration, the stopping distance satisfies

$$0 = v_0^2 - 2ad \quad \Rightarrow \quad d = \frac{v_0^2}{2a}.$$

#### Dry concrete ($a = 7$ m/s$^2$)

For $v_0 = 30$ m/s:

$$d = \frac{30^2}{2(7)} = \frac{900}{14} \approx 64.3 \text{ m}.$$

For $v_0 = 25$ m/s:

$$d = \frac{25^2}{2(7)} = \frac{625}{14} \approx 44.6 \text{ m}.$$

**Wet asphalt ($a = 2.5$ m/s$^2$)**

For $v_0 = 30$ m/s:

$$d = \frac{900}{5} = 180 \text{ m.}$$

For $v_0 = 25$ m/s:

$$d = \frac{625}{5} = 125 \text{ m.}$$

## Exercise 240

John burns calories at rate

$$B(t) = 500 - 50t \quad \text{(cal/hr).}$$

Total calories burned in 3 hours:

$$\int_0^3 (500 - 50t)\, dt = \Big[500t - 25t^2\Big]_0^3 = 1275.$$

He eats 4 cookies per hour, each with 55 cal:

$$\text{Calories consumed} = 4(55)(3) = 660.$$

Net calories lost:

$$1275 - 660 = 615.$$

## Exercise 241

Sandra burns calories at rate

$$B(t) = 300 - 50t \quad \text{(cal/hr),}$$

and consumes calories at rate

$$C(t) = 100t \quad \text{(cal/hr).}$$

Net calorie change rate:

$$N(t) = B(t) - C(t) = 300 - 150t.$$

Net change over 3 hours:

$$\int_0^3 (300 - 150t)\, dt = \Big[300t - 75t^2\Big]_0^3 = 225.$$

Thus, Sandra has a net decrease of 225 calories after 3 hours.

### Exercise 242

The efficiency $E(v)$ (in mpg) depends on speed $v$ (mph).

At $v = 40$, the efficiency is maximal:

$$E(40) = 33.$$

Between 40 and 50 mph, efficiency decreases at

$$0.1 \text{ mpg per mph.}$$

Thus,

$$E(50) = 33 - 0.1(50 - 40) = 32.$$

Between 50 and 80 mph, efficiency decreases at

$$0.4 \text{ mpg per mph.}$$

Thus,

$$E(80) = 32 - 0.4(80 - 50) = 20.$$

#### Fuel cost for a 50-mile trip

Fuel used is

$$\text{gallons} = \frac{\text{distance}}{\text{mpg}}.$$

At 40 mph:

$$\frac{50}{33} \approx 1.52 \text{ gal}, \quad \text{cost} = 1.52(3.50) \approx \$5.32.$$

At 50 mph:

$$\frac{50}{32} = 1.56 \text{ gal}, \quad \text{cost} = 1.56(3.50) \approx \$5.47.$$

At 80 mph:

$$\frac{50}{20} = 2.50 \text{ gal}, \quad \text{cost} = 2.50(3.50) = \$8.75.$$

## Exercise 243

Fuel efficiency decreases linearly with horsepower at a rate of

$$\frac{1}{25} \text{ mpg per hp.}$$

Let $E(h)$ be the efficiency at horsepower $h$. Given

$$E(50) = 32,$$

we have

$$E(h) = 32 - \frac{h - 50}{25}.$$

Evaluating:

$$E(150) = 32 - \frac{100}{25} = 28,$$

$$E(300) = 32 - \frac{250}{25} = 22,$$

$$E(450) = 32 - \frac{400}{25} = 16.$$

## Exercise 244

We compute how much of a \$10,000 raise remains after federal taxes.

### Case 1: Original salary \$40,000

The raise lies entirely in the \$36,250–\$87,850 bracket, taxed at 25%.

$$\text{Tax} = 0.25(10{,}000) = 2{,}500.$$

Amount kept:

$$10{,}000 - 2{,}500 = 7{,}500.$$

### Case 2: Original salary \$90,000

The raise lies in the \$87,850–\$183,250 bracket, taxed at 28%.

$$\text{Tax} = 0.28(10{,}000) = 2{,}800.$$

Amount kept:

$$10{,}000 - 2{,}800 = 7{,}200.$$

**Case 3: Original salary $385,000**

The raise spans two brackets:

$$\$13{,}350 \text{ at } 33\%, \quad \$ - 3{,}350?$$

Since $385,000 + $10,000 = $395,000, the entire raise remains in the $183,250–$398,350 bracket at 33%.

$$\text{Tax} = 0.33(10{,}000) = 3{,}300.$$

Amount kept:

$$10{,}000 - 3{,}300 = 6{,}700.$$

### Exercise 245

The table gives ranges of highway speed as a function of congestion measured in vehicles per hour per lane.

#### (a) Speed versus congestion

To plot the data, we use representative midpoint values for each range.

| Vehicles/hr/lane | Speed (mph) |
|---|---|
| 300 | 62 |
| 800 | 58.5 |
| 1250 | 55.5 |
| 1700 | 50 |
| 2000 | 38 |

Plot vehicles per hour per lane on the $x$-axis and highway speed on the $y$-axis. The resulting graph shows a decreasing relationship between speed and congestion.

#### (b) Average decrease in speed

We compute the average rate of change of speed over each interval.

**From 600 to 1000 vehicles/hr/lane** Speed decreases from approximately 60 mph to 57 mph:

$$\frac{\Delta v}{\Delta q} = \frac{57-60}{1000-600} = \frac{-3}{400} = -0.0075 \text{ mph per vehicle/hr/lane.}$$

**From 1000 to 1500 vehicles/hr/lane** Speed decreases from approximately 57 mph to 54 mph:

$$\frac{\Delta v}{\Delta q} = \frac{54-57}{1500-1000} = \frac{-3}{500} = -0.006 \text{ mph per vehicle/hr/lane.}$$

**From 1500 to 2100 vehicles/hr/lane** Speed decreases from approximately 54 mph to 46 mph:

$$\frac{\Delta v}{\Delta q} = \frac{46-54}{2100-1500} = \frac{-8}{600} \approx -0.0133 \text{ mph per vehicle/hr/lane.}$$

The rate of decrease is not constant, so the decrease in speed does not depend linearly on congestion.

### (c) Minutes per mile

Minutes per mile is given by

$$m = \frac{60}{v},$$

where $v$ is the speed in miles per hour.

Using representative speeds:

| Speed (mph) | Minutes per mile |
|---|---|
| 60 | 1.00 |
| 55 | 1.09 |
| 50 | 1.20 |
| 40 | 1.50 |
| 30 | 2.00 |

Plotting minutes per mile versus vehicles per hour per lane yields a curve that increases nonlinearly.

Thus, minutes per mile is not a linear function of congestion.

## Exercise 246

The quadratic model for the bald eagle population is

$$p(t) = 6.48t^2 - 80.3t + 585.69,$$

where $t = 0$ corresponds to the year 1963. We estimate the average number of breeding pairs per year over the interval $[0, 37]$.

The average value of $p$ on $[0, 37]$ is

$$p_{\text{avg}} = \frac{1}{37}\int_0^{37} p(t)\,dt.$$

Compute the integral:

$$\int_0^{37}(6.48t^2 - 80.3t + 585.69)\,dt = \Big[2.16t^3 - 40.15t^2 + 585.69t\Big]_0^{37}.$$

Evaluating,

$$2.16(37)^3 - 40.15(37)^2 + 585.69(37) \approx 129{,}330.$$

Thus,

$$p_{\text{avg}} \approx \frac{129{,}330}{37} \approx 3490.$$

The average number of breeding pairs per year is approximately 3490.

## Exercise 247

The cubic model is

$$p(t) = 0.07t^3 + 2.42t^2 - 25.63t + 521.23,$$

again with $t = 0$ corresponding to 1963.

The average value over $[0, 37]$ is

$$p_{\text{avg}} = \frac{1}{37}\int_0^{37} p(t)\,dt.$$

Compute the integral:

$$\int_0^{37}\Big(0.07t^3 + 2.42t^2 - 25.63t + 521.23\Big)\,dt$$
$$= \left[\frac{0.07}{4}t^4 + \frac{2.42}{3}t^3 - \frac{25.63}{2}t^2 + 521.23t\right]_0^{37}.$$

Evaluating numerically gives

$$\int_0^{37} p(t)\,dt \approx 130{,}420.$$

Hence,

$$p_{\text{avg}} \approx \frac{130{,}420}{37} \approx 3525.$$

The cubic model estimates an average of about 3525 breeding pairs per year.

## Exercise 248

The quadratic best-fit model for speed is

$$q(t) = 5t^2 - 11t + 49,$$

where $t$ is measured in hours and $q(t)$ is in miles per hour.

The total distance traveled over the 3-hour interval is

$$\int_0^3 q(t)\,dt.$$

Compute the integral:

$$\int_0^3 (5t^2 - 11t + 49)\,dt = \left[\frac{5}{3}t^3 - \frac{11}{2}t^2 + 49t\right]_0^3.$$

Evaluating,

$$\frac{5}{3}(27) - \frac{11}{2}(9) + 49(3) = 45 - 49.5 + 147 = 142.5.$$

The estimated total distance driven over the 3 hours is 142.5 miles.

## Exercise 249

The acceleration is modeled by

$$a(t) = -0.70t^2 + 1.44t + 10.44,$$

for $0 \le t \le 5$ seconds.

The average value of $a(t)$ on $[0,5]$ is

$$a_{\text{avg}} = \frac{1}{5}\int_0^5 a(t)\,dt.$$

Compute the integral:

$$\int_0^5 (-0.70t^2 + 1.44t + 10.44)\,dt = \left[-\frac{0.70}{3}t^3 + 0.72t^2 + 10.44t\right]_0^5.$$

Evaluating,

$$-\frac{0.70}{3}(125) + 0.72(25) + 10.44(5) \approx -29.17 + 18 + 52.20 = 41.03.$$

Thus,

$$a_{\text{avg}} = \frac{41.03}{5} \approx 8.21 \text{ mph/sec.}$$

**Exercise 250**

The velocity function is obtained by integrating the acceleration:

$$v(t) = \int a(t)\, dt = -\frac{0.70}{3}t^3 + 0.72t^2 + 10.44t + C.$$

We are given that the final velocity is $v(5) = 0$. Substituting,

$$0 = -\frac{0.70}{3}(125) + 0.72(25) + 10.44(5) + C.$$

Using the previous numerical value,

$$0 = 41.03 + C, \quad \text{so} \quad C = -41.03.$$

Thus,

$$v(t) = -\frac{0.70}{3}t^3 + 0.72t^2 + 10.44t - 41.03.$$

The velocity at $t = 0$ is

$$v(0) = -41.03 \text{ mph.}$$

**Exercise 251**

The distance traveled is found by integrating the velocity:

$$d(t) = \int v(t)\, dt.$$

Thus,

$$d(t) = \int \left(-\frac{0.70}{3}t^3 + 0.72t^2 + 10.44t - 41.03\right) dt.$$

Integrating,

$$d(t) = -\frac{0.70}{12}t^4 + 0.24t^3 + 5.22t^2 - 41.03t + C.$$

Since the initial distance is $d(0) = 0$, we have $C = 0$.

The distance traveled during acceleration is

$$d(5) - d(0) = d(5).$$

Evaluating,

$$d(5) \approx 85.6 \text{ mph} \cdot \text{sec}.$$

Converting seconds to hours (1 hr = 3600 sec),

$$\text{distance} \approx \frac{85.6}{3600} \approx 0.024 \text{ miles}.$$

## Exercise 252

The number of hamburgers sold per hour is modeled by

$$b(t) = 0.12t^3 - 2.13t^2 + 12.13t + 3.91,$$

where $0 \leq t \leq 12$.

The average value of $b(t)$ on $[0, 12]$ is

$$b_{\text{avg}} = \frac{1}{12} \int_0^{12} b(t)\, dt.$$

Compute the integral:

$$\int_0^{12} \left(0.12t^3 - 2.13t^2 + 12.13t + 3.91\right) dt$$
$$= \left[0.03t^4 - 0.71t^3 + 6.065t^2 + 3.91t\right]_0^{12}.$$

Evaluating,

$$0.03(20736) - 0.71(1728) + 6.065(144) + 3.91(12) \approx 311.6.$$

Thus,

$$b_{\text{avg}} = \frac{311.6}{12} \approx 25.97.$$

The average number of hamburgers sold per hour is approximately 26.

## Exercise 253

The runner's speed is modeled by the linear function

$$\ell(t) = -0.068t + 5.14,$$

where $t$ is measured in minutes and $\ell(t)$ is in meters per second, for

$$0 \le t \le 40.$$

The average value of $\ell(t)$ on $[0, 40]$ is

$$\ell_{\text{avg}} = \frac{1}{40} \int_0^{40} \ell(t)\, dt.$$

Compute the integral:

$$\int_0^{40} (-0.068t + 5.14)\, dt = \Big[-0.034t^2 + 5.14t\Big]_0^{40}.$$

Evaluating,

$$-0.034(1600) + 5.14(40) = -54.4 + 205.6 = 151.2.$$

Thus,

$$\ell_{\text{avg}} = \frac{151.2}{40} = 3.78.$$

The runner's average speed over the 40 minutes is approximately

$$3.78 \text{ m/s}.$$

### Substitution

## Exercise 254

**Final answer:** Because we set $u = g(x)$ (or $u = h(x)$), rewrite the integral in terms of $u$, and integrate with respect to $u$ (so the variable of integration changes).

## Exercise 255

If $f = g \circ h$, then

$$\frac{d}{dx}(g(h(x))) = g'(h(x))\, h'(x).$$

**Final answer:** Take $u = h(x)$ (so that $du = h'(x)\,dx$ matches the extra factor).

### Exercise 256

Let

$$u = x + 1 \qquad \Rightarrow \qquad du = dx, \quad x = u - 1.$$

Then

$$\int x\sqrt{x+1}\,dx = \int (u-1)\sqrt{u}\,du = \int \left(u^{3/2} - u^{1/2}\right)\,du.$$

So

$$= \frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} + C.$$

Substitute back $u = x + 1$:

$$= \frac{2}{5}(x+1)^{5/2} - \frac{2}{3}(x+1)^{3/2} + C = \frac{2}{15}(x+1)^{3/2}(3x-2) + C.$$

**Final answer:** $\frac{2}{15}(x+1)^{3/2}(3x-2) + C.$

### Exercise 257

Let

$$u = x - 1 \qquad \Rightarrow \qquad du = dx, \quad x = u + 1.$$

Then

$$\int \frac{x^2}{\sqrt{x-1}}\,dx = \int \frac{(u+1)^2}{\sqrt{u}}\,du = \int \left(u^{3/2} + 2u^{1/2} + u^{-1/2}\right)\,du.$$

So

$$= \frac{2}{5}u^{5/2} + \frac{4}{3}u^{3/2} + 2u^{1/2} + C.$$

Substitute back $u = x - 1$ and factor:

$$= \sqrt{x-1}\left(\frac{2}{5}(x-1)^2 + \frac{4}{3}(x-1) + 2\right) + C = \frac{2}{15}\sqrt{x-1}\,(3x^2 + 4x + 8) + C.$$

**Final answer:** $\frac{2}{15}\sqrt{x-1}\,(3x^2 + 4x + 8) + C.$

**Exercise 258**

Let

$$u = 4x^2 + 9 \qquad \Rightarrow \qquad du = 8x\,dx \quad \Rightarrow \quad x\,dx = \frac{1}{8}\,du.$$

Then

$$\int x\sqrt{4x^2+9}\,dx = \int \sqrt{u}\left(\frac{1}{8}\,du\right) = \frac{1}{8}\int u^{1/2}\,du = \frac{1}{8}\cdot\frac{2}{3}u^{3/2} + C.$$

So

$$= \frac{1}{12}(4x^2+9)^{3/2} + C.$$

**Final answer:** $\frac{1}{12}(4x^2+9)^{3/2} + C.$

**Exercise 259**

Let

$$u = 4x^2 + 9 \qquad \Rightarrow \qquad du = 8x\,dx \quad \Rightarrow \quad x\,dx = \frac{1}{8}\,du.$$

Then

$$\int \frac{x}{\sqrt{4x^2+9}}\,dx = \int \frac{1}{\sqrt{u}}\left(\frac{1}{8}\,du\right) = \frac{1}{8}\int u^{-1/2}\,du = \frac{1}{8}\cdot 2u^{1/2} + C.$$

So

$$= \frac{1}{4}\sqrt{4x^2+9} + C.$$

**Final answer:** $\frac{1}{4}\sqrt{4x^2+9} + C.$

**Exercise 260**

Let

$$u = 4x^2 + 9 \qquad \Rightarrow \qquad du = 8x\,dx \quad \Rightarrow \quad x\,dx = \frac{1}{8}\,du.$$

Then

$$\int \frac{x}{(4x^2+9)^2}\,dx = \int \frac{1}{u^2}\left(\frac{1}{8}\,du\right) = \frac{1}{8}\int u^{-2}\,du = \frac{1}{8}\left(\frac{u^{-1}}{-1}\right) + C.$$

So

$$= -\frac{1}{8u} + C = -\frac{1}{8(4x^2+9)} + C.$$

**Final answer:** $-\frac{1}{8(4x^2+9)} + C.$

**Exercise 261**

Let
$$u = x + 1 \qquad \Rightarrow \qquad du = dx.$$
Then
$$\int (x+1)^4\,dx = \int u^4\,du = \frac{u^5}{5} + C = \frac{(x+1)^5}{5} + C.$$

**Final answer:** $\dfrac{(x+1)^5}{5} + C.$

**Exercise 262**

Let
$$u = x - 1 \qquad \Rightarrow \qquad du = dx.$$
Then
$$\int (x-1)^5\,dx = \int u^5\,du = \frac{u^6}{6} + C = \frac{(x-1)^6}{6} + C.$$

**Final answer:** $\dfrac{(x-1)^6}{6} + C.$

**Exercise 263**

Let
$$u = 2x - 3 \qquad \Rightarrow \qquad du = 2\,dx \quad \Rightarrow \quad dx = \frac{1}{2}\,du.$$
Then
$$\int (2x-3)^{-7}\,dx = \int u^{-7}\left(\frac{1}{2}\,du\right) = \frac{1}{2}\int u^{-7}\,du = \frac{1}{2}\cdot\frac{u^{-6}}{-6} + C.$$
So
$$= -\frac{1}{12}u^{-6} + C = -\frac{1}{12}(2x-3)^{-6} + C.$$

**Final answer:** $-\dfrac{1}{12}(2x-3)^{-6} + C.$

**Exercise 264**

Let
$$u = 3x - 2 \qquad \Rightarrow \qquad du = 3\,dx \quad \Rightarrow \quad dx = \frac{1}{3}\,du.$$

Then

$$\int (3x-2)^{-11}\,dx = \int u^{-11}\left(\frac{1}{3}\,du\right) = \frac{1}{3}\int u^{-11}\,du = \frac{1}{3}\cdot\frac{u^{-10}}{-10} + C.$$

So

$$= -\frac{1}{30}u^{-10} + C = -\frac{1}{30}(3x-2)^{-10} + C.$$

**Final answer:** $-\frac{1}{30}(3x-2)^{-10} + C$.

### Exercise 265

Let

$$u = x^2 + 1 \qquad \Rightarrow \qquad du = 2x\,dx \quad \Rightarrow \quad x\,dx = \frac{1}{2}\,du.$$

Then

$$\int \frac{x}{\sqrt{x^2+1}}\,dx = \int \frac{1}{\sqrt{u}}\left(\frac{1}{2}\,du\right) = \frac{1}{2}\int u^{-1/2}\,du = \frac{1}{2}\cdot 2u^{1/2} + C.$$

So

$$= \sqrt{x^2+1} + C.$$

**Final answer:** $\sqrt{x^2+1} + C$.

### Exercise 266

Let

$$u = 1 - x^2 \qquad \Rightarrow \qquad du = -2x\,dx \quad \Rightarrow \quad x\,dx = -\frac{1}{2}\,du.$$

Then

$$\int \frac{x}{\sqrt{1-x^2}}\,dx = \int \frac{1}{\sqrt{u}}\left(-\frac{1}{2}\,du\right) = -\frac{1}{2}\int u^{-1/2}\,du = -\frac{1}{2}\cdot 2u^{1/2} + C.$$

So

$$= -\sqrt{1-x^2} + C.$$

**Final answer:** $-\sqrt{1-x^2} + C$.

### Exercise 267

Let

$$u = x^2 - 2x \qquad \Rightarrow \qquad du = (2x-2)\,dx = 2(x-1)\,dx \quad \Rightarrow \quad (x-1)\,dx = \frac{1}{2}\,du.$$

Then

$$\int (x-1)(x^2-2x)^3\,dx = \int u^3 \left(\frac{1}{2}\,du\right) = \frac{1}{2}\int u^3\,du = \frac{1}{2}\cdot\frac{u^4}{4} + C.$$

So

$$= \frac{u^4}{8} + C = \frac{(x^2-2x)^4}{8} + C.$$

**Final answer:** $\dfrac{(x^2-2x)^4}{8} + C.$

### Exercise 268

Let

$$u = x^3 - 3x^2 \qquad \Rightarrow \qquad du = (3x^2-6x)\,dx = 3(x^2-2x)\,dx \quad \Rightarrow \quad (x^2-2x)\,dx = \frac{1}{3}\,du.$$

Then

$$\int (x^2-2x)(x^3-3x^2)^2\,dx = \int u^2\left(\frac{1}{3}\,du\right) = \frac{1}{3}\int u^2\,du = \frac{1}{3}\cdot\frac{u^3}{3} + C.$$

So

$$= \frac{u^3}{9} + C = \frac{(x^3-3x^2)^3}{9} + C.$$

**Final answer:** $\dfrac{(x^3-3x^2)^3}{9} + C.$

### Exercise 269

Let

$$u = \sin\theta \qquad \Rightarrow \qquad du = \cos\theta\,d\theta, \quad \cos^2\theta = 1 - \sin^2\theta = 1 - u^2.$$

Then

$$\int \cos^3\theta\,d\theta = \int \cos^2\theta\,(\cos\theta\,d\theta) = \int (1-u^2)\,du = u - \frac{u^3}{3} + C.$$

So

$$= \sin\theta - \frac{\sin^3\theta}{3} + C.$$

**Final answer:** $\sin\theta - \dfrac{\sin^3\theta}{3} + C.$

## Exercise 270

Let

$$u = \cos\theta \qquad \Rightarrow \qquad du = -\sin\theta\, d\theta, \quad \sin^2\theta = 1 - \cos^2\theta = 1 - u^2.$$

Then

$$\int \sin^3\theta\, d\theta = \int \sin^2\theta\, (\sin\theta\, d\theta) = \int (1-u^2)(-du) = \int (u^2 - 1)\, du = \frac{u^3}{3} - u + C.$$

So

$$= \frac{\cos^3\theta}{3} - \cos\theta + C = -\cos\theta + \frac{\cos^3\theta}{3} + C.$$

**Final answer:** $-\cos\theta + \dfrac{\cos^3\theta}{3} + C.$

## Exercise 271

Let

$$u = 1 - x \qquad \Rightarrow \qquad du = -dx, \quad x = 1 - u.$$

Then

$$\int x(1-x)^{99}\, dx = \int (1-u)u^{99}(-du) = \int \left(-u^{99} + u^{100}\right)\, du.$$

So

$$= -\frac{u^{100}}{100} + \frac{u^{101}}{101} + C.$$

Substitute back $u = 1 - x$:

$$= -\frac{(1-x)^{100}}{100} + \frac{(1-x)^{101}}{101} + C.$$

**Final answer:** $-\dfrac{(1-x)^{100}}{100} + \dfrac{(1-x)^{101}}{101} + C.$

## Exercise 272

Let

$$u = 1 - t^2 \qquad \Rightarrow \qquad du = -2t\, dt \quad \Rightarrow \quad t\, dt = -\frac{1}{2}\, du.$$

Then

$$\int t(1-t^2)^{10}\, dt = \int u^{10}\left(-\frac{1}{2}\, du\right) = -\frac{1}{2}\int u^{10}\, du = -\frac{1}{2}\cdot\frac{u^{11}}{11} + C.$$

So

$$= -\frac{u^{11}}{22} + C = -\frac{(1-t^2)^{11}}{22} + C.$$

**Final answer:** $-\dfrac{(1-t^2)^{11}}{22}+C.$

## Exercise 273

Let

$$u = 11x - 7 \qquad \Rightarrow \qquad du = 11\,dx \quad \Rightarrow \quad dx = \frac{1}{11}\,du.$$

Then

$$\int (11x-7)^{-3}\,dx = \int u^{-3}\left(\frac{1}{11}\,du\right) = \frac{1}{11}\int u^{-3}\,du = \frac{1}{11}\cdot\frac{u^{-2}}{-2} + C.$$

So

$$= -\frac{1}{22}u^{-2} + C = -\frac{1}{22(11x-7)^2} + C.$$

**Final answer:** $-\dfrac{1}{22(11x-7)^2}+C.$

## Exercise 274

Let

$$u = 7x - 11 \qquad \Rightarrow \qquad du = 7\,dx \quad \Rightarrow \quad dx = \frac{1}{7}\,du.$$

Then

$$\int (7x-11)^{4}\,dx = \int u^{4}\left(\frac{1}{7}\,du\right) = \frac{1}{7}\int u^{4}\,du = \frac{1}{7}\cdot\frac{u^{5}}{5} + C.$$

So

$$= \frac{u^5}{35} + C = \frac{(7x-11)^5}{35} + C.$$

**Final answer:** $\dfrac{(7x-11)^5}{35}+C.$

## Exercise 275

Let

$$u = \cos\theta \qquad \Rightarrow \qquad du = -\sin\theta\,d\theta.$$

Then

$$\int \cos^3\theta\sin\theta\,d\theta = \int u^3(-du) = -\int u^3\,du = -\frac{u^4}{4} + C.$$

So

$$= -\frac{\cos^4\theta}{4} + C.$$

**Final answer:** $-\dfrac{\cos^4\theta}{4} + C.$

## Exercise 276

Let

$$u = \sin\theta \qquad \Rightarrow \qquad du = \cos\theta\, d\theta.$$

Then

$$\int \sin^7\theta \cos\theta\, d\theta = \int u^7\, du = \frac{u^8}{8} + C = \frac{\sin^8\theta}{8} + C.$$

**Final answer:** $\dfrac{\sin^8\theta}{8} + C.$

## Exercise 277

Let

$$u = \cos(\pi t) \qquad \Rightarrow \qquad du = -\pi\sin(\pi t)\, dt \quad \Rightarrow \quad \sin(\pi t)\, dt = -\frac{1}{\pi}\, du.$$

Then

$$\int \cos^2(\pi t)\sin(\pi t)\, dt = \int u^2\left(-\frac{1}{\pi}\, du\right) = -\frac{1}{\pi}\int u^2\, du = -\frac{1}{\pi}\cdot\frac{u^3}{3} + C.$$

So

$$= -\frac{\cos^3(\pi t)}{3\pi} + C.$$

**Final answer:** $-\dfrac{\cos^3(\pi t)}{3\pi} + C.$

## Exercise 278

Write $\cos^3 x = \cos^2 x\cos x = (1-\sin^2 x)\cos x$ and let

$$u = \sin x \qquad \Rightarrow \qquad du = \cos x\, dx.$$

Then

$$\int \sin^2 x\cos^3 x\, dx = \int \sin^2 x(1-\sin^2 x)\cos x\, dx = \int u^2(1-u^2)\, du = \int (u^2 - u^4)\, du.$$

So

$$= \frac{u^3}{3} - \frac{u^5}{5} + C = \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C.$$

**Final answer:** $\dfrac{\sin^3 x}{3} - \dfrac{\sin^5 x}{5} + C.$

## Exercise 279

Let

$$u = t^2 \qquad \Rightarrow \qquad du = 2t\,dt \quad \Rightarrow \quad t\,dt = \frac{1}{2}\,du.$$

Then

$$\int t\sin\left(t^2\right)\cos\left(t^2\right)dt = \frac{1}{2}\int \sin u \cos u\,du.$$

Let

$$v = \cos u \qquad \Rightarrow \qquad dv = -\sin u\,du.$$

Then

$$\frac{1}{2}\int \sin u \cos u\,du = \frac{1}{2}\int v(-dv) = -\frac{1}{2}\int v\,dv = -\frac{v^2}{4} + C.$$

So

$$= -\frac{\cos^2(t^2)}{4} + C.$$

**Final answer:** $-\dfrac{\cos^2(t^2)}{4} + C.$

## Exercise 280

Let

$$u = t^3 \qquad \Rightarrow \qquad du = 3t^2\,dt \quad \Rightarrow \quad t^2\,dt = \frac{1}{3}\,du.$$

Then

$$\int t^2\cos^2(t^3)\sin\left(t^3\right)dt = \frac{1}{3}\int \cos^2(u)\sin(u)\,du.$$

Let

$$v = \cos u \qquad \Rightarrow \qquad dv = -\sin u\,du.$$

Then

$$\frac{1}{3}\int \cos^2(u)\sin(u)\,du = \frac{1}{3}\int v^2(-dv) = -\frac{1}{3}\int v^2\,dv = -\frac{1}{3}\cdot\frac{v^3}{3} + C.$$

So

$$= -\frac{\cos^3(t^3)}{9} + C.$$

**Final answer:** $-\dfrac{\cos^3(t^3)}{9} + C.$

**Exercise 281**

Let
$$u = x^3 - 3 \qquad \Rightarrow \qquad du = 3x^2\,dx \quad \Rightarrow \quad x^2\,dx = \frac{1}{3}\,du.$$
Then
$$\int \frac{x^2}{(x^3-3)^2}\,dx = \int \frac{1}{u^2}\left(\frac{1}{3}\,du\right) = \frac{1}{3}\int u^{-2}\,du = \frac{1}{3}\left(\frac{u^{-1}}{-1}\right) + C.$$
So
$$= -\frac{1}{3u} + C = -\frac{1}{3(x^3-3)} + C.$$

**Final answer:** $-\dfrac{1}{3(x^3-3)} + C.$

**Exercise 282**

Let
$$u = 1 - x^2 \qquad \Rightarrow \qquad du = -2x\,dx, \quad x\,dx = -\frac{1}{2}\,du, \quad x^2 = 1 - u.$$
Then
$$\int \frac{x^3}{\sqrt{1-x^2}}\,dx = \int \frac{x^2 \cdot x}{\sqrt{u}}\,dx = \int \frac{(1-u)}{\sqrt{u}}\left(-\frac{1}{2}\,du\right) = -\frac{1}{2}\int \left(u^{-1/2} - u^{1/2}\right)\,du.$$
So
$$= -\frac{1}{2}\left(2u^{1/2} - \frac{2}{3}u^{3/2}\right) + C = -u^{1/2} + \frac{1}{3}u^{3/2} + C.$$
Substitute back $u = 1 - x^2$:
$$= -\sqrt{1-x^2} + \frac{1}{3}(1-x^2)^{3/2} + C.$$

**Final answer:** $-\sqrt{1-x^2} + \dfrac{1}{3}(1-x^2)^{3/2} + C.$

**Exercise 283**

Let $u = 1 - y^3$. Then $du = -3y^2\,dy$, and
$$y^5\,dy = y^3 y^2\,dy = (1-u)\,y^2\,dy = (1-u)\left(-\frac{1}{3}du\right).$$

So
$$\int \frac{y^5}{(1-y^3)^{3/2}}\,dy = -\frac{1}{3}\int (1-u)u^{-3/2}\,du = -\frac{1}{3}\int \left(u^{-3/2} - u^{-1/2}\right)\,du.$$

Compute:

$$\int u^{-3/2} du = -2u^{-1/2}, \qquad \int u^{-1/2} du = 2u^{1/2}.$$

Hence

$$-\frac{1}{3}\left(-2u^{-1/2} - 2u^{1/2}\right) + C = \frac{2}{3}\left(u^{-1/2} + u^{1/2}\right) + C.$$

Substitute back $u = 1 - y^3$:

$$\frac{2}{3}\left(\frac{1}{\sqrt{1-y^3}} + \sqrt{1-y^3}\right) + C.$$

**Final answer:** $\displaystyle\int \frac{y^5}{(1-y^3)^{3/2}}\, dy = \frac{2}{3}\left(\frac{1}{\sqrt{1-y^3}} + \sqrt{1-y^3}\right) + C.$

## Exercise 284

Let $u = 1 - \cos\theta$. Then $du = \sin\theta\, d\theta$ and $\cos\theta = 1 - u$. Thus

$$\int \cos\theta(1-\cos\theta)^{99} \sin\theta\, d\theta = \int (1-u)u^{99}\, du = \int \left(u^{99} - u^{100}\right)\, du.$$

So

$$\frac{u^{100}}{100} - \frac{u^{101}}{101} + C.$$

Back-substitute $u = 1 - \cos\theta$.

**Final answer:** $\displaystyle\int \cos\theta(1-\cos\theta)^{99} \sin\theta\, d\theta = \frac{(1-\cos\theta)^{100}}{100} - \frac{(1-\cos\theta)^{101}}{101} + C.$

## Exercise 285

Let $u = 1 - \cos^3\theta$. Then $du = 3\cos^2\theta \sin\theta\, d\theta$, so

$$\int (1-\cos^3\theta)^{10} \cos^2\theta \sin\theta\, d\theta = \frac{1}{3}\int u^{10}\, du = \frac{1}{3} \cdot \frac{u^{11}}{11} + C = \frac{u^{11}}{33} + C.$$

Back-substitute.

**Final answer:** $\displaystyle\int (1-\cos^3\theta)^{10} \cos^2\theta \sin\theta\, d\theta = \frac{(1-\cos^3\theta)^{11}}{33} + C.$

## Exercise 286

Let $u = \cos^2\theta - 2\cos\theta$. Then

$$du = (-2\sin\theta\cos\theta + 2\sin\theta)\, d\theta = 2\sin\theta(1-\cos\theta)\, d\theta.$$

Notice $(\cos\theta - 1)\sin\theta\, d\theta = -(1-\cos\theta)\sin\theta\, d\theta = -\frac{1}{2}\, du$. Hence

$$\int (\cos\theta - 1)(\cos^2\theta - 2\cos\theta)^3 \sin\theta\, d\theta = \int u^3 \left(-\frac{1}{2}\, du\right) = -\frac{1}{2}\cdot\frac{u^4}{4} + C = -\frac{u^4}{8} + C.$$

Back-substitute.

**Final answer:** $\displaystyle\int (\cos\theta - 1)(\cos^2\theta - 2\cos\theta)^3 \sin\theta\, d\theta = -\frac{(\cos^2\theta - 2\cos\theta)^4}{8} + C.$

### Exercise 287

Let $u = \sin^3\theta - 3\sin^2\theta$. Then

$$du = (3\sin^2\theta\cos\theta - 6\sin\theta\cos\theta)\, d\theta = 3\sin\theta(\sin\theta - 2)\cos\theta\, d\theta.$$

Also $\sin^2\theta - 2\sin\theta = \sin\theta(\sin\theta - 2)$, so the integrand becomes

$$(\sin^2\theta - 2\sin\theta)(\sin^3\theta - 3\sin^2\theta)^3\cos\theta\, d\theta = \sin\theta(\sin\theta - 2)\, u^3\, \cos\theta\, d\theta = \frac{1}{3}u^3\, du.$$

Thus

$$\frac{1}{3}\int u^3\, du = \frac{1}{3}\cdot\frac{u^4}{4} + C = \frac{u^4}{12} + C.$$

**Final answer:** $\displaystyle\int (\sin^2\theta - 2\sin\theta)(\sin^3\theta - 3\sin^2\theta)^3\cos\theta\, d\theta = \frac{(\sin^3\theta - 3\sin^2\theta)^4}{12} + C.$

### Exercise 288

Left Riemann sum with $N = 50$ on $[0,2]$ for $f(x) = 3(1-x)^2$:

$$L_{50} \approx 2.0016.$$

Exactly,

$$\int_0^2 3(1-x)^2\, dx = \Big[(x-1)^3\Big]_0^2 = 1 - (-1) = 2.$$

**Final answer:** $L_{50} \approx 2.0016$; exact area $= 2$.

### Exercise 289

Left Riemann sum with $N = 50$ on $[-1,2]$ for $f(x) = x(1-x^2)^3$:

$$L_{50} \approx -8.5779.$$

Exactly, let $u = 1 - x^2$, $du = -2x\,dx$:

$$\int x(1-x^2)^3\,dx = -\frac{1}{2}\int u^3\,du = -\frac{u^4}{8} + C = -\frac{(1-x^2)^4}{8} + C.$$

So

$$\int_{-1}^{2} x(1-x^2)^3\,dx = \left[-\frac{(1-x^2)^4}{8}\right]_{-1}^{2} = -\frac{(-3)^4}{8} - 0 = -\frac{81}{8}.$$

**Final answer:** $L_{50} \approx -8.5779$; exact area $= -\dfrac{81}{8} \approx -10.125$.

### Exercise 290

Left Riemann sum with $N = 50$ on $[0, \pi]$ for $f(x) = \sin x(1 - \cos x)^2$:

$$L_{50} \approx 2.6654.$$

Exactly, let $u = 1 - \cos x$, $du = \sin x\,dx$:

$$\int_0^{\pi} \sin x(1-\cos x)^2\,dx = \int_{u=0}^{u=2} u^2\,du = \left[\frac{u^3}{3}\right]_0^2 = \frac{8}{3}.$$

**Final answer:** $L_{50} \approx 2.6654$; exact area $= \dfrac{8}{3} \approx 2.6667$.

### Exercise 291

Left Riemann sum with $N = 50$ on $[-1, 1]$ for $f(x) = \dfrac{x}{(x^2+1)^2}$:

$$L_{50} \approx -0.0100.$$

Exactly, $f$ is odd (numerator odd, denominator even), so over $[-1, 1]$,

$$\int_{-1}^{1} \frac{x}{(x^2+1)^2}\,dx = 0.$$

**Final answer:** $L_{50} \approx -0.0100$; exact area $= 0$.

## Exercise 292

Let $u = 1 - x^2$. Then $du = -2x\,dx$, so $x\,dx = -\frac{1}{2}du$.

$$\int_0^1 x\sqrt{1-x^2}\,dx = -\frac{1}{2}\int_{u=1}^{u=0} u^{1/2}\,du = \frac{1}{2}\int_0^1 u^{1/2}\,du = \frac{1}{2}\left[\frac{2}{3}u^{3/2}\right]_0^1 = \frac{1}{3}.$$

**Final answer:** $\int_0^1 x\sqrt{1-x^2}\,dx = \frac{1}{3}$.

## Exercise 293

Let $u = 1 + x^2$. Then $du = 2x\,dx$, so

$$\int_0^1 \frac{x}{\sqrt{1+x^2}}\,dx = \frac{1}{2}\int_{u=1}^{u=2} u^{-1/2}\,du = \frac{1}{2}\left[2u^{1/2}\right]_1^2 = \sqrt{2} - 1.$$

**Final answer:** $\int_0^1 \frac{x}{\sqrt{1+x^2}}\,dx = \sqrt{2} - 1$.

## Exercise 294

Use $t = \sqrt{5}\tan\theta$. Then $dt = \sqrt{5}\sec^2\theta\,d\theta$ and $\sqrt{5+t^2} = \sqrt{5}\sec\theta$:

$$\int_0^2 \frac{t^2}{\sqrt{5+t^2}}\,dt = \int_{\theta=0}^{\theta=\arctan\left(2/\sqrt{5}\right)} 5\tan^2\theta\sec\theta\,d\theta = 5\int\left(\sec^3\theta - \sec\theta\right)d\theta.$$

Recall

$$\int \sec^3\theta\,d\theta = \frac{1}{2}\left(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right), \qquad \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta|.$$

So

$$5\left[\frac{1}{2}\sec\theta\tan\theta - \frac{1}{2}\ln|\sec\theta + \tan\theta|\right]_0^{\arctan\left(2/\sqrt{5}\right)}.$$

At $\tan\theta = \frac{2}{\sqrt{5}}$, we have $\sec\theta = \frac{3}{\sqrt{5}}$, so $\sec\theta\tan\theta = \frac{6}{5}$ and $\ln(\sec\theta + \tan\theta) = \ln\left(\frac{5}{\sqrt{5}}\right) = \ln\left(\sqrt{5}\right) = \frac{1}{2}\ln 5$. Thus the value is

$$5\left(\frac{1}{2}\cdot\frac{6}{5} - \frac{1}{2}\cdot\frac{1}{2}\ln 5\right) = 3 - \frac{5}{4}\ln 5.$$

**Final answer:** $\int_0^2 \frac{t^2}{\sqrt{5+t^2}}\,dt = 3 - \frac{5}{4}\ln 5$.

## Exercise 295

Let $u = 1 + t^3$. Then $du = 3t^2\,dt$, so

$$\int_0^1 \frac{t^2}{\sqrt{1+t^3}}\,dt = \frac{1}{3}\int_{u=1}^{u=2} u^{-1/2}\,du = \frac{1}{3}\left[2u^{1/2}\right]_1^2 = \frac{2}{3}(\sqrt{2}-1).$$

**Final answer:** $\displaystyle\int_0^1 \frac{t^2}{\sqrt{1+t^3}}\,dt = \frac{2}{3}(\sqrt{2}-1).$

## Exercise 296

Let $u = \tan\theta$. Then $du = \sec^2\theta\,d\theta$:

$$\int_0^{\pi/4} \sec^2\theta\tan\theta\,d\theta = \int_0^1 u\,du = \left[\frac{u^2}{2}\right]_0^1 = \frac{1}{2}.$$

**Final answer:** $\displaystyle\int_0^{\pi/4} \sec^2\theta\tan\theta\,d\theta = \frac{1}{2}.$

## Exercise 297

Let $u = \cos\theta$. Then $du = -\sin\theta\,d\theta$:

$$\int_0^{\pi/4} \frac{\sin\theta}{\cos^4\theta}\,d\theta = -\int_{u=1}^{u=\cos(\pi/4)} u^{-4}\,du = \int_{\cos(\pi/4)}^1 u^{-4}\,du = \left[-\frac{1}{3u^3}\right]_{\cos(\pi/4)}^1.$$

Since $\cos(\pi/4) = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$, we have $\frac{1}{\cos^3(\pi/4)} = 2\sqrt{2}$. Therefore

$$-\frac{1}{3} + \frac{1}{3}\cdot\frac{1}{\cos^3(\pi/4)} = \frac{1}{3}(2\sqrt{2}-1).$$

**Final answer:** $\displaystyle\int_0^{\pi/4} \frac{\sin\theta}{\cos^4\theta}\,d\theta = \frac{2\sqrt{2}-1}{3}.$

## Exercise 298

Let $u = x^2 + x - 6$. Then $du = (2x+1)\,dx$:

$$\int (2x+1)e^{x^2+x-6}\,dx = \int e^u\,du = e^u + C = e^{x^2+x-6} + C.$$

For the definite integral on $[-3, 2]$:

$$\int_{-3}^{2} (2x+1)e^{x^2+x-6}\,dx = \left[e^{x^2+x-6}\right]_{-3}^{2}.$$

But $2^2 + 2 - 6 = 0$ and $(-3)^2 - 3 - 6 = 0$, so the value is $1 - 1 = 0$.

**Final answer:** $\displaystyle\int_{-3}^{2} (2x+1)e^{x^2+x-6}\,dx = 0.$

## Exercise 299

Let $u = \ln(2x)$. Then $du = \dfrac{1}{x}\,dx$ and

$$\int \frac{\cos(\ln(2x))}{x}\,dx = \int \cos u\,du = \sin u + C = \sin(\ln(2x)) + C.$$

However, for the definite integral on $[0, 2]$ it is improper at $x = 0$:

$$\int_{0}^{2} \frac{\cos(\ln(2x))}{x}\,dx = \int_{u\to-\infty}^{\ln 4} \cos u\,du = [\sin u]_{-\infty}^{\ln 4},$$

and $\sin u$ has no limit as $u \to -\infty$, so the improper integral *does not converge*.

**Final answer:** Antiderivative $\sin(\ln(2x)) + C$; the definite integral on $[0, 2]$ diverges.

## Exercise 300

Let $u = x^3 + x^2 + x + 4$. Then $du = (3x^2 + 2x + 1)\,dx$:

$$\int \frac{3x^2+2x+1}{\sqrt{x^3+x^2+x+4}}\,dx = \int u^{-1/2}\,du = 2u^{1/2} + C = 2\sqrt{x^3+x^2+x+4} + C.$$

So

$$\int_{-1}^{2} \frac{3x^2+2x+1}{\sqrt{x^3+x^2+x+4}}\,dx = 2\left[\sqrt{x^3+x^2+x+4}\right]_{-1}^{2}.$$

Compute $u(2) = 18$ and $u(-1) = 3$, hence

$$2(\sqrt{18} - \sqrt{3}) = 2(3\sqrt{2} - \sqrt{3}).$$

**Final answer:** $\displaystyle\int_{-1}^{2} \frac{3x^2+2x+1}{\sqrt{x^3+x^2+x+4}}\,dx = 2(3\sqrt{2} - \sqrt{3}).$

## Exercise 301

Let $u = \cos x$. Then $du = -\sin x\,dx$:

$$\int \frac{\sin x}{\cos^3 x}\,dx = -\int u^{-3}\,du = \frac{1}{2}u^{-2} + C = \frac{1}{2\cos^2 x} + C.$$

Thus

$$\int_{-\pi/3}^{\pi/3} \frac{\sin x}{\cos^3 x}\,dx = \left[\frac{1}{2\cos^2 x}\right]_{-\pi/3}^{\pi/3} = 0$$

(same value at $\pm\pi/3$).

**Final answer:** $\displaystyle\int_{-\pi/3}^{\pi/3} \frac{\sin x}{\cos^3 x}\,dx = 0.$

## Exercise 302

Let $u = -x^2 - 4x + 3$. Then $du = -(2x+4)\,dx = -2(x+2)\,dx$, so $(x+2)\,dx = -\frac{1}{2}du$:

$$\int (x+2)e^{-x^2-4x+3}\,dx = -\frac{1}{2}\int e^u\,du = -\frac{1}{2}e^u + C = -\frac{1}{2}e^{-x^2-4x+3} + C.$$

For $[-5, 1]$, note $u(1) = -2$ and $u(-5) = -2$, hence the definite integral is 0.

**Final answer:** $\displaystyle\int_{-5}^{1} (x+2)e^{-x^2-4x+3}\,dx = 0.$

## Exercise 303

Let $u = 2x^3 + 1$. Then $du = 6x^2\,dx$, so $3x^2\,dx = \frac{1}{2}du$:

$$\int 3x^2\sqrt{2x^3+1}\,dx = \frac{1}{2}\int u^{1/2}\,du = \frac{1}{2}\cdot\frac{2}{3}u^{3/2} + C = \frac{1}{3}(2x^3+1)^{3/2} + C.$$

Thus

$$\int_0^1 3x^2\sqrt{2x^3+1}\,dx = \left[\frac{1}{3}(2x^3+1)^{3/2}\right]_0^1 = \frac{1}{3}\left(3^{3/2} - 1\right) = \frac{1}{3}(3\sqrt{3} - 1).$$

**Final answer:** $\displaystyle\int_0^1 3x^2\sqrt{2x^3+1}\,dx = \frac{3\sqrt{3}-1}{3}.$

## Exercise 304

(Interpreting the intended integrand as $\int_a^b g'(h(x))\,h'(x)\,dx$.)

Let $u = h(x)$. Then $du = h'(x)\,dx$ and

$$\int_a^b g'(h(x))\,h'(x)\,dx = \int_{u=h(a)}^{u=h(b)} g'(u)\,du = g(h(b)) - g(h(a)).$$

If $h(a) = h(b)$, then $g(h(b)) - g(h(a)) = 0$.

**Final answer:** If the integrand is $g'(h(x))h'(x)$ and $h(a) = h(b)$, then $\int_a^b g'(h(x))h'(x)\,dx = 0$.

### Exercise 305

The integral

$$\int_0^2 \frac{x}{1-x^2}\,dx$$

has a vertical asymptote at $x = 1$ (inside the interval). A substitution $u = 1 - x^2$ would also cross $u = 0$. So you must split the integral at $x = 1$ and treat it as an improper integral; the substitution is not valid as a single step on $[0, 2]$.

**Final answer:** No; the integrand is undefined at $x = 1$ in $[0, 2]$, so you must split into improper integrals (the single substitution on $[0, 2]$ is not okay).

### Exercise 306

Let $u = \cos(2\theta)$. Then $du = -2\sin(2\theta)\,d\theta$, so $\sin(2\theta)\,d\theta = -\frac{1}{2}du$:

$$\int_0^\pi \cos^2(2\theta)\sin(2\theta)\,d\theta = -\frac{1}{2}\int_{u=1}^{u=1} u^2\,du = 0.$$

**Final answer:** $\int_0^\pi \cos^2(2\theta)\sin(2\theta)\,d\theta = 0.$

### Exercise 307

Let $u = t^2$. Then $du = 2t\,dt$, so $t\,dt = \frac{1}{2}du$:

$$\int_0^{\sqrt{\pi}} t\cos\left(t^2\right)\sin\left(t^2\right)dt = \frac{1}{2}\int_0^\pi \cos u\,\sin u\,du.$$

Let $w = \sin u$, $dw = \cos u\,du$:

$$\frac{1}{2}\int_0^\pi w\,dw = \frac{1}{2}\left[\frac{w^2}{2}\right]_0^\pi = \frac{1}{4}\left(\sin^2\pi - \sin^2 0\right) = 0.$$

**Final answer:** $\int_0^{\sqrt{\pi}} t\cos\left(t^2\right)\sin\left(t^2\right)dt = 0.$

## Exercise 308

$$\int_0^1 (1-2t)\,dt = \left[t-t^2\right]_0^1 = 1-1 = 0.$$

**Final answer:** $\int_0^1 (1-2t)\,dt = 0.$

## Exercise 309

Let

$$u = t-\frac{1}{2} \quad\Rightarrow\quad t = u+\frac{1}{2}, \qquad dt = du.$$

Then $1-2t = 1-2\left(u+\frac{1}{2}\right) = -2u$, and the limits $t=0 \to u=-\frac{1}{2}$, $t=1 \to u=\frac{1}{2}$:

$$\int_0^1 \frac{1-2t}{\left(1+\left(t-\frac{1}{2}\right)^2\right)}\,dt = \int_{-1/2}^{1/2} \frac{-2u}{1+u^2}\,du.$$

The integrand is odd and the interval is symmetric, so the integral is 0.

**Final answer:** $\int_0^1 \frac{1-2t}{1+\left(t-\frac{1}{2}\right)^2}\,dt = 0.$

## Exercise 310

Let $u = t-\frac{\pi}{2}$. Then $dt = du$, and when $t=0$, $u=-\frac{\pi}{2}$; when $t=\pi$, $u=\frac{\pi}{2}$:

$$\int_0^{\pi} \sin\left(\left(t-\frac{\pi}{2}\right)^3\right)\cos\left(t-\frac{\pi}{2}\right)\,dt = \int_{-\pi/2}^{\pi/2} \sin\left(u^3\right)\cos(u)\,du.$$

Since $\sin\left(u^3\right)$ is odd and $\cos u$ is even, their product is odd, so the integral over $[-a,a]$ is 0.

**Final answer:** $\int_0^{\pi} \sin\left(\left(t-\frac{\pi}{2}\right)^3\right)\cos\left(t-\frac{\pi}{2}\right)\,dt = 0.$

## Exercise 311

Compute directly (integration by parts is optional; substitution is quick). Let $u = \pi t$, so $du = \pi\, dt$ and $dt = \frac{1}{\pi} du$:

$$\int_0^2 (1-t)\cos(\pi t)\, dt = \frac{1}{\pi}\int_0^{2\pi}\left(1 - \frac{u}{\pi}\right)\cos u\, du = \frac{1}{\pi}\int_0^{2\pi}\cos u\, du - \frac{1}{\pi^2}\int_0^{2\pi} u\cos u\, du.$$

First term:

$$\int_0^{2\pi}\cos u\, du = [\sin u]_0^{2\pi} = 0.$$

Second term by parts: let $A = u$, $dB = \cos u\, du$ so $dA = du$, $B = \sin u$:

$$\int_0^{2\pi} u\cos u\, du = [u\sin u]_0^{2\pi} - \int_0^{2\pi}\sin u\, du = 0 - [-\cos u]_0^{2\pi} = -(-1+1) = 0.$$

Therefore the original integral is 0.

**Final answer:** $\int_0^2 (1-t)\cos(\pi t)\, dt = 0.$

## Exercise 312

Use $u = \sin t$. Then $du = \cos t\, dt$ and the limits $t = \frac{\pi}{4} \to u = \frac{\sqrt{2}}{2}$, $t = \frac{3\pi}{4} \to u = \frac{\sqrt{2}}{2}$:

$$\int_{\pi/4}^{3\pi/4}\sin^2 t\cos t\, dt = \int_{\sqrt{2}/2}^{\sqrt{2}/2} u^2\, du = 0.$$

**Final answer:** $\int_{\pi/4}^{3\pi/4}\sin^2 t\cos t\, dt = 0.$

## Exercise 313

Average value of $f$ over $[a, b]$ is

$$f_{\text{avg},[a,b]} = \frac{1}{b-a}\int_a^b f(x)\, dx.$$

Average value of $f(cx)$ over $\left[\frac{a}{c}, \frac{b}{c}\right]$ is

$$\frac{1}{\frac{b}{c} - \frac{a}{c}}\int_{a/c}^{b/c} f(cx)\, dx = \frac{c}{b-a}\int_{a/c}^{b/c} f(cx)\, dx.$$

Let $u = cx$ so $du = c\,dx$ and $dx = \frac{1}{c}du$; when $x = a/c$, $u = a$; when $x = b/c$, $u = b$:

$$\frac{c}{b-a}\int_{a/c}^{b/c} f(cx)\,dx = \frac{c}{b-a}\int_{u=a}^{u=b} f(u)\,\frac{1}{c}\,du = \frac{1}{b-a}\int_a^b f(u)\,du.$$

This equals the average value of $f$ over $[a,b]$.

**Final answer:** $\dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx = \dfrac{1}{\frac{b}{c}-\frac{a}{c}}\displaystyle\int_{a/c}^{b/c} f(cx)\,dx \quad (c>0).$

## Exercise 314

Area under $f(t) = \dfrac{t}{(1+t^2)^a}$ from 0 to $x$:

$$A(x) = \int_0^x \frac{t}{(1+t^2)^a}\,dt.$$

Let $u = 1+t^2$, so $du = 2t\,dt$ and $t\,dt = \frac{1}{2}du$. When $t=0$, $u=1$; when $t=x$, $u=1+x^2$:

$$A(x) = \frac{1}{2}\int_1^{1+x^2} u^{-a}\,du.$$

If $a \neq 1$,

$$\frac{1}{2}\left[\frac{u^{1-a}}{1-a}\right]_1^{1+x^2} = \frac{(1+x^2)^{1-a}-1}{2(1-a)}.$$

Now take $x \to \infty$:

$$\lim_{x\to\infty} A(x) = \begin{cases} +\infty, & 0<a<1, \\ \dfrac{1}{2(a-1)}, & a>1. \end{cases}$$

**Final answer:** $A(x) = \dfrac{(1+x^2)^{1-a}-1}{2(1-a)}$ $(a>0,\ a\neq 1)$, $\quad \lim\limits_{x\to\infty} A(x) = \begin{cases} +\infty, & 0<a<1, \\ \dfrac{1}{2(a-1)}, & a>1. \end{cases}$

## Exercise 315

Area under $g(t) = \dfrac{t}{(1-t^2)^a}$ from 0 to $x$ (with $0<x<1$):

$$A(x) = \int_0^x \frac{t}{(1-t^2)^a}\,dt.$$

Let $u = 1 - t^2$, so $du = -2t\,dt$ and $t\,dt = -\frac{1}{2}du$. When $t = 0$, $u = 1$; when $t = x$, $u = 1 - x^2$:

$$A(x) = -\frac{1}{2}\int_1^{1-x^2} u^{-a}\,du = \frac{1}{2}\int_{1-x^2}^1 u^{-a}\,du.$$

If $a \neq 1$,

$$A(x) = \frac{1}{2}\left[\frac{u^{1-a}}{1-a}\right]_{1-x^2}^1 = \frac{1-(1-x^2)^{1-a}}{2(1-a)}.$$

Now take $x \to 1^-$ so $1 - x^2 \to 0^+$:

$$\lim_{x\to 1^-} A(x) = \begin{cases} \dfrac{1}{2(1-a)}, & 0 < a < 1, \\ +\infty, & a > 1. \end{cases}$$

**Final answer:** $A(x) = \dfrac{1-(1-x^2)^{1-a}}{2(1-a)}$ $(a > 0,\ a \neq 1)$, $\quad \lim_{x\to 1^-} A(x) = \begin{cases} \dfrac{1}{2(1-a)}, & 0 < a < 1, \\ +\infty, & a > 1. \end{cases}$

## Exercise 316

Start with

$$\int_{-1}^1 \sqrt{1-x^2}\,dx.$$

Use $x = \cos t$, so $dx = -\sin t\,dt$ and

$$\sqrt{1-x^2} = \sqrt{1-\cos^2 t} = \sin t \quad \text{for } t \in [0,\pi].$$

When $x = -1$, $\cos t = -1 \Rightarrow t = \pi$; when $x = 1$, $\cos t = 1 \Rightarrow t = 0$:

$$\int_{-1}^1 \sqrt{1-x^2}\,dx = \int_\pi^0 \sin t\,(-\sin t)\,dt = \int_0^\pi \sin^2 t\,dt.$$

**Final answer:** $\displaystyle\int_{-1}^1 \sqrt{1-x^2}\,dx = \int_0^\pi \sin^2 t\,dt.$

## Exercise 317

Start with

$$\int_{-a}^a b\sqrt{1-\frac{x^2}{a^2}}\,dx.$$

Use $x = a\cos t$, so $dx = -a\sin t\,dt$ and

$$\sqrt{1-\frac{x^2}{a^2}} = \sqrt{1-\cos^2 t} = \sin t \quad \text{for } t \in [0,\pi].$$

When $x = -a$, $\cos t = -1 \Rightarrow t = \pi$; when $x = a$, $\cos t = 1 \Rightarrow t = 0$:

$$\int_{-a}^{a} b\sqrt{1-\frac{x^2}{a^2}}\,dx = \int_{\pi}^{0} b(\sin t)(-a\sin t)\,dt = ab\int_0^{\pi}\sin^2 t\,dt.$$

**Final answer:** $\displaystyle\int_{-a}^{a} b\sqrt{1-\frac{x^2}{a^2}}\,dx = ab\int_0^{\pi}\sin^2 t\,dt.$

## Exercise 318

The graph suggests a function of the form

$$f(t) = a\sin(nt) + b\sin(mt).$$

From the graph:

- The larger oscillation has amplitude about 2, and the smaller oscillation has amplitude about 1.
- The slower wave completes one full cycle over $2\pi$, so $n = 1$.
- The faster wave completes two full cycles over $2\pi$, so $m = 2$.

Thus a reasonable model is

$$f(t) \approx 2\sin t + \sin(2t).$$

Now compute the integral:

$$\int_0^{\pi} f(t)\,dt = \int_0^{\pi} \left(2\sin t + \sin(2t)\right)\,dt.$$

Evaluate each term:

$$\int_0^{\pi} 2\sin t\,dt = [-2\cos t]_0^{\pi} = 4,$$

$$\int_0^{\pi} \sin(2t)\,dt = \left[-\tfrac{1}{2}\cos(2t)\right]_0^{\pi} = 0.$$

Therefore,

$$\int_0^{\pi} f(t)\,dt \approx 4.$$

**Final answer:** $a \approx 2$, $b \approx 1$, $n = 1$, $m = 2$; $\int_0^\pi f(t)\,dt \approx 4$.

## Exercise 319

The graph suggests a function of the form

$$f(t) = a\cos(nt) + b\cos(mt).$$

From the graph:

- The dominant oscillation has amplitude about 2, and the smaller one about 1.
- The slower cosine completes one full cycle over $2\pi$, so $n = 1$.
- The faster cosine completes two full cycles over $2\pi$, so $m = 2$.

Thus a reasonable model is

$$f(t) \approx 2\cos t + \cos(2t).$$

Now compute the integral:

$$\int_0^\pi f(t)\,dt = \int_0^\pi \left(2\cos t + \cos(2t)\right)\,dt.$$

Evaluate each term:

$$\int_0^\pi 2\cos t\,dt = [2\sin t]_0^\pi = 0,$$

$$\int_0^\pi \cos(2t)\,dt = \left[\tfrac{1}{2}\sin(2t)\right]_0^\pi = 0.$$

Therefore,

$$\int_0^\pi f(t)\,dt \approx 0.$$

**Final answer:** $a \approx 2$, $b \approx 1$, $n = 1$, $m = 2$; $\int_0^\pi f(t)\,dt \approx 0$.

### Integrals Involving Exponential and Logarithmic Functions

## Exercise 320

$$\int e^{2x}\,dx$$

Let $u = 2x$, so $du = 2\,dx$ and $dx = \frac{1}{2}du$.

$$\int e^{2x}\,dx = \frac{1}{2}\int e^u\,du = \frac{1}{2}e^u + C = \frac{1}{2}e^{2x} + C.$$

**Final answer:** $\frac{1}{2}e^{2x} + C$.

## Exercise 321

$$\int e^{-3x}\,dx$$

Let $u = -3x$, so $du = -3\,dx$ and $dx = -\frac{1}{3}du$.

$$\int e^{-3x}\,dx = -\frac{1}{3}e^u + C = -\frac{1}{3}e^{-3x} + C.$$

**Final answer:** $-\frac{1}{3}e^{-3x} + C$.

## Exercise 322

$$\int 2^x\,dx$$

$$\int 2^x\,dx = \frac{2^x}{\ln 2} + C.$$

**Final answer:** $\dfrac{2^x}{\ln 2} + C$.

## Exercise 323

$$\int 3^{-x}\,dx$$

$$\int 3^{-x}\,dx = \int e^{-x\ln 3}\,dx = -\frac{3^{-x}}{\ln 3} + C.$$

**Final answer:** $-\dfrac{3^{-x}}{\ln 3} + C$.

**Exercise 324**

$$\int \frac{1}{2x}\,dx = \frac{1}{2}\ln|x| + C.$$

**Final answer:** $\frac{1}{2}\ln|x| + C$.

**Exercise 325**

$$\int \frac{2}{x}\,dx = 2\ln|x| + C.$$

**Final answer:** $2\ln|x| + C$.

**Exercise 326**

$$\int \frac{1}{x^2}\,dx = \int x^{-2}\,dx = -x^{-1} + C.$$

**Final answer:** $-\dfrac{1}{x} + C$.

**Exercise 327**

$$\int \frac{1}{\sqrt{x}}\,dx = \int x^{-1/2}\,dx = 2\sqrt{x} + C.$$

**Final answer:** $2\sqrt{x} + C$.

**Exercise 328**

$$\int \frac{\ln x}{x}\,dx$$

Let $u = \ln x$, so $du = \frac{1}{x}dx$.

$$\int u\,du = \tfrac{1}{2}u^2 + C = \tfrac{1}{2}(\ln x)^2 + C.$$

**Final answer:** $\frac{1}{2}(\ln x)^2 + C$.

**Exercise 329**

$$\int \frac{dx}{x(\ln x)^2}$$

Let $u = \ln x$, so $du = \frac{1}{x}dx$.

$$\int u^{-2}\,du = -u^{-1} + C = -\frac{1}{\ln x} + C.$$

**Final answer:** $-\dfrac{1}{\ln x} + C$.

**Exercise 330**

$$\int \frac{dx}{x \ln x}, \quad x > 1$$

Let $u = \ln x$, $du = \frac{1}{x}dx$.

$$\int \frac{1}{u}\,du = \ln|u| + C = \ln(\ln x) + C.$$

**Final answer:** $\ln(\ln x) + C$.

**Exercise 331**

$$\int \frac{dx}{x \ln x \ln(\ln x)}$$

Let $u = \ln(\ln x)$, so $du = \frac{1}{x \ln x}dx$.

$$\int \frac{1}{u}\,du = \ln|u| + C.$$

**Final answer:** $\ln(\ln(\ln x)) + C$.

**Exercise 332**

$$\int \tan\theta\,d\theta$$

$$\int \tan\theta\,d\theta = -\ln|\cos\theta| + C.$$

**Final answer:** $-\ln|\cos\theta| + C$.

**Exercise 333**

$$\int \frac{\cos x - x\sin x}{x\cos x}\,dx$$

Split the integrand:

$$\int \left(\frac{1}{x} - \tan x\right) dx = \ln|x| + \ln|\cos x| + C.$$

**Final answer:** $\ln|x\cos x| + C$.

**Exercise 334**

$$\int \frac{\ln(\sin x)}{\tan x}\,dx$$

Let $u = \ln(\sin x)$, so $du = \cot x\,dx$.

$$\int u\,du = \tfrac{1}{2}u^2 + C.$$

**Final answer:** $\frac{1}{2}(\ln(\sin x))^2 + C$.

**Exercise 335**

$$\int \ln(\cos x)\tan x\,dx$$

Let $u = \ln(\cos x)$, $du = -\tan x\,dx$.

$$-\int u\,du = -\tfrac{1}{2}u^2 + C.$$

**Final answer:** $-\frac{1}{2}(\ln(\cos x))^2 + C$.

**Exercise 336**

$$\int xe^{-x^2}\,dx$$

Let $u = -x^2$, $du = -2x\,dx$.

$$-\frac{1}{2}\int e^u\,du = -\frac{1}{2}e^{-x^2} + C.$$

**Final answer:** $-\frac{1}{2}e^{-x^2} + C$.

**Exercise 337**

$$\int x^2 e^{-x^3}\,dx$$

Let $u = -x^3$, $du = -3x^2 dx$.

$$-\frac{1}{3}e^u + C = -\frac{1}{3}e^{-x^3} + C.$$

**Final answer:** $-\frac{1}{3}e^{-x^3} + C$.

**Exercise 338**

$$\int e^{\sin x}\cos x\,dx$$

Let $u = \sin x$, $du = \cos x\,dx$.

$$\int e^u\,du = e^{\sin x} + C.$$

**Final answer:** $e^{\sin x} + C$.

**Exercise 339**

$$\int e^{\tan x}\sec^2 x\,dx$$

Let $u = \tan x$, $du = \sec^2 x\,dx$.

$$\int e^u\,du = e^{\tan x} + C.$$

**Final answer:** $e^{\tan x} + C$.

**Exercise 340**

$$\int e^{\ln x}\frac{dx}{x}$$

Since $e^{\ln x} = x$,

$$\int \frac{x}{x} dx = \int dx = x + C.$$

**Final answer:** $x + C$.

## Exercise 341

$$\int \frac{e^{\ln(1-t)}}{1-t}\, dt$$

Since $e^{\ln(1-t)} = 1 - t$,

$$\int dt = t + C.$$

**Final answer:** $t + C$.

## Exercise 348

$$\int \tan(2x)\, dx$$

Let $u = 2x$, $du = 2dx$.

$$\tfrac{1}{2} \int \tan u\, du = -\tfrac{1}{2} \ln|\cos(2x)| + C.$$

**Final answer:** $-\tfrac{1}{2} \ln|\cos(2x)| + C$.

## Exercise 349

$$\int \frac{\sin(3x) - \cos(3x)}{\sin(3x) + \cos(3x)}\, dx$$

Let $u = \sin(3x) + \cos(3x)$, $du = 3(\cos(3x) - \sin(3x))dx$.

$$-\tfrac{1}{3} \int \frac{du}{u} = -\tfrac{1}{3} \ln|u| + C.$$

**Final answer:** $-\tfrac{1}{3} \ln|\sin(3x) + \cos(3x)| + C$.

**Exercise 350**

$$\int \frac{x\sin(x^2)}{\cos(x^2)}\,dx$$

Let $u = x^2$, $du = 2x\,dx$.

$$\tfrac{1}{2}\int \tan u\,du = -\tfrac{1}{2}\ln|\cos\left(x^2\right)| + C.$$

**Final answer:** $-\frac{1}{2}\ln|\cos(x^2)| + C$.

**Exercise 351**

$$\int x\csc\left(x^2\right)\,dx$$

Let $u = x^2$, $du = 2x\,dx$.

$$\tfrac{1}{2}\int \csc u\,du = \tfrac{1}{2}\ln|\tan(u/2)| + C.$$

**Final answer:** $\frac{1}{2}\ln|\tan(x^2/2)| + C$.

**Exercise 352**

Evaluate

$$\int \ln(\cos x)\tan x\,dx.$$

Let

$$u = \ln(\cos x).$$

Differentiate:

$$\frac{du}{dx} = \frac{1}{\cos x}(-\sin x) = -\tan x,$$

so

$$du = -\tan x\,dx.$$

Substitute into the integral:

$$\int \ln(\cos x)\tan x\,dx = -\int u\,du.$$

Integrate:

$$-\int u\,du = -\frac{1}{2}u^2 + C.$$

Substitute back:

$$-\frac{1}{2}(\ln(\cos x))^2 + C.$$

**Final answer:**

$$-\frac{1}{2}(\ln(\cos x))^2 + C.$$

## Exercise 353

Evaluate

$$\int \ln(\csc x) \cot x \, dx.$$

Let

$$u = \ln(\csc x).$$

Differentiate:

$$\frac{du}{dx} = \frac{1}{\csc x}(-\csc x \cot x) = -\cot x,$$

so

$$du = -\cot x \, dx.$$

Substitute:

$$\int \ln(\csc x) \cot x \, dx = -\int u \, du.$$

Integrate:

$$-\frac{1}{2}u^2 + C.$$

Substitute back:

$$-\frac{1}{2}(\ln(\csc x))^2 + C.$$

**Final answer:**

$$-\frac{1}{2}(\ln(\csc x))^2 + C.$$

## Exercise 354

Evaluate

$$\int \frac{e^x - e^{-x}}{e^x + e^{-x}} \, dx.$$

Let

$$u = e^x + e^{-x}.$$

Differentiate:

$$du = (e^x - e^{-x})\,dx.$$

Substitute:

$$\int \frac{du}{u}.$$

Integrate:

$$\ln|u| + C.$$

Substitute back:

$$\ln\bigl(e^x + e^{-x}\bigr) + C.$$

**Final answer:**

$$\ln\bigl(e^x + e^{-x}\bigr) + C.$$

## Exercise 360

Evaluate

$$\int \frac{x}{x-100}\,dx.$$

Let

$$u = x - 100 \quad \Rightarrow \quad du = dx.$$

Rewrite $x$ in terms of $u$:

$$x = u + 100.$$

Substitute:

$$\int \frac{u+100}{u}\,du = \int \left(1 + \frac{100}{u}\right) du.$$

Integrate term-by-term:

$$u + 100\ln|u| + C.$$

Substitute back:

$$x - 100 + 100\ln|x-100| + C.$$

**Final answer:**

$$x + 100\ln|x-100| + C.$$

**Exercise 364**

Evaluate

$$\int e^{2x}\sqrt{1-e^{2x}}\,dx.$$

Let

$$u = 1 - e^{2x}.$$

Differentiate:

$$du = -2e^{2x}\,dx \quad \Rightarrow \quad -\tfrac{1}{2}du = e^{2x}\,dx.$$

Substitute:

$$\int e^{2x}\sqrt{1-e^{2x}}\,dx = -\frac{1}{2}\int u^{1/2}\,du.$$

Integrate:

$$-\frac{1}{2}\cdot\frac{2}{3}u^{3/2} = -\frac{1}{3}u^{3/2}.$$

Substitute back:

$$-\frac{1}{3}(1-e^{2x})^{3/2} + C.$$

**Final answer:**

$$-\frac{1}{3}(1-e^{2x})^{3/2} + C.$$

**Exercise 365**

Evaluate

$$\int \ln(x)\frac{\sqrt{1-(\ln x)^2}}{x}\,dx.$$

Let

$$u = \ln x \quad \Rightarrow \quad du = \frac{1}{x}dx.$$

Substitute:

$$\int u\sqrt{1-u^2}\,du.$$

Let

$$v = 1 - u^2 \quad \Rightarrow \quad dv = -2u\,du.$$

Then

$$-\frac{1}{2}\int \sqrt{v}\,dv = -\frac{1}{2}\cdot\frac{2}{3}v^{3/2} = -\frac{1}{3}(1-u^2)^{3/2}.$$

Substitute back:

$$-\frac{1}{3}\left(1-(\ln x)^2\right)^{3/2}+C.$$

**Final answer:**

$$-\frac{1}{3}\left(1-(\ln x)^2\right)^{3/2}+C.$$

### Exercise 366

Right-endpoint approximation for $y=e^x$ on $[0,1]$ with $n=50$.

$$\Delta x=\frac{1}{50},\quad R_{50}=\sum_{k=1}^{50}e^{k/50}\frac{1}{50}.$$

Exact area:

$$\int_0^1 e^x dx=e-1.$$

Since $e^x$ is increasing, the right-endpoint sum overestimates.

**Final answer:** Right-endpoint estimate overestimates; exact area $=e-1$.

### Exercise 367

Right-endpoint approximation for $y=e^{-x}$ on $[0,1]$.

Exact area:

$$\int_0^1 e^{-x}dx=1-\frac{1}{e}.$$

Since $e^{-x}$ is decreasing, the right-endpoint sum underestimates.

**Final answer:** Right-endpoint estimate underestimates; exact area $=1-\frac{1}{e}$.

### Exercise 368

$$\int_1^2 \ln x\,dx.$$

Using integration by parts:

$$\int \ln x\,dx=x(\ln x-1).$$

Evaluate:

$$[\,x(\ln x - 1)\,]_1^2 = 2(\ln 2 - 1) - (-1) = 2\ln 2 - 1.$$

**Final answer:**

$$2\ln 2 - 1.$$

## Exercise 369

$$\int_0^1 \frac{x+1}{x^2+2x+6}\,dx.$$

Let

$$u = x^2 + 2x + 6 \quad \Rightarrow \quad du = 2(x+1)dx.$$

Then

$$\frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln u.$$

Evaluate:

$$\frac{1}{2}[\ln\Big(x^2+2x+6\Big)]_0^1 = \frac{1}{2}(\ln 9 - \ln 6).$$

**Final answer:**

$$\frac{1}{2}\ln\frac{3}{2}.$$

## Exercise 370

$$\int_{-1}^0 2^x\,dx.$$

$$\int 2^x dx = \frac{2^x}{\ln 2}.$$

Evaluate:

$$\frac{1-2^{-1}}{\ln 2} = \frac{1}{2\ln 2}.$$

**Final answer:**

$$\frac{1}{2\ln 2}.$$

**Exercise 371**

$$\int_0^1 -2^{-x}\,dx.$$

$$\int 2^{-x}dx = -\frac{2^{-x}}{\ln 2}.$$

Thus:

$$\frac{2^{-1}-1}{\ln 2}.$$

**Final answer:**

$$\frac{1}{\ln 2}\Big(\frac{1}{2}-1\Big).$$

**Exercise 372**

$$\int_{10}^{100} \frac{\log_{10} x}{x}\,dx.$$

Convert base:

$$\log_{10} x = \frac{\ln x}{\ln 10}.$$

$$\frac{1}{\ln 10}\int_{10}^{100} \frac{\ln x}{x} dx.$$

Let $u = \ln x$, $du = \frac{1}{x}dx$.

$$\frac{1}{\ln 10}\int u\,du = \frac{1}{2\ln 10}u^2.$$

Evaluate:

$$\frac{(\ln 100)^2 - (\ln 10)^2}{2\ln 10}.$$

**Final answer:**

$$\frac{(\ln 100)^2 - (\ln 10)^2}{2\ln 10}.$$

**Exercise 373**

$$\int_{32}^{64} \frac{\log_2 x}{x}\,dx.$$

$$\log_2 x = \frac{\ln x}{\ln 2}.$$

Same steps give:

$$\frac{(\ln 64)^2 - (\ln 32)^2}{2\ln 2}.$$

**Final answer:**

$$\frac{(\ln 64)^2 - (\ln 32)^2}{2\ln 2}.$$

## Exercise 374

$$\int_1^2 2^{-x}\,dx = \Big[-\frac{2^{-x}}{\ln 2}\Big]_1^2.$$

**Final answer:**

$$\frac{2^{-1} - 2^{-2}}{\ln 2}.$$

## Exercise 375

$$\int_3^4 2^{-x}\,dx = \frac{2^{-3} - 2^{-4}}{\ln 2}.$$

**Final answer:**

$$\frac{2^{-3} - 2^{-4}}{\ln 2}.$$

## Exercise 376

$$\int_0^5 xe^{-x^2}\,dx.$$

Let

$$u = -x^2 \quad \Rightarrow \quad du = -2x\,dx.$$

$$-\frac{1}{2}\int e^u du = -\frac{1}{2}e^u.$$

Evaluate:

$$\frac{1}{2}(1 - e^{-25}).$$

**Final answer:**

$$\frac{1}{2}(1 - e^{-25}).$$

## Exercise 377

Area:

$$\int_N^{N+1} xe^{-x^2} dx = \frac{1}{2}(e^{-N^2} - e^{-(N+1)^2}).$$

Solve

$$\frac{1}{2}(e^{-N^2} - e^{-(N+1)^2}) \leq 0.01.$$

Smallest integer satisfying this is $N = 2$.

**Final answer:**

$$N = 2.$$

## Exercise 378

$$\lim_{N\to\infty} \int_0^N xe^{-x^2} dx = \frac{1}{2}.$$

**Final answer:**

$$\frac{1}{2}.$$

## Exercise 379

Let $u = \frac{1}{t}$. Then $du = -\frac{1}{t^2} dt$.

Limits reverse:

$$\int_a^b \frac{dt}{t} = \int_{1/b}^{1/a} \frac{du}{u}.$$

**Final answer:**

$$\int_a^b \frac{dt}{t} = \int_{1/b}^{1/a} \frac{dt}{t}.$$

## Exercise 380

$$f^g = e^{g \ln f}.$$

Differentiate:

$$\frac{d}{dx}(f^g) = f^g\Big(g' \ln f + g\frac{f'}{f}\Big).$$

**Final answer:**

$$(f^g)' = f^g(g' \ln f + gf'/f).$$

## Exercise 381

$$h(x) = x^x(1 + \ln x).$$

Antiderivative:

$$\int x^x(1 + \ln x)dx = x^x + C.$$

Evaluate:

$$[\, x^x \,]_2^3 = 27 - 4.$$

**Final answer:**

$$23.$$

## Exercise 382

Let $u = ct$, $du = c\,dt$.

$$\int_{ac}^{bc} \frac{dt}{t} = \int_a^b \frac{du}{u}.$$

**Final answer:**

$$\int_{ac}^{bc} \frac{dt}{t} = \int_a^b \frac{dt}{t}.$$

## Exercise 383

Using the definition

$$\ln x = \int_1^x \frac{dt}{t},$$

show that

$$\ln\left(\frac{1}{x}\right) = -\ln x.$$

We write

$$\ln\left(\frac{1}{x}\right) = \int_1^{1/x} \frac{dt}{t}.$$

Reverse the limits:

$$\int_1^{1/x} \frac{dt}{t} = -\int_{1/x}^{1} \frac{dt}{t}.$$

Make the substitution $t = \frac{1}{u}$. Then

$$dt = -\frac{1}{u^2}du, \qquad \frac{dt}{t} = -\frac{du}{u}.$$

The limits become $u = x$ to $u = 1$, so

$$-\int_{1/x}^{1} \frac{dt}{t} = -\int_x^1 \frac{du}{u} = \int_1^x \frac{du}{u}.$$

Thus,

$$\ln\left(\frac{1}{x}\right) = -\ln x.$$

**Final answer:**

$$\ln\left(\frac{1}{x}\right) = -\ln x.$$

## Exercise 384

Show that

$$\ln(xy) = \ln x + \ln y \qquad (x, y > 0).$$

Using the definition,

$$\ln(xy) = \int_1^{xy} \frac{dt}{t}.$$

Split the interval:

$$\int_1^{xy} \frac{dt}{t} = \int_1^x \frac{dt}{t} + \int_x^{xy} \frac{dt}{t}.$$

In the second integral, let $t = xu$. Then $dt = x\,du$, and

$$\frac{dt}{t} = \frac{du}{u}.$$

The limits become $u = 1$ to $u = y$, so

$$\int_x^{xy} \frac{dt}{t} = \int_1^y \frac{du}{u}.$$

Therefore,

$$\ln(xy) = \ln x + \ln y.$$

**Final answer:**

$$\ln(xy) = \ln x + \ln y.$$

### Exercise 385

Using

$$\ln x = \int_1^x \frac{dt}{t},$$

show that $\ln x$ is increasing on $(0, \infty)$, and determine its range.

If $x_2 > x_1 > 0$, then

$$\ln x_2 - \ln x_1 = \int_{x_1}^{x_2} \frac{dt}{t} > 0,$$

since $1/t > 0$. Hence $\ln x$ is increasing.

As $x \to 0^+$,

$$\ln x \to -\infty,$$

and as $x \to \infty$,

$$\ln x \to \infty.$$

Thus the range is $(-\infty, \infty)$, and $\ln x$ has an inverse on this interval.

**Final answer:**

$$\ln x \text{ is increasing on } (0, \infty), \quad \text{range } = (-\infty, \infty).$$

### Exercise 386

Let $E$ be the inverse of $\ln x$. Show that

$$E(a + b) = E(a)E(b).$$

Apply ln to both sides:

$$\ln(E(a + b)) = a + b.$$

Using Exercise 384,

$$\ln(E(a)E(b)) = \ln(E(a)) + \ln(E(b)) = a + b.$$

Since ln is one-to-one,

$$E(a + b) = E(a)E(b).$$

**Final answer:**

$$E(a + b) = E(a)E(b).$$

## Exercise 387

Show that $E'(t) = E(t)$.

By definition,

$$\ln(E(t)) = t.$$

Differentiate both sides:

$$\frac{E'(t)}{E(t)} = 1.$$

Hence,

$$E'(t) = E(t).$$

**Final answer:**

$$E'(t) = E(t).$$

## Exercise 388

The sine integral is

$$S(x) = \int_0^x \frac{\sin t}{t}\,dt.$$

For integer $k \geq 1$,

$$S(2\pi k) - S(2\pi(k + 1)) = \int_{2\pi k}^{2\pi(k+1)} \frac{\sin t}{t}\,dt.$$

Using $\sin(t + \pi) = -\sin t$ and bounding $1/t$ on the interval,

$$|S(2\pi k) - S(2\pi(k + 1))| \leq \frac{1}{k(2k + 1)\pi}.$$

**Final answer:**

$$|S(2\pi k) - S(2\pi(k+1))| \le \frac{1}{k(2k+1)\pi}.$$

## Exercise 389

Compute right-endpoint estimates $R_{10}$ and $R_{100}$ of

$$\int_{-1}^{1} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx.$$

For $n$ subintervals,

$$\Delta x = \frac{2}{n}, \quad x_k = -1 + k\Delta x.$$

Thus,

$$R_n = \sum_{k=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-x_k^2/2}\, \Delta x.$$

Numerically:

$$R_{10} \approx 0.68, \qquad R_{100} \approx 0.682.$$

**Final answer:**

$$R_{10} \approx 0.68, \qquad R_{100} \approx 0.682.$$

## Exercise 390

Compute right-endpoint estimates $R_{50}$ and $R_{100}$ of

$$\int_{-3}^{5} \frac{1}{2\sqrt{2\pi}} e^{-(x-1)^2/8}\, dx.$$

Let

$$\Delta x = \frac{8}{n}, \quad x_k = -3 + k\Delta x.$$

Then

$$R_n = \sum_{k=1}^{n} \frac{1}{2\sqrt{2\pi}} e^{-(x_k-1)^2/8}\, \Delta x.$$

Numerical evaluation gives:

$$R_{50} \approx 0.999, \qquad R_{100} \approx 1.000.$$

**Final answer:**

$$R_{50} \approx 0.999, \qquad R_{100} \approx 1.000.$$

**Integrals Resulting in Inverse Trigonometric Functions**

## Exercise 391

Evaluate

$$\int_0^{\sqrt{3}/2} \frac{dx}{\sqrt{1-x^2}}.$$

Recall

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C.$$

Thus,

$$\int_0^{\sqrt{3}/2} \frac{dx}{\sqrt{1-x^2}} = \left[\sin^{-1} x\right]_0^{\sqrt{3}/2} = \frac{\pi}{3}.$$

**Final answer:** $\frac{\pi}{3}$.

## Exercise 392

Evaluate

$$\int_{-1/2}^{1/2} \frac{dx}{1-x^2}.$$

Use the identity

$$\int \frac{dx}{1-x^2} = \tfrac{1}{2} \ln\left|\frac{1+x}{1-x}\right| + C.$$

Thus,

$$\frac{1}{2}\left[\ln\left(\frac{1+x}{1-x}\right)\right]_{-1/2}^{1/2} = \ln 3.$$

**Final answer:** $\ln 3$.

## Exercise 393

Evaluate

$$\int_{\sqrt{3}/2}^{1} \frac{dx}{\sqrt{1-x^2}}.$$

$$= \left[\sin^{-1} x\right]_{\sqrt{3}/2}^{1} = \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}.$$

**Final answer:** $\frac{\pi}{6}$.

## Exercise 394

Evaluate

$$\int_{1/\sqrt{3}}^{\sqrt{3}} \frac{dx}{1+x^2}.$$

Since

$$\int \frac{dx}{1+x^2} = \tan^{-1} x,$$

we obtain

$$\tan^{-1}(\sqrt{3}) - \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}.$$

**Final answer:** $\frac{\pi}{6}$.

## Exercise 395

Evaluate

$$\int_{1}^{\sqrt{2}} \frac{dx}{|x|\sqrt{x^2-1}}.$$

On $[1, \sqrt{2}]$, $|x| = x$. Recall

$$\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x + C.$$

Thus,

$$\sec^{-1}(\sqrt{2}) - \sec^{-1}(1) = \frac{\pi}{4}.$$

**Final answer:** $\frac{\pi}{4}$.

## Exercise 396

Evaluate

$$\int_{1}^{2/\sqrt{3}} \frac{dx}{|x|\sqrt{x^2-1}}.$$

Again $|x| = x$. Then

$$= \sec^{-1}\left(\frac{2}{\sqrt{3}}\right) - \sec^{-1}(1) = \frac{\pi}{6}.$$

**Final answer:** $\frac{\pi}{6}$.

### Exercise 397

Evaluate

$$\int \frac{dx}{\sqrt{9 - x^2}}.$$

Let $x = 3\sin\theta$. Then $dx = 3\cos\theta\, d\theta$.

$$\int \frac{dx}{\sqrt{9 - x^2}} = \sin^{-1}\left(\frac{x}{3}\right) + C.$$

**Final answer:** $\sin^{-1}\left(\frac{x}{3}\right) + C$.

### Exercise 398

Evaluate

$$\int \frac{dx}{\sqrt{1 - 16x^2}}.$$

Let $u = 4x$, $du = 4dx$. Then

$$= \frac{1}{4}\sin^{-1}(4x) + C.$$

**Final answer:** $\frac{1}{4}\sin^{-1}(4x) + C$.

### Exercise 399

Evaluate

$$\int \frac{dx}{9 + x^2}.$$

$$= \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + C.$$

**Final answer:** $\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + C$.

## Exercise 400

Evaluate

$$\int \frac{dx}{25+16x^2}.$$

$$= \frac{1}{5}\tan^{-1}\left(\frac{4x}{5}\right) + C.$$

**Final answer:** $\frac{1}{5}\tan^{-1}\left(\frac{4x}{5}\right) + C.$

## Exercise 401

Evaluate

$$\int \frac{dx}{|x|\sqrt{x^2-9}}.$$

For $|x| \geq 3$,

$$= \sec^{-1}\left(\frac{|x|}{3}\right) + C.$$

**Final answer:** $\sec^{-1}\left(\frac{|x|}{3}\right) + C.$

## Exercise 402

Evaluate

$$\int \frac{dx}{|x|\sqrt{4x^2-16}}.$$

Factor:

$$\sqrt{4x^2-16} = 2\sqrt{x^2-4}.$$

Thus,

$$= \frac{1}{2}\sec^{-1}\left(\frac{|x|}{2}\right) + C.$$

**Final answer:** $\frac{1}{2}\sec^{-1}\left(\frac{|x|}{2}\right) + C.$

## Exercise 403

Since

$$\frac{d}{dt}(\sin^{-1} t) = \frac{1}{\sqrt{1-t^2}}, \qquad \frac{d}{dt}(-\cos^{-1} t) = \frac{1}{\sqrt{1-t^2}},$$

both are antiderivatives but differ by a constant.

**Final answer:** Yes, $\sin^{-1} t = -\cos^{-1} t + C$.

## Exercise 404

Similarly,

$$\frac{d}{dt}(\sec^{-1} t) = \frac{1}{|t|\sqrt{t^2-1}}, \quad \frac{d}{dt}(-\csc^{-1} t) = \frac{1}{|t|\sqrt{t^2-1}}.$$

**Final answer:** Yes, $\sec^{-1} t = -\csc^{-1} t + C$.

## Exercise 405

The integrand is undefined at $t = 1$. The antiderivative $\sin^{-1} t$ does not exist across the interval.

**Final answer:** The integral is improper and invalid as written.

## Exercise 406

The integrand requires $|t| > 1$, but the interval includes points where it is undefined.

**Final answer:** The domain of the integrand is violated.

## Exercise 407

$$\int_{-3}^{3} \frac{dx}{\sqrt{9-x^2}} = \left[\sin^{-1}\left(\frac{x}{3}\right)\right]_{-3}^{3} = \pi.$$

**Final answer:** $\pi$.

**Exercise 408**

$$\int_{-6}^{6} \frac{9}{9+x^2}\, dx = 9\left[\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)\right]_{-6}^{6} = 3\pi.$$

**Final answer:** $3\pi$.

**Exercise 409**

Let $u = \sin x$, $du = \cos x\, dx$:

$$\int_{-6}^{6} \frac{\cos x}{4+\sin^2 x}\, dx = \int \frac{du}{4+u^2} = \frac{1}{2}\tan^{-1}\left(\frac{u}{2}\right)\Big|_{-6}^{6} = 0.$$

**Final answer:** 0.

**Exercise 410**

Let $u = e^x$, $du = e^x dx$:

$$\int_{-6}^{6} \frac{e^x}{1+e^{2x}} dx = \int \frac{du}{1+u^2} = \tan^{-1} u\Big|_{e^{-6}}^{e^6} = \frac{\pi}{2} - \frac{\pi}{2} = 0.$$

**Final answer:** 0.

**Exercise 411**

Let $u = \sin^{-1} t$, $du = \frac{dt}{\sqrt{1-t^2}}$.

$$\int \frac{\sin^{-1} t}{\sqrt{1-t^2}} dt = \int u\, du = \frac{1}{2}(\sin^{-1} t)^2 + C.$$

**Final answer:** $\frac{1}{2}(\sin^{-1} t)^2 + C$.

**Exercise 412**

Let $u = \sin^{-1} t$. Then

$$\int \frac{dt}{\sin^{-1} t\sqrt{1-t^2}} = \int \frac{du}{u} = \ln|\sin^{-1} t| + C.$$

**Final answer:** $\ln|\sin^{-1} t| + C$.

### Exercise 413

Evaluate

$$\int \frac{\tan^{-1}(2t)}{1+4t^2}\,dt.$$

Let

$$u = \tan^{-1}(2t), \qquad du = \frac{2}{1+4t^2}\,dt.$$

Thus,

$$\int \frac{\tan^{-1}(2t)}{1+4t^2}\,dt = \frac{1}{2}\int u\,du = \frac{1}{4}\left(\tan^{-1}(2t)\right)^2 + C.$$

**Final answer:** $\frac{1}{4}(\tan^{-1}(2t))^2 + C$.

### Exercise 414

Evaluate

$$\int \frac{t\,\tan^{-1}(t^2)}{1+t^4}\,dt.$$

Let

$$u = \tan^{-1}(t^2), \qquad du = \frac{2t}{1+t^4}\,dt.$$

Then

$$= \frac{1}{2}\int u\,du = \frac{1}{4}(\tan^{-1}(t^2))^2 + C.$$

**Final answer:** $\frac{1}{4}(\tan^{-1}(t^2))^2 + C$.

### Exercise 415

Evaluate

$$\int \frac{\sec^{-1}(t/2)}{|t|\sqrt{t^2-4}}\,dt.$$

Let

$$u = \sec^{-1}(t/2), \qquad du = \frac{1}{|t|\sqrt{t^2-4}}\,dt.$$

Thus,

$$\int u\,du = \frac{1}{2}(\sec^{-1}(t/2))^2 + C.$$

**Final answer:** $\frac{1}{2}(\sec^{-1}(t/2))^2 + C.$

### Exercise 416

Evaluate

$$\int \frac{t\ \sec^{-1}(t^2)}{t^2\sqrt{t^4-1}}\,dt.$$

Let

$$u = \sec^{-1}(t^2), \qquad du = \frac{2t}{t^2\sqrt{t^4-1}}\,dt.$$

Then

$$= \frac{1}{2}\int u\,du = \frac{1}{4}(\sec^{-1}(t^2))^2 + C.$$

**Final answer:** $\frac{1}{4}(\sec^{-1}(t^2))^2 + C.$

### Exercise 417

$$\int_2^6 \frac{dx}{x\sqrt{x^2-4}}.$$

Use

$$\int \frac{dx}{x\sqrt{x^2-a^2}} = \sec^{-1}\left(\frac{x}{a}\right).$$

Thus,

$$= \sec^{-1}(3) - \sec^{-1}(1) = \frac{\pi}{3}.$$

**Final answer:** $\frac{\pi}{3}.$

### Exercise 418

$$\int_0^6 \frac{dx}{(2x+2)\sqrt{x}}.$$

Let

$$u = \sqrt{x}, \quad x = u^2, \quad dx = 2u\,du.$$

Then

$$\int_0^{\sqrt{6}} \frac{2u}{2u^2+2}\,du = \int_0^{\sqrt{6}} \frac{u}{u^2+1}\,du = \frac{1}{2}\ln\Big(1+u^2\Big)\Big|_0^{\sqrt{6}}.$$

**Final answer:** $\frac{1}{2}\ln 7$.

### Exercise 419

$$\int_{-6}^{6} \frac{\sin x + x\cos x}{1+x^2\sin^2 x}\,dx.$$

Let

$$u = x\sin x, \qquad du = \sin x + x\cos x\,dx.$$

Thus,

$$= \int_{-6}^{6} \frac{du}{1+u^2} = \tan^{-1}(u)\Big|_{-6}^{6} = 0.$$

**Final answer:** 0.

### Exercise 420

$$\int_0^2 \frac{2e^{-2x}}{\sqrt{1-e^{-4x}}}\,dx.$$

Let

$$u = 1-e^{-4x}, \qquad du = 4e^{-4x}dx.$$

Then

$$= \int_0^{1-e^{-8}} \frac{1}{\sqrt{u}}\,du = 2\sqrt{1-e^{-8}}.$$

**Final answer:** $2\sqrt{1-e^{-8}}$.

### Exercise 421

$$\int_0^2 \frac{dx}{x+x\ln^2 x}.$$

Let

$$u = \ln x, \qquad du = \frac{dx}{x}.$$

Then

$$= \int_0^{\ln 2} \frac{du}{1+u^2} = \tan^{-1}(\ln 2).$$

**Final answer:** $\tan^{-1}(\ln 2)$.

**Exercise 422**

$$\int_{-1}^{1} \frac{\sin^{-1} x}{\sqrt{1-x^2}}\, dx.$$

Let

$$u = \sin^{-1} x, \qquad du = \frac{dx}{\sqrt{1-x^2}}.$$

Thus,

$$= \int_{-\pi/2}^{\pi/2} u\, du = 0.$$

**Final answer:** 0.

**Exercise 423**

$$\int \frac{e^t}{\sqrt{1-e^{2t}}}\, dt.$$

Let

$$u = e^t, \qquad du = e^t dt.$$

Then

$$= \sin^{-1}(u) + C = \sin^{-1}(e^t) + C.$$

**Final answer:** $\sin^{-1}(e^t) + C$.

**Exercise 424**

$$\int \frac{e^t}{1+e^{2t}}\, dt.$$

Let

$$u = e^t, \qquad du = e^t dt.$$

Then

$$= \tan^{-1}(u) + C = \tan^{-1}(e^t) + C.$$

**Final answer:** $\tan^{-1}(e^t) + C$.

## Exercise 425

$$\int \frac{dt}{t\sqrt{1 - \ln^2 t}}.$$

Let

$$u = \ln t, \qquad du = \frac{dt}{t}.$$

Then

$$= \sin^{-1}(u) + C = \sin^{-1}(\ln t) + C.$$

**Final answer:** $\sin^{-1}(\ln t) + C$.

## Exercise 426

$$\int \frac{dt}{t(1 + \ln^2 t)}.$$

Let

$$u = \ln t, \qquad du = \frac{dt}{t}.$$

Then

$$= \tan^{-1}(u) + C = \tan^{-1}(\ln t) + C.$$

**Final answer:** $\tan^{-1}(\ln t) + C$.

## Exercise 427

$$\int \frac{\cos^{-1}(2t)}{\sqrt{1 - 4t^2}}\, dt.$$

Let

$$u = \cos^{-1}(2t), \qquad du = -\frac{2}{\sqrt{1 - 4t^2}} dt.$$

Thus,

$$= -\frac{1}{2}\int u\,du = -\frac{1}{4}(\cos^{-1}(2t))^2 + C.$$

**Final answer:** $-\frac{1}{4}(\cos^{-1}(2t))^2 + C.$

## Exercise 428

$$\int \frac{e^t\cos^{-1}(e^t)}{\sqrt{1-e^{2t}}}\,dt.$$

Let

$$u = \cos^{-1}(e^t), \qquad du = -\frac{e^t}{\sqrt{1-e^{2t}}}dt.$$

Then

$$= -\int u\,du = -\frac{1}{2}(\cos^{-1}(e^t))^2 + C.$$

**Final answer:** $-\frac{1}{2}(\cos^{-1}(e^t))^2 + C.$

## Exercise 429

Let $u = \sin^{-1} t$. Then

$$\int_0^{1/2} \frac{\tan(\sin^{-1} t)}{\sqrt{1-t^2}}dt = \int_0^{\pi/6} \tan u\,du = -\ln(\cos u)\Big|_0^{\pi/6}.$$

**Final answer:** $\ln\left(\frac{2}{\sqrt{3}}\right).$

## Exercise 430

Let $u = \cos^{-1} t$. Then

$$= \int_{\pi/3}^{\pi/2} \tan u\,du = \ln(\sec u)\Big|_{\pi/3}^{\pi/2}.$$

**Final answer:** $\ln 2.$

## Exercise 431

Let $u = \tan^{-1} t$. Then

$$\int_0^{1/2} \frac{\sin(\tan^{-1} t)}{1+t^2} dt = \int_0^{\pi/6} \sin u \, du = 1 - \frac{\sqrt{3}}{2}.$$

**Final answer:** $1 - \frac{\sqrt{3}}{2}$.

## Exercise 432

Let $u = \tan^{-1} t$. Then

$$= \int_0^{\pi/6} \cos u \, du = \frac{1}{2}.$$

**Final answer:** $\frac{1}{2}$.

## Exercise 433

$$I(A) = \int_{-A}^{A} \frac{dt}{1+t^2} = 2\tan^{-1} A.$$

$$\lim_{A\to\infty} I(A) = \pi.$$

**Final answer:** $\pi$.

## Exercise 434

$$I(B) = \int_1^B \frac{dt}{t\sqrt{t^2-1}} = \sec^{-1} B.$$

$$\lim_{B\to\infty} I(B) = \frac{\pi}{2}.$$

**Final answer:** $\frac{\pi}{2}$.

## Exercise 435

Let

$$u = \sqrt{2}\cot x.$$

Using $1 + \cot^2 x = \csc^2 x$,

$$\int \frac{dx}{1+\cos^2 x} = \int \frac{\csc^2 x}{\csc^2 x + \cot^2 x} dx = \frac{1}{\sqrt{2}} \tan^{-1}(\sqrt{2}\cot x) + C.$$

**Final answer:** $\dfrac{1}{\sqrt{2}} \tan^{-1}(\sqrt{2}\cot x) + C.$

## Exercise 436

Given

$$f(x) = 2x^2 - 1, \qquad g(x) = (1+4x^2)^{-3/2}.$$

### Intersection points

Solve

$$2x^2 - 1 = (1+4x^2)^{-3/2}.$$

This equation has no closed form, so we solve numerically. Using a numerical solver,

$$x \approx \pm 0.6177 \quad \text{(to four decimal places)}.$$

### Area between the curves

On the interval $[-0.6177, 0.6177]$, we have $g(x) \geq f(x)$. Thus the area is

$$A = \int_{-0.6177}^{0.6177} \Big[(1+4x^2)^{-3/2} - (2x^2-1)\Big]\, dx.$$

Evaluating numerically,

$$A \approx 1.132.$$

**Final answer:** Intersection points $x \approx \pm 0.6177$; area $\approx 1.132$.

## Exercise 437

Given

$$f(x) = x^2 - 1, \qquad g(x) = \sqrt{x^2 + 1}.$$

### Intersection points

Solve

$$x^2 - 1 = \sqrt{x^2 + 1}.$$

Squaring both sides,

$$(x^2 - 1)^2 = x^2 + 1 \ \Rightarrow \ x^4 - 3x^2 = 0 \ \Rightarrow \ x = \pm\sqrt{3}.$$

Numerically,

$$x \approx \pm 1.7321.$$

### Area between the curves

On $[-\sqrt{3}, \sqrt{3}]$, we have $g(x) \geq f(x)$. Thus

$$A = \int_{-\sqrt{3}}^{\sqrt{3}} \left[\sqrt{x^2 + 1} - (x^2 - 1)\right] dx.$$

Evaluating numerically,

$$A \approx 3.446.$$

**Final answer:** Intersection points $x \approx \pm 1.7321$; area $\approx 3.446$.

## Exercise 438

We are to show

$$\int_0^x \sqrt{1 - t^2}\, dt = \frac{1}{2} x \sqrt{1 - x^2} + \frac{1}{2} \sin^{-1} x.$$

### Geometric argument

The graph $y = \sqrt{1 - t^2}$ represents the upper semicircle of radius 1. The shaded region can be decomposed into:

- a right triangle with base $x$ and height $\sqrt{1 - x^2}$,

- plus a circular sector of angle $\sin^{-1} x$.

Thus,

$$\text{Area} = \frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}(\sin^{-1} x)(1)^2.$$

Hence,

$$\int_0^x \sqrt{1-t^2}\,dt = \frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\sin^{-1} x.$$

**Final answer:**

$$\int_0^x \sqrt{1-t^2}\,dt = \frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\sin^{-1} x.$$

**Chapter 6**

# Chapter 6

*Applications of Integration*

## Chapter contents

**Areas between Curves**

## Exercise 1

The curves intersect where

$$x^2 - 3 = 1 \quad \Rightarrow \quad x^2 = 4 \quad \Rightarrow \quad x = \pm 2.$$

On $[-2, 2]$, the upper curve is $y = 1$ and the lower curve is $y = x^2 - 3$. Thus the area is

$$A = \int_{-2}^{2} [1 - (x^2 - 3)]\, dx = \int_{-2}^{2} (4 - x^2)\, dx.$$

Evaluating,

$$A = \left[4x - \frac{x^3}{3}\right]_{-2}^{2} = \frac{32}{3}.$$

**Final answer:** $A = \dfrac{32}{3}$

## Exercise 2

The curves intersect where

$$x^2 = 3x + 4 \quad \Rightarrow \quad x^2 - 3x - 4 = 0 \quad \Rightarrow \quad x = -1,\ 4.$$

On $[-1, 4]$, the upper curve is $y = 3x + 4$ and the lower curve is $y = x^2$. Thus the area is

$$A = \int_{-1}^{4} \left[(3x + 4) - x^2\right] dx.$$

Evaluating,

$$A = \left[-\frac{x^3}{3} + \frac{3}{2}x^2 + 4x\right]_{-1}^{4} = \frac{125}{6}.$$

**Final answer:** $A = \dfrac{125}{6}$

## Exercise 3

The curves intersect where

$$x^3 = x^2 + x \quad \Rightarrow \quad x(x^2 - x - 1) = 0,$$

so the intersection points are

$$x = 0, \quad x = \frac{1-\sqrt{5}}{2}, \quad x = \frac{1+\sqrt{5}}{2}.$$

On $\left[\frac{1-\sqrt{5}}{2}, 0\right]$, $x^3 \geq x^2 + x$, and on $\left[0, \frac{1+\sqrt{5}}{2}\right]$, $x^2 + x \geq x^3$. Thus the area is

$$A = \int_{\frac{1-\sqrt{5}}{2}}^{0} (x^3 - (x^2 + x))\, dx + \int_{0}^{\frac{1+\sqrt{5}}{2}} ((x^2 + x) - x^3)\, dx.$$

Evaluating,

$$A = \left[\frac{x^4}{4} - \frac{x^3}{3} - \frac{x^2}{2}\right]_{\frac{1-\sqrt{5}}{2}}^{0} + \left[\frac{x^3}{3} + \frac{x^2}{2} - \frac{x^4}{4}\right]_{0}^{\frac{1+\sqrt{5}}{2}}.$$

**Final answer:** $A = \dfrac{5\sqrt{5}}{12}$

## Exercise 4

The curves intersect when

$$\cos\theta = 0.5 \quad \Rightarrow \quad \theta = \frac{\pi}{3}, \ \frac{5\pi}{3}.$$

On $0 \leq \theta \leq \pi$, the relevant intersection is $\theta = \pi/3$.

On $[0, \pi/3]$, $\cos\theta \geq 0.5$; on $[\pi/3, \pi]$, $0.5 \geq \cos\theta$. Thus,

$$A = \int_0^{\pi/3} (\cos\theta - 0.5)\, d\theta + \int_{\pi/3}^{\pi} (0.5 - \cos\theta)\, d\theta.$$

Evaluating,

$$A = \left[\sin\theta - \frac{\theta}{2}\right]_0^{\pi/3} + \left[\frac{\theta}{2} - \sin\theta\right]_{\pi/3}^{\pi}.$$

**Final answer:** $A = \sqrt{3} - \frac{\pi}{6}$

## Exercise 5

The region is bounded on the right by $x = 9$ and on the left by $x = y^2$. The curves intersect when $y^2 = 9$, so $y = \pm 3$.

The area is

$$A = \int_{-3}^{3} (9 - y^2)\, dy.$$

Evaluating,

$$A = \left[9y - \frac{y^3}{3}\right]_{-3}^{3} = 36.$$

**Final answer:** $A = 36$

## Exercise 6

The curves intersect where

$$y = x \quad \text{and} \quad x = y^2 \quad \Rightarrow \quad y = y^2,$$

so $y = 0, 1$.

For $0 \leq y \leq 1$, the right curve is $x = y$ and the left curve is $x = y^2$. Thus,

$$A = \int_0^1 (y - y^2)\, dy.$$

Evaluating,

$$A = \left[\frac{y^2}{2} - \frac{y^3}{3}\right]_0^1.$$

**Final answer:** $A = \frac{1}{6}$

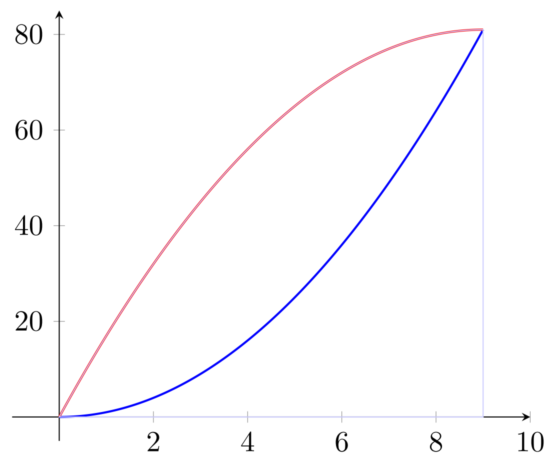
## Exercise 7

Intersection points:

$$x^2 = -x^2 + 18x \;\Rightarrow\; 2x^2 - 18x = 0 \;\Rightarrow\; x = 0, 9.$$

$$A = \int_0^9 \left[(-x^2 + 18x) - x^2\right] dx = \int_0^9 (18x - 2x^2)\, dx = \left[9x^2 - \frac{2}{3}x^3\right]_0^9 = 243.$$

**Final answer:** $A = 243$

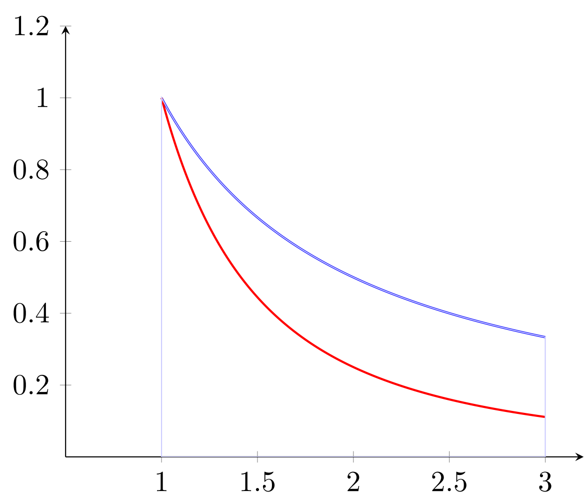



### Exercise 8

Intersections: $\frac{1}{x} = \frac{1}{x^2} \Rightarrow x = 1$.

$$A = \int_1^3 \left( \frac{1}{x} - \frac{1}{x^2} \right) dx = \left[ \ln x + \frac{1}{x} \right]_1^3 = \ln 3 - \frac{2}{3}.$$

**Final answer:** $A = \ln 3 - \dfrac{2}{3}$

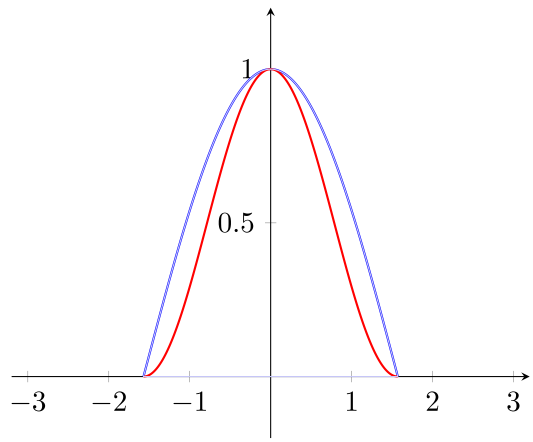



### Exercise 9

Intersections: $\cos x = \cos^2 x \Rightarrow x = \pm\frac{\pi}{2}$.

$$A = 2\int_0^{\pi/2} (\cos x - \cos^2 x)\, dx = 2\left[ \sin x - \frac{1}{2}\left(x + \sin x \cos x\right) \right]_0^{\pi/2} = 2 - \frac{\pi}{2}.$$

**Final answer:** $A = 2 - \frac{\pi}{2}$

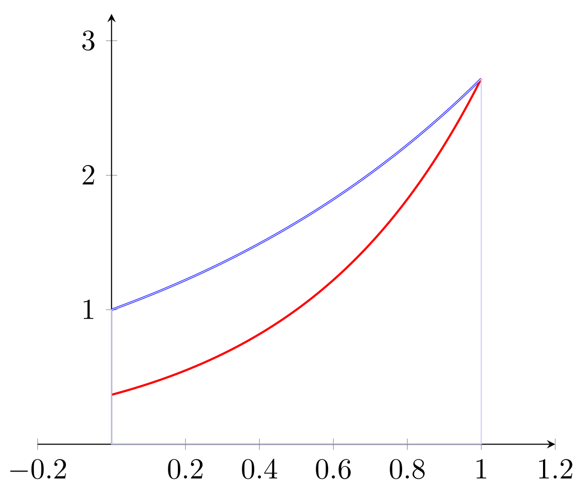



## Exercise 10

Intersection: $e^x = e^{2x-1} \Rightarrow x = 1$.

$$A = \int_0^1 (e^x - e^{2x-1})dx = \left[e^x - \frac{1}{2}e^{2x-1}\right]_0^1 = \frac{1}{2}(e-1).$$

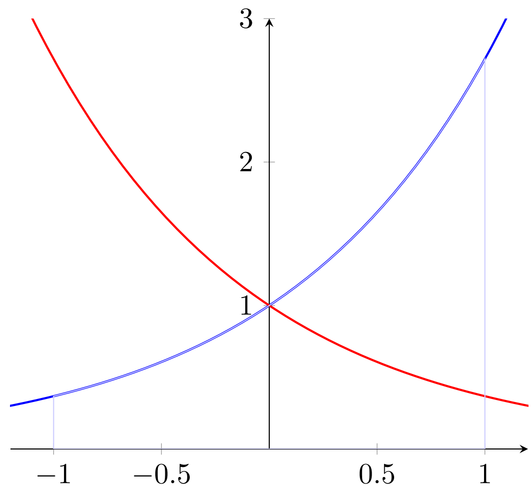
**Final answer:** $A = \frac{e-1}{2}$




## Exercise 11

$$A = \int_{-1}^1 (e^x - e^{-x})dx = [e^x + e^{-x}]_{-1}^1 = 2\left(e - \frac{1}{e}\right).$$

**Final answer:** $A = 2\left(e - \frac{1}{e}\right)$

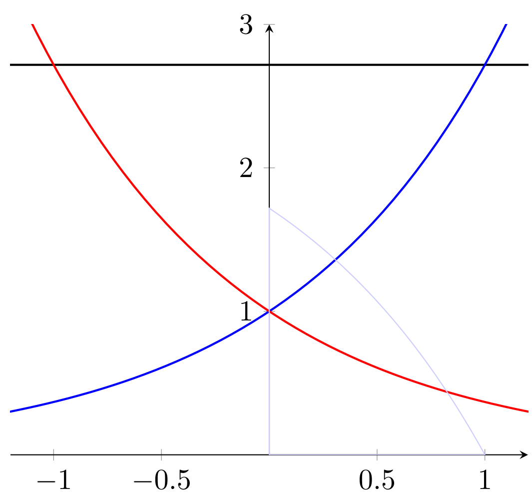



## Exercise 12

Intersections: $e = e^x = e^{-x} \Rightarrow x = \pm 1$.

$$A = \int_{-1}^{1} \left(e - \max(e^x, e^{-x})\right) dx = 2\int_0^1 (e - e^x)\, dx = 2.$$

**Final answer:** $A = 2$

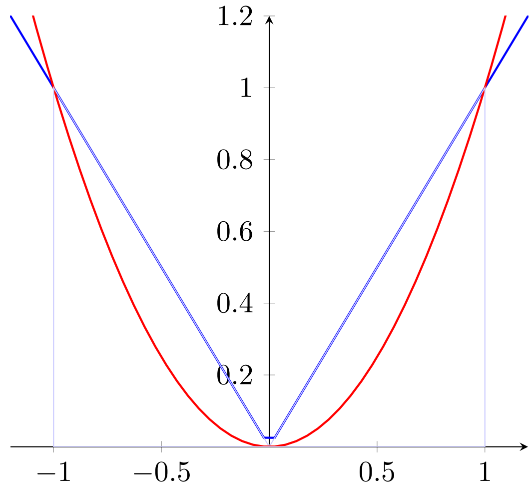



## Exercise 13

Intersections: $|x| = x^2 \Rightarrow x = -1, 0, 1$.

$$A = 2\int_0^1 (x - x^2)\,dx = 2\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{3}.$$

**Final answer:** $A = \dfrac{1}{3}$



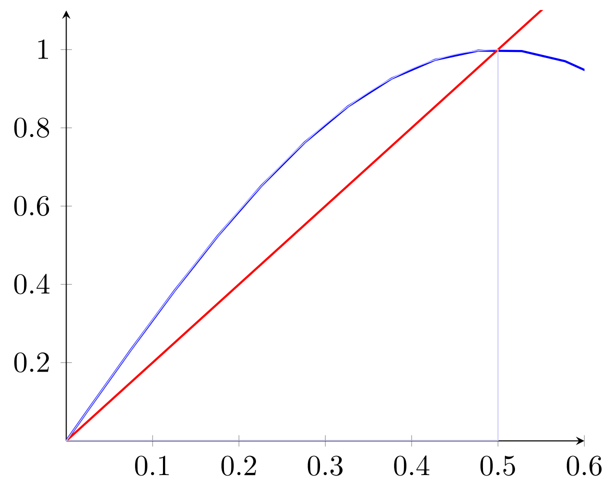

## Exercise 14

Intersections for $x > 0$:

$$\sin(\pi x) = 2x \quad \Rightarrow \quad x = 0,\ x = \tfrac{1}{2}.$$

$$A = \int_0^{1/2} (\sin(\pi x) - 2x)\,dx = \left[-\frac{\cos(\pi x)}{\pi} - x^2\right]_0^{1/2} = \frac{1}{\pi} - \frac{1}{4}.$$

**Final answer:** $A = \dfrac{1}{\pi} - \dfrac{1}{4}$




## Exercise 15

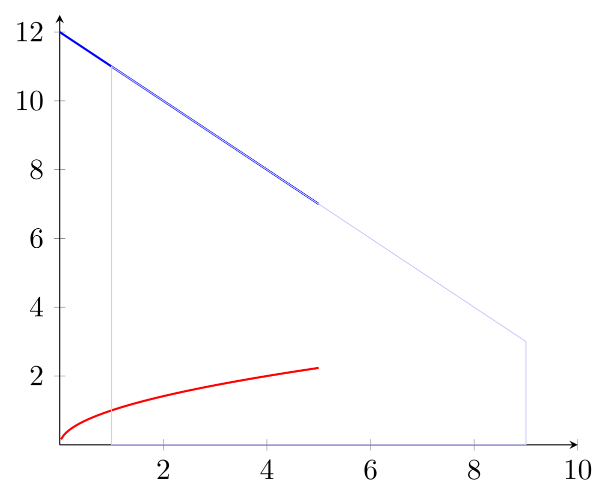
Intersections:

$$\sqrt{x} = 1 \Rightarrow x = 1, \qquad 12 - x = \sqrt{x} \Rightarrow x = 9.$$

Split at $x = 1$:

$$A = \int_1^9 \left[(12 - x) - \sqrt{x}\right] dx = \left[12x - \frac{x^2}{2} - \frac{2}{3}x^{3/2}\right]_1^9 = \frac{250}{3}.$$

**Final answer:** $A = \dfrac{250}{3}$




## Exercise 16

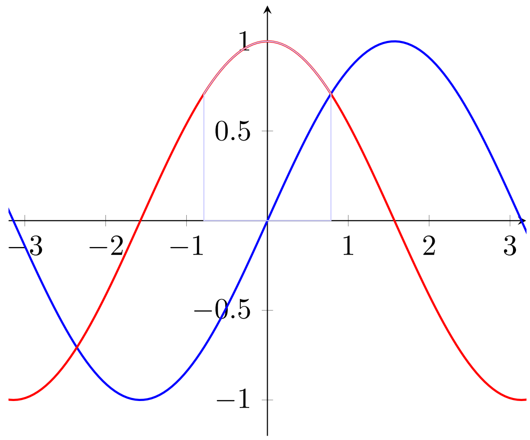
Intersections:

$$\sin x = \cos x \Rightarrow x = \pm\frac{\pi}{4}.$$

By symmetry,

$$A = 2\int_{-\pi/4}^{\pi/4} (\cos x - \sin x)\, dx = 2\left[\sin x + \cos x\right]_{-\pi/4}^{\pi/4} = 2\sqrt{2}.$$

**Final answer:** $A = 2\sqrt{2}$




## Exercise 17

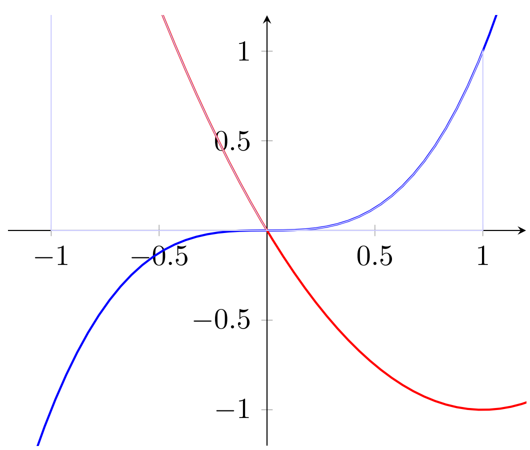
Intersections:

$$x^3 = x^2 - 2x \Rightarrow x = -1, 0, 1.$$

Split into two regions:

$$A = \int_{-1}^{0} [(x^2 - 2x) - x^3]\, dx + \int_{0}^{1} [x^3 - (x^2 - 2x)]\, dx = \frac{5}{6}.$$

**Final answer:** $A = \frac{5}{6}$



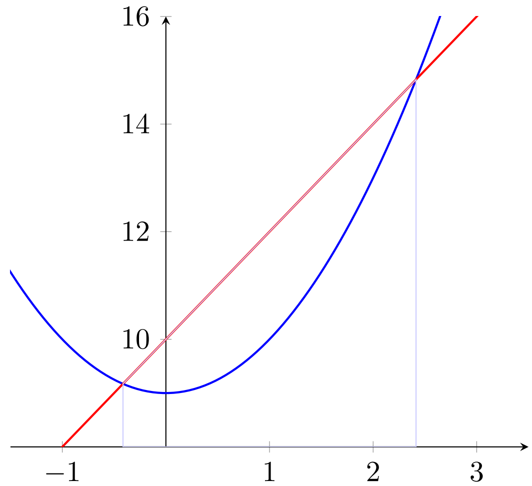
## Exercise 18

Intersections:

$$x^2 + 9 = 10 + 2x \Rightarrow (x-1)^2 = 2 \Rightarrow x = 1 \pm \sqrt{2}.$$

$$A = \int_{1-\sqrt{2}}^{1+\sqrt{2}} [(10+2x) - (x^2+9)]\, dx = \frac{8\sqrt{2}}{3}.$$

**Final answer:** $A = \dfrac{8\sqrt{2}}{3}$



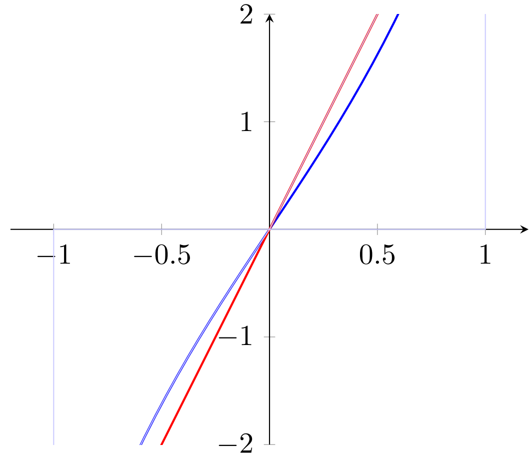

## Exercise 19

Intersections:

$$x^3 + 3x = 4x \Rightarrow x(x^2-1) = 0 \Rightarrow x = -1, 0, 1.$$

$$A = \int_{-1}^{0} [(x^3+3x) - 4x]\, dx + \int_{0}^{1} [4x - (x^3+3x)]\, dx = 1.$$

**Final answer:** $A = 1$




## Exercise 20

Intersections:

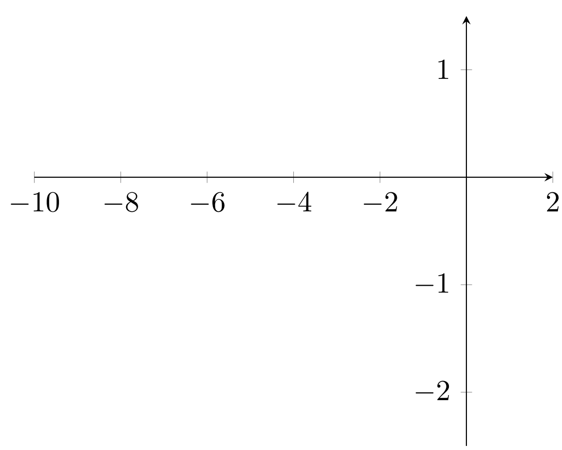
$$y^3 = 3y - 2 \;\Rightarrow\; y^3 - 3y + 2 = 0 \;\Rightarrow\; (y-1)^2(y+2) = 0$$

So $y = -2, 1$.

Right curve: $x = 3y - 2$, Left curve: $x = y^3$.

$$A = \int_{-2}^{1} \left[(3y-2) - y^3\right] dy = \left[\tfrac{3}{2}y^2 - 2y - \tfrac{1}{4}y^4\right]_{-2}^{1} = \frac{27}{4}.$$

**Final answer:** $A = \dfrac{27}{4}$



## Exercise 21

Intersections:

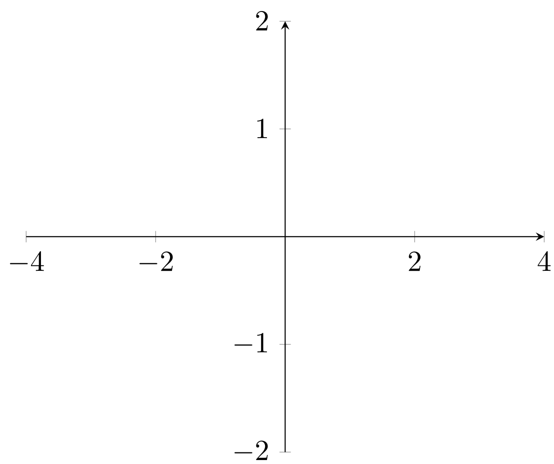
$$2y = y^3 - y \Rightarrow y(y^2 - 3) = 0 \Rightarrow y = 0, \pm\sqrt{3}.$$

Right curve: $x = 2y$, Left curve: $x = y^3 - y$.

By symmetry,

$$A = 2\int_0^{\sqrt{3}} [2y - (y^3 - y)]\, dy = 2\left[\tfrac{3}{2}y^2 - \tfrac{1}{4}y^4\right]_0^{\sqrt{3}} = \frac{9}{2}.$$

**Final answer:** $A = \dfrac{9}{2}$




## Exercise 22

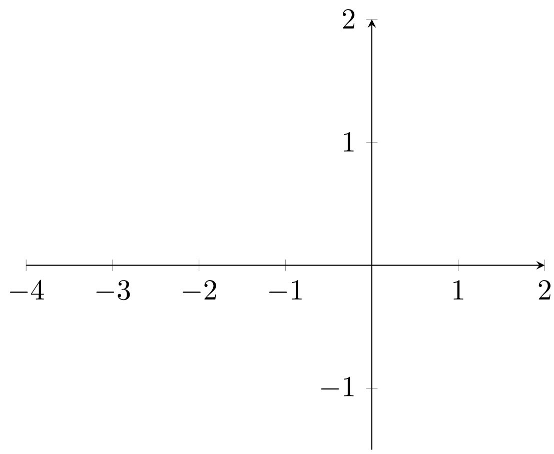
Intersections:

$$-3 + y^2 = y - y^2 \Rightarrow 2y^2 - y - 3 = 0 \Rightarrow y = -1, \tfrac{3}{2}.$$

Right curve: $x = y - y^2$, Left curve: $x = -3 + y^2$.

$$A = \int_{-1}^{3/2} [(y - y^2) - (-3 + y^2)]\, dy = \int_{-1}^{3/2} (y - 2y^2 + 3)\, dy = \frac{125}{24}.$$

**Final answer:** $A = \dfrac{125}{24}$




### Exercise 23

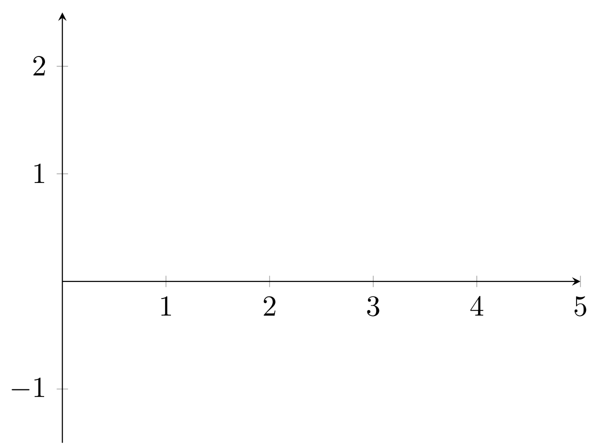
Intersections:

$$y^2 = y + 2 \Rightarrow y^2 - y - 2 = 0 \Rightarrow y = -1, 2.$$

Right curve: $x = y + 2$, Left curve: $x = y^2$.

$$A = \int_{-1}^{2} \left[(y+2) - y^2\right] dy = \left[\tfrac{1}{2}y^2 + 2y - \tfrac{1}{3}y^3\right]_{-1}^{2} = \frac{9}{2}.$$

**Final answer:** $A = \dfrac{9}{2}$




### Exercise 24

From $2x = -y^2 + 2 \Rightarrow x = 1 - \frac{1}{2}y^2$.

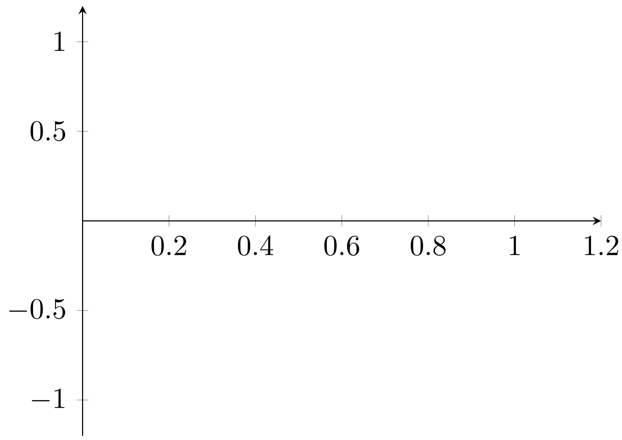
Intersections with $x = |y|$:

$$|y| = 1 - \tfrac{1}{2}y^2 \Rightarrow y = \pm 1.$$

By symmetry,

$$A = 2\int_0^1 \big[(1 - \tfrac{1}{2}y^2) - y\big]\,dy = 2\left[y - \tfrac{1}{6}y^3 - \tfrac{1}{2}y^2\right]_0^1 = \frac{1}{3}.$$

**Final answer:** $A = \dfrac{1}{3}$



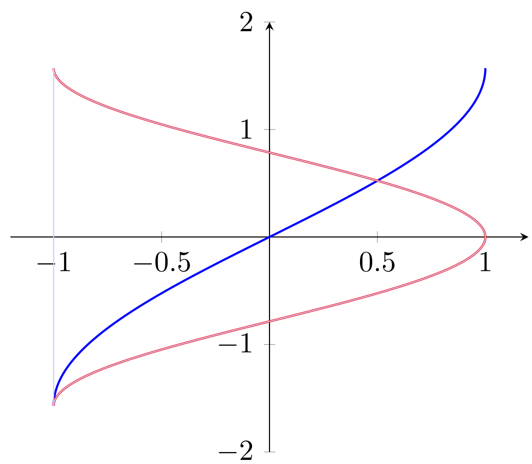

## Exercise 25

Bounds: $y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

Right curve: $x = \cos(2y)$, Left curve: $x = \sin y$.

$$A = \int_{-\pi/2}^{\pi/2} \big[\cos(2y) - \sin y\big]\,dy = \left[\tfrac{1}{2}\sin(2y) + \cos y\right]_{-\pi/2}^{\pi/2} = 2.$$

**Final answer:** $A = 2$



## Exercise 26

Curves: $x = y^4$ and $x = y^5$.

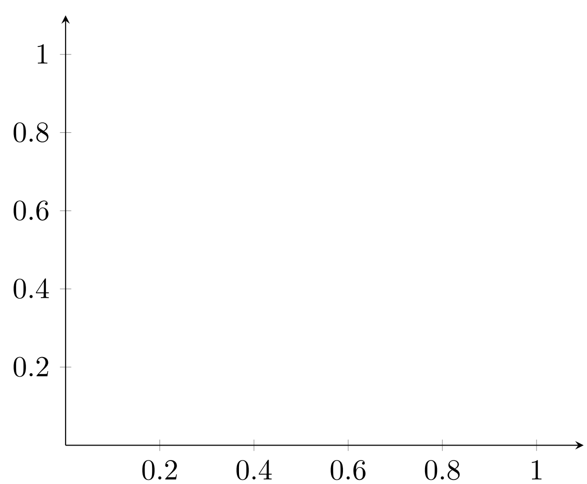
Intersections:

$$y^4 = y^5 \Rightarrow y^4(1-y) = 0 \Rightarrow y = 0, 1.$$

Right curve: $x = y^4$, Left curve: $x = y^5$.

$$A = \int_0^1 (y^4 - y^5)\,dy = \left[\tfrac{1}{5}y^5 - \tfrac{1}{6}y^6\right]_0^1 = \frac{1}{30}.$$

**Final answer:** $A = \dfrac{1}{30}$



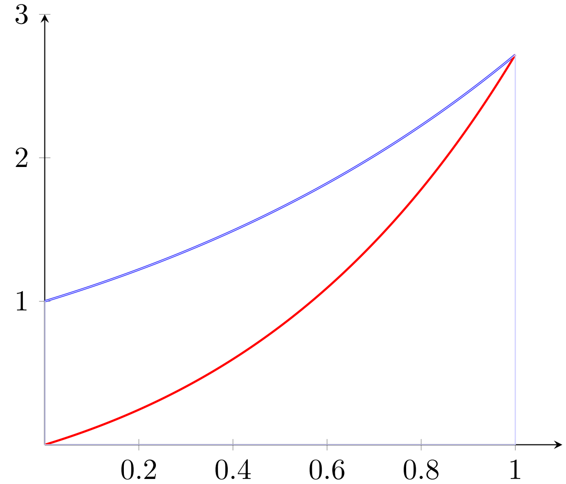

## Exercise 27

Curves: $y = e^x$ and $y = xe^x$, with $0 \le x \le 1$.

Top: $e^x$, Bottom: $xe^x$.

$$A = \int_0^1 (e^x - xe^x)\,dx = \int_0^1 e^x(1-x)\,dx = [e^x(2-x)]_0^1 = e - 2.$$

**Final answer:** $A = e - 2$




## Exercise 28

Curves: $y = x^6$ and $y = x^4$.

Intersections:

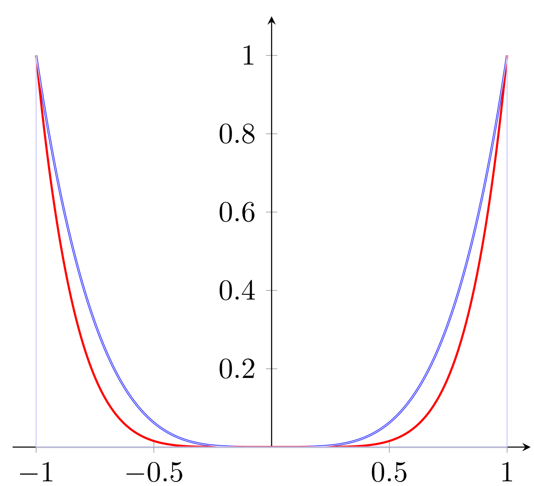
$$x^6 = x^4 \Rightarrow x^4(x^2 - 1) = 0 \Rightarrow x = -1, 0, 1.$$

Top: $x^4$, Bottom: $x^6$.

By symmetry,

$$A = 2\int_0^1 (x^4 - x^6)\,dx = 2\left[\tfrac{1}{5}x^5 - \tfrac{1}{7}x^7\right]_0^1 = \frac{4}{35}.$$

**Final answer:** $A = \dfrac{4}{35}$



### Exercise 29

Curves: $x = y^3 + 2y^2 + 1$ and $x = -y^2 + 1$.

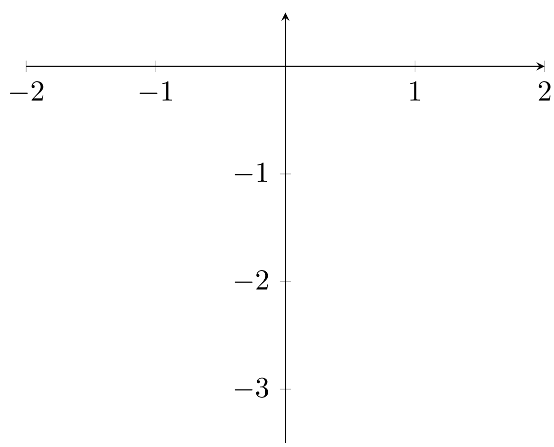
Intersections:

$$y^3 + 3y^2 = 0 \Rightarrow y^2(y+3) = 0 \Rightarrow y = -3, 0.$$

Right: $x = y^3 + 2y^2 + 1$, Left: $x = -y^2 + 1$.

$$A = \int_{-3}^{0} \left[(y^3 + 2y^2 + 1) - (-y^2 + 1)\right] dy = \int_{-3}^{0} (y^3 + 3y^2)\, dy = \frac{27}{4}.$$

**Final answer:** $A = \dfrac{27}{4}$




### Exercise 30

Curves: $y = |x|$ and $y = x^2 - 1$.

Intersections:

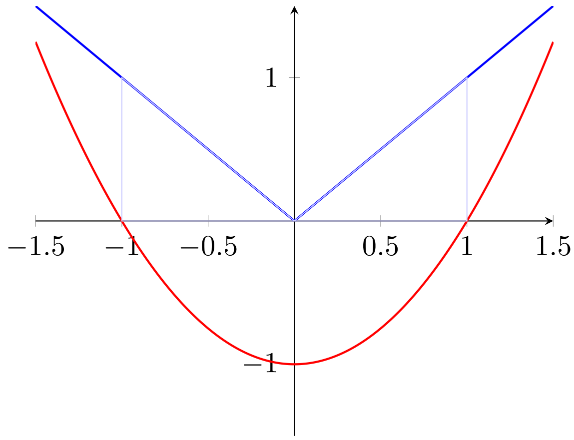
$$|x| = x^2 - 1 \Rightarrow x = \pm 1.$$

Top: $|x|$, Bottom: $x^2 - 1$.

By symmetry,

$$A = 2\int_0^1 \left[x - (x^2 - 1)\right] dx = 2\left[\tfrac{1}{2}x^2 - \tfrac{1}{3}x^3 + x\right]_0^1 = \frac{7}{3}.$$

**Final answer:** $A = \dfrac{7}{3}$




**Exercise 31**

Curves: $y = 4 - 3x$ and $y = \frac{1}{x}$.

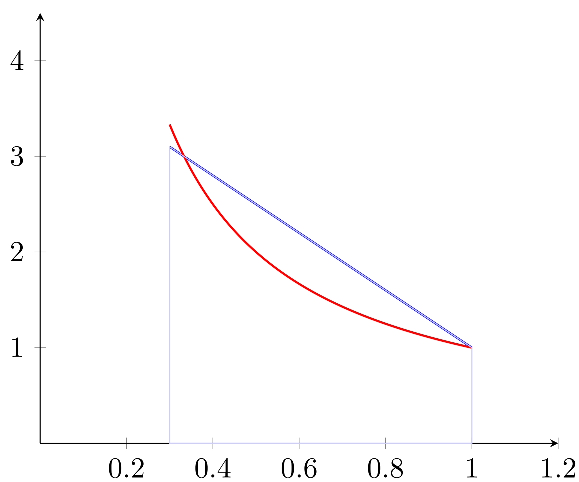
Intersections:

$$4 - 3x = \frac{1}{x} \Rightarrow 3x^2 - 4x + 1 = 0 \Rightarrow x = \frac{1}{3}, 1.$$

Top: $4 - 3x$, Bottom: $\frac{1}{x}$.

$$A = \int_{1/3}^{1} \left[ (4 - 3x) - \frac{1}{x} \right] dx = \left[ 4x - \frac{3}{2}x^2 - \ln x \right]_{1/3}^{1} = \frac{5}{2} - \ln 3.$$

**Final answer:** $A = \frac{5}{2} - \ln 3$



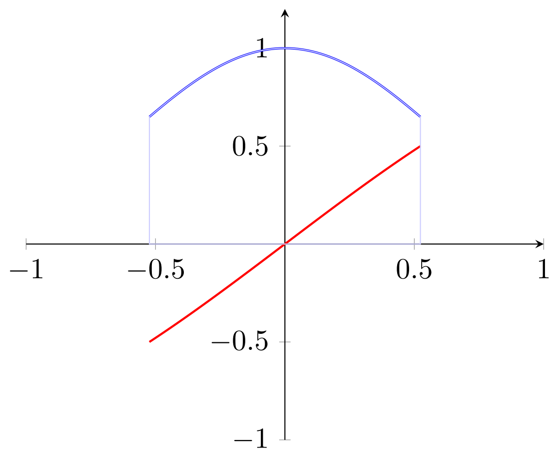
### Exercise 32

Curves: $y = \sin x$ and $y = \cos^3 x$, $-\frac{\pi}{6} \le x \le \frac{\pi}{6}$.

Top: $\cos^3 x$, Bottom: $\sin x$.

$$A = \int_{-\pi/6}^{\pi/6} (\cos^3 x - \sin x)\, dx = \left[\sin x - \tfrac{1}{3}\sin^3 x + \cos x\right]_{-\pi/6}^{\pi/6} = \sqrt{3} - \frac{1}{3}.$$

**Final answer:** $A = \sqrt{3} - \dfrac{1}{3}$




### Exercise 33

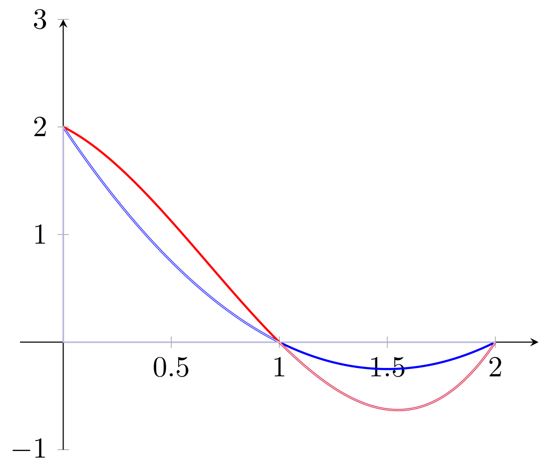
Curves: $y = x^2 - 3x + 2$ and $y = x^3 - 2x^2 - x + 2$.

Intersections:

$$x(x-1)(x-2) = 0 \Rightarrow x = 0, 1, 2.$$

$$A = \int_0^1 [(x^2 - 3x + 2) - (x^3 - 2x^2 - x + 2)]dx + \int_1^2 [(x^3 - 2x^2 - x + 2) - (x^2 - 3x + 2)]dx = \frac{1}{4}.$$

**Final answer:** $A = \dfrac{1}{4}$

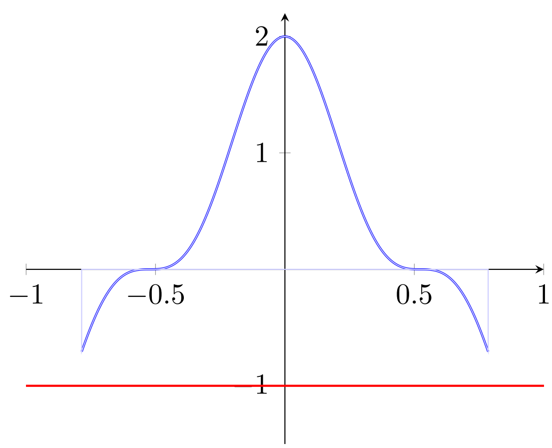



**Exercise 34**

Curves: $y = 2\cos^3(3x)$ and $y = -1$, $-\frac{\pi}{4} \leq x \leq \frac{\pi}{4}$.

$$A = \int_{-\pi/4}^{\pi/4} [2\cos^3(3x) + 1]\, dx = \frac{\pi}{2}.$$

**Final answer:** $A = \dfrac{\pi}{2}$




**Exercise 35**

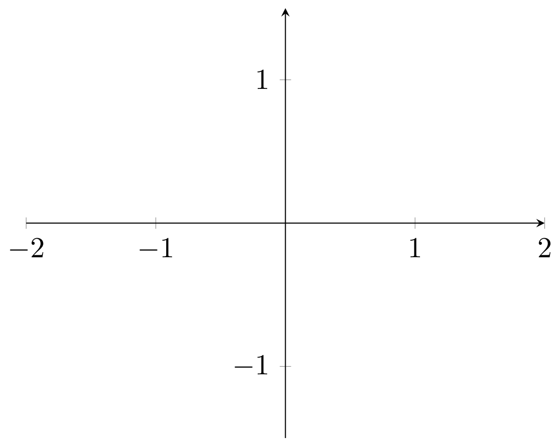
Curves: $x = y + y^3$ and $x = 2y$.

Intersections:

$$y(y^2 - 1) = 0 \Rightarrow y = -1, 0, 1.$$

$$A = \int_{-1}^{1} [(2y) - (y + y^3)]dy = \int_{-1}^{1} (y - y^3)dy = \frac{1}{2}.$$

**Final answer:** $A = \frac{1}{2}$



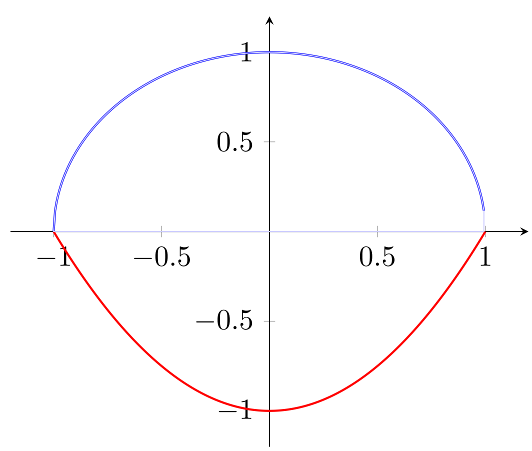

## Exercise 36

Curves: $y = \sqrt{1 - x^2}$ and $y = x^2 - 1$.

Intersections: $x = \pm 1$.

$$A = \int_{-1}^{1} \left[\sqrt{1 - x^2} - (x^2 - 1)\right] dx = \frac{\pi}{2} + \frac{4}{3}.$$

**Final answer:** $A = \frac{\pi}{2} + \frac{4}{3}$

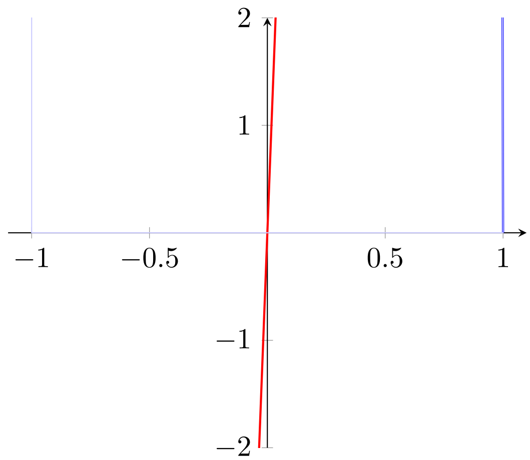


## Exercise 37

Curves: $y = \cos^{-1} x$ and $y = \sin^{-1} x$, $-1 \le x \le 1$.

$$A = \int_{-1}^{1} (\cos^{-1} x - \sin^{-1} x)\, dx = \int_{-1}^{1} \left(\frac{\pi}{2} - 2\sin^{-1} x\right) dx = \pi.$$

**Final answer:** $A = \pi$




## Exercise 38

Curves: $x = e^y$ and $y = x - 2 \Rightarrow x = y + 2$.

Intersections satisfy $e^y = y + 2$, which has no closed-form solution. Numerically,

$$y \approx -1.146, \quad y \approx 1.256.$$

$$A = \int_{-1.146}^{1.256} \left[(y + 2) - e^y\right] dy \approx 0.814.$$

**Final answer:** $A \approx 0.814$

## Exercise 39

Curves: $y = x^2$ and $y = \sqrt{1 - x^2}$.

Intersections:

$$x^2 = \sqrt{1 - x^2} \Rightarrow x^2 = \frac{\sqrt{5} - 1}{2}.$$

$$A = 2\int_0^{\sqrt{\frac{\sqrt{5}-1}{2}}} \left(\sqrt{1-x^2} - x^2\right) dx = \frac{\pi}{4}.$$

**Final answer:** $A = \dfrac{\pi}{4}$

## Exercise 40

Curves: $y = 3x^2 + 8x + 9$ and $y = \frac{x+24}{3}$.

Intersections:

$$3x^2 + 8x + 9 = \frac{x+24}{3} \Rightarrow x = -3, -1.$$

$$A = \int_{-3}^{-1} \left(\frac{x+24}{3} - (3x^2 + 8x + 9)\right) dx = \frac{8}{3}.$$

**Final answer:** $A = \dfrac{8}{3}$

## Exercise 41

Curves: $x = \sqrt{4-y^2}$ and $y^2 = 1 + x^2 \Rightarrow x = \sqrt{y^2-1}$.

Intersections at $y = \pm\sqrt{5/2}$.

$$A = 2\int_1^{\sqrt{5/2}} \left(\sqrt{4-y^2} - \sqrt{y^2-1}\right) dy = \frac{\pi}{2}.$$

**Final answer:** $A = \dfrac{\pi}{2}$

## Exercise 42

Curves: $x = y^{3/2}$ and $x = 3y$.

Intersections:

$$y^{3/2} = 3y \Rightarrow y = 0, 9.$$

$$A = \int_0^9 (3y - y^{3/2})\, dy = \frac{81}{2}.$$

**Final answer:** $A = \dfrac{81}{2}$

### Exercise 43

Curves: $y = \sin^3 x + 2$ and $y = \tan x$, $-1.5 \le x \le 1.5$.

No analytic intersections; numeric integration used.

$$A = \int_{-1.5}^{1.5} [(\sin^3 x + 2) - \tan x] dx \approx 5.998.$$

**Final answer:** $A \approx 5.998$

### Exercise 44

Curves: $y = \sqrt{1 - x^2}$ and $y = |x|$.

Intersections: $x = \pm\frac{1}{\sqrt{2}}$.

$$A = 2 \int_0^{1/\sqrt{2}} (\sqrt{1 - x^2} - x)\, dx = \frac{\pi}{4} - \frac{1}{2}.$$

**Final answer:** $A = \dfrac{\pi}{4} - \dfrac{1}{2}$

### Exercise 45

Curves: $y = \sqrt{1 - x^2}$ and $y = (x + 1)^2$.

Intersections at $x = 0$.

$$A = \int_{-1}^{0} (\sqrt{1 - x^2} - (x + 1)^2) dx = \frac{\pi}{4} - \frac{1}{3}.$$

**Final answer:** $A = \dfrac{\pi}{4} - \dfrac{1}{3}$

### Exercise 46

Curves: $x = 4 - y^2$ and $x = 1 + 3y + y^2$.

Intersections:

$$4 - y^2 = 1 + 3y + y^2 \Rightarrow y = -2, \tfrac{1}{2}.$$

$$A = \int_{-2}^{1/2} [(4 - y^2) - (1 + 3y + y^2)] dy = \frac{27}{4}.$$

**Final answer:** $A = \dfrac{27}{4}$

### Exercise 47

Curves: $y = \cos x$ and $y = e^x$, $-\pi \le x \le 0$.

$$A = \int_{-\pi}^{0} (\cos x - e^x)\, dx = \big[\sin x - e^x\big]_{-\pi}^{0} = 1 - e^{-\pi}.$$

**Final answer:** $A = 1 - e^{-\pi}$

### Exercise 48

The semicircle is $x^2 + y^2 = 1$ with $y \ge 0$. The triangle is bounded by $y = 1 + x$, $y = 1 - x$, and the $x$-axis.

Area of the semicircle:

$$A_{\text{semi}} = \frac{\pi}{2}.$$

The triangle has base length 2 and height 1, so

$$A_{\triangle} = \frac{1}{2}(2)(1) = 1.$$

**Final answer:**

$$A = \frac{\pi}{2} - 1.$$

### Exercise 49

Marginal cost: $C(x) = 0.01x^2 - 3x + 229$

Marginal revenue: $R(x) = 429 - 2x$

Intersection points satisfy $C(x) = R(x)$:

$$0.01x^2 - x - 200 = 0 \quad \Rightarrow \quad x = 200.$$

The area between the curves and $x = 0$ is

$$A = \int_{0}^{200} (R(x) - C(x))\, dx = \int_{0}^{200} (-0.01x^2 + x + 200)\, dx.$$

$$A = \left[-\frac{0.01}{3}x^3 + \frac{1}{2}x^2 + 200x\right]_0^{200} = \frac{40{,}000}{3}.$$

**Final answer:**

$$A = \frac{40{,}000}{3}.$$

This area represents the **total profit**.

### Exercise 50

Marginal cost: $C(x) = 1000e^{-x} + 5$

Marginal revenue: $R(x) = 60 - 0.1x$

Profit is

$$P = \int_0^{550} \big(R(x) - C(x)\big)\,dx = \int_0^{550} \big(55 - 0.1x - 1000e^{-x}\big)\,dx.$$

$$P = \Big[55x - 0.05x^2 + 1000e^{-x}\Big]_0^{550}.$$

Since $e^{-550} \approx 0$,

$$P \approx 30{,}250 - 15{,}125 - 1000 = 14{,}125.$$

**Final answer:**

$$P \approx 14{,}125.$$

This represents the **total profit** from selling 550 tickets.

### Exercise 51

Hare speed:

$$H(t) = 1 - \cos\left(\frac{\pi t}{2}\right)$$

Tortoise speed:

$$T(t) = \frac{1}{2}\tan^{-1}\left(\frac{t}{4}\right).$$

The first intersection after $t = 0$ occurs at

$$t \approx 2.782.$$

The area between the curves is

$$A = \int_0^{2.782} \big(H(t) - T(t)\big)\,dt \approx 1.984.$$

**Final answer:**

$$A \approx 1.984.$$

This area represents the **difference in total distance traveled**.

## Exercise 52

Hare speed:

$$H(t) = \frac{1}{2} - \frac{1}{2}\cos(2\pi t)$$

Tortoise speed:

$$T(t) = \sqrt{t}.$$

They intersect at

$$t \approx 0.611.$$

Distance traveled in 1 hour:

$$D_H = \int_0^1 H(t)\,dt = \frac{1}{2}, \quad D_T = \int_0^1 \sqrt{t}\,dt = \frac{2}{3}.$$

**Final answer:**

$$\text{Tortoise wins by } \frac{1}{6} \text{ km.}$$

## Exercise 53

The curves are

$$y_1 = x^2 + 2x + 1 = (x+1)^2, \qquad y_2 = -x^2 - 3x + 4.$$

Intersection points:

$$(x+1)^2 = -x^2 - 3x + 4 \;\Rightarrow\; 2x^2 + 5x - 3 = 0 \;\Rightarrow\; x = -3,\; x = \tfrac{1}{2}.$$

**Integrating with respect to $x$**

On $[-3, \frac{1}{2}]$, $y_2 \geq y_1$. Hence

$$A = \int_{-3}^{1/2} \Big[(-x^2 - 3x + 4) - (x^2 + 2x + 1)\Big]\,dx = \int_{-3}^{1/2} (-2x^2 - 5x + 3)\,dx.$$

$$A = \Big[-\tfrac{2}{3}x^3 - \tfrac{5}{2}x^2 + 3x\Big]_{-3}^{1/2} = \frac{343}{24}.$$

**Integrating with respect to $y$**

Solve for $x$:

$$y = (x+1)^2 \Rightarrow x = -1 \pm \sqrt{y}, \qquad y = -x^2 - 3x + 4 \Rightarrow x = \frac{-3 \pm \sqrt{25 - 4y}}{2}.$$

Using left/right branches over $y \in [0, \frac{25}{4}]$,

$$A = \int_0^{25/4} \left[\frac{-3 + \sqrt{25-4y}}{2} - (-1 + \sqrt{y})\right] dy - \int_0^{25/4} \left[\frac{-3 - \sqrt{25-4y}}{2} - (-1 - \sqrt{y})\right] dy = \frac{343}{24}.$$

**Final answer:** $A = \dfrac{343}{24}$. Both methods give the same area; integrating with respect to $x$ is simpler.

## Exercise 54

The curves are $y = x^4$ and $x = y^5 \Rightarrow y = x^{1/5}$. Intersections satisfy $x^4 = x^{1/5}$, giving $x = 0, 1$.

**Integrating with respect to $x$**

For $x \in [0, 1]$, $x^{1/5} \geq x^4$:

$$A = \int_0^1 (x^{1/5} - x^4)\,dx = \Big[\tfrac{5}{6}x^{6/5} - \tfrac{1}{5}x^5\Big]_0^1 = \frac{19}{30}.$$

**Integrating with respect to $y$**

Write both as $x$ in terms of $y$:

$$x = y^{1/4}, \qquad x = y^5,$$

with $y \in [0, 1]$ and $y^{1/4} \geq y^5$:

$$A = \int_0^1 (y^{1/4} - y^5)\,dy = \Big[\tfrac{4}{5}y^{5/4} - \tfrac{1}{6}y^6\Big]_0^1 = \frac{19}{30}.$$

**Final answer:** $A = \dfrac{19}{30}$. Both methods agree; integrating with respect to $x$ is slightly simpler.

### Exercise 55

The curves are $x = y^2 - 2$ and $x = 2y$. Intersections:

$$y^2 - 2 = 2y \Rightarrow y^2 - 2y - 2 = 0 \Rightarrow y = 1 \pm \sqrt{3}.$$

#### Integrating with respect to $y$

Right minus left gives

$$A = \int_{1-\sqrt{3}}^{1+\sqrt{3}} \Big[2y - (y^2 - 2)\Big]\,dy = \int_{1-\sqrt{3}}^{1+\sqrt{3}} (-y^2 + 2y + 2)\,dy.$$

Let $u = y - 1$. Then limits are $-\sqrt{3}$ to $\sqrt{3}$:

$$A = \int_{-\sqrt{3}}^{\sqrt{3}} (3 - u^2)\,du = \Big[3u - \tfrac{1}{3}u^3\Big]_{-\sqrt{3}}^{\sqrt{3}} = 4\sqrt{3}.$$

#### Integrating with respect to $x$

Solve for $y$:

$$x = 2y \Rightarrow y = \tfrac{x}{2}, \qquad x = y^2 - 2 \Rightarrow y = \pm\sqrt{x+2}.$$

The region splits at $x = -1$ (where $y = \frac{x}{2}$ meets the lower branch).

$$A = \int_{-2}^{-1} (\sqrt{x+2} - (-\sqrt{x+2}))\,dx + \int_{-1}^{4} (\tfrac{x}{2} - (-\sqrt{x+2}))\,dx = 4\sqrt{3}.$$

**Final answer:** $A = 4\sqrt{3}$. Both methods give the same area; integrating with respect to $y$ is easier.

### Exercise 56

The unit circle is

$$x^2 + y^2 = 1.$$

A square that *touches* the unit circle on all four sides must be centered at the origin, with each side tangent to the circle. The top side touches the circle at $(0, 1)$, bottom at $(0, -1)$, left at $(-1, 0)$, and right at $(1, 0)$.

Hence the equations of the four sides are

$$x = 1, \quad x = -1, \quad y = 1, \quad y = -1.$$

**Area using calculus**

The area between the square and the circle equals

$$(\text{area of square}) - (\text{area of circle}).$$

Using calculus, the area of the unit circle can be computed as

$$A_{\text{circle}} = 4 \int_0^1 \sqrt{1 - x^2}\, dx = 4 \cdot \frac{\pi}{4} = \pi.$$

The area of the square is

$$A_{\text{square}} = (2)(2) = 4.$$

Therefore,

$$A = 4 - \pi.$$

**Final answer:** $A = 4 - \pi$.

**Geometric check**

Without calculus, the area follows immediately from geometry: a square of side length 2 minus a circle of radius 1. This confirms the result.

## Exercise 57

The unit circle is

$$x^2 + y^2 = 1.$$

The triangle is formed by the lines

$$y = 2x + 1, \qquad y = 1 - 2x, \qquad y = -\frac{3}{5}.$$

**Triangle area**

The intersection of $y = 2x + 1$ and $y = 1 - 2x$ is

$$2x + 1 = 1 - 2x \Rightarrow x = 0,\ y = 1.$$

The base of the triangle lies on $y = -\frac{3}{5}$. Intersecting with the sides:

$$-\frac{3}{5} = 2x + 1 \Rightarrow x = -\frac{4}{5}, \qquad -\frac{3}{5} = 1 - 2x \Rightarrow x = \frac{4}{5}.$$

Thus the base length is

$$\frac{4}{5} - \left(-\frac{4}{5}\right) = \frac{8}{5},$$

and the height is

$$1 - \left(-\frac{3}{5}\right) = \frac{8}{5}.$$

Hence the triangle area is

$$A_{\text{triangle}} = \frac{1}{2} \cdot \frac{8}{5} \cdot \frac{8}{5} = \frac{32}{25}.$$

**Area of the circle inside the triangle**

The triangle completely contains the unit circle above $y = -\frac{3}{5}$, but cuts off the bottom portion of the circle.

The area of the circular segment below $y = -\frac{3}{5}$ is

$$A_{\text{cap}} = \int_{-1}^{-3/5} 2\sqrt{1 - y^2}\, dy.$$

Thus the area of the circle *inside* the triangle is

$$A_{\text{circle in triangle}} = \pi - \int_{-1}^{-3/5} 2\sqrt{1 - y^2}\, dy.$$

Evaluating,

$$\int 2\sqrt{1 - y^2}\, dy = y\sqrt{1 - y^2} + \arcsin(y).$$

Hence

$$A_{\text{cap}} = \left[ y\sqrt{1 - y^2} + \arcsin(y) \right]_{-1}^{-3/5} = \arcsin\left(-\frac{3}{5}\right) + \frac{3}{5} \cdot \frac{4}{5} + \frac{\pi}{2}.$$

**Area between triangle and circle**

The desired area is

$$A = A_{\text{triangle}} - A_{\text{circle in triangle}} = \frac{32}{25} - \pi + \arcsin\left(\frac{3}{5}\right) + \frac{12}{25}.$$

Simplifying,

$$A = \frac{44}{25} - \pi + \arcsin\left(\frac{3}{5}\right).$$

**Final answer:**

$$A = \frac{44}{25} - \pi + \arcsin\left(\frac{3}{5}\right).$$

**Geometric remark**

There is no purely elementary geometric formula for the circular segment here; calculus is the most natural approach.

**Determining Volumes by Slicing**

## Exercise 58

Consider a sphere of radius $r$ centered at the origin. A cross-sectional slice perpendicular to the $x$-axis at position $x$ is a disk of radius

$$y = \sqrt{r^2 - x^2}.$$

The area of the slice is

$$A(x) = \pi y^2 = \pi(r^2 - x^2).$$

Thus the volume is

$$V = \int_{-r}^{r} \pi(r^2 - x^2)\, dx = \pi \left[ r^2 x - \frac{x^3}{3} \right]_{-r}^{r} = \frac{4}{3}\pi r^3.$$

**Final answer:** $V = \frac{4}{3}\pi r^3$

## Exercise 59

At height $x$ from the tip of the cone, similar triangles give the radius

$$\rho(x) = \frac{r}{h} x.$$

The cross-sectional area is

$$A(x) = \pi \rho(x)^2 = \pi \frac{r^2}{h^2} x^2.$$

Integrating from $x = 0$ to $x = h$,

$$V = \int_0^h \pi \frac{r^2}{h^2} x^2\, dx = \pi \frac{r^2}{h^2} \cdot \frac{h^3}{3} = \frac{1}{3}\pi r^2 h.$$

**Final answer:** $V = \frac{1}{3}\pi r^2 h$

## Exercise 60

The height of a regular tetrahedron with side length $a$ is

$$h = \sqrt{\frac{2}{3}}\, a.$$

At height $x$, the side length of the equilateral triangular cross-section is

$$s(x) = a\left(1 - \frac{x}{h}\right).$$

The area of the cross-section is

$$A(x) = \frac{\sqrt{3}}{4} s(x)^2 = \frac{\sqrt{3}}{4} a^2 \left(1 - \frac{x}{h}\right)^2.$$

Thus,

$$V = \int_0^h A(x)\,dx = \frac{\sqrt{3}}{4} a^2 \int_0^h \left(1 - \frac{x}{h}\right)^2 dx = \frac{\sqrt{3}}{4} a^2 \cdot \frac{h}{3}.$$

Substituting $h = \sqrt{\frac{2}{3}} a$,

$$V = \frac{\sqrt{2}}{12} a^3.$$

**Final answer:** $V = \frac{\sqrt{2}}{12} a^3$

## Exercise 61

Each cross-section perpendicular to the axis has constant area

$$A = \frac{1}{2}(b_1 + b_2)h.$$

Integrating along the length $L$,

$$V = \int_0^L A\,dx = \frac{1}{2}(b_1 + b_2)hL.$$

**Final answer:** $V = \frac{1}{2}(b_1 + b_2)hL$

## Exercise 62

The disk method applies when the region being rotated touches the axis of rotation, producing solid cross-sections with no holes. The washer method applies when the region does not touch the axis, producing cross-sections with an inner and outer radius.

The two methods are interchangeable when the inner radius is zero everywhere, in which case the washer reduces to a disk.

**Final answer:** The disk method is used when there is no hole, the washer method when there is; they coincide when the inner radius is zero.

## Exercise 63

Using horizontal slices at height $y$ from the apex.

The square base has side 2, and the height is 6. By similarity, the side length at height $y$ is

$$s(y) = \frac{2}{6}y = \frac{y}{3}.$$

Thus the cross-sectional area is

$$A(y) = s(y)^2 = \frac{y^2}{9}.$$

The volume is

$$V = \int_0^6 \frac{y^2}{9}\,dy = \frac{1}{9} \cdot \frac{6^3}{3} = 8.$$

**Final answer:** $V = 8$

## Exercise 64

The rectangular base is $3 \times 2$ and the height is 4. Using horizontal slices, similarity gives

$$\text{length}(y) = \frac{3}{4}y, \quad \text{width}(y) = \frac{2}{4}y = \frac{y}{2}.$$

So

$$A(y) = \frac{3y}{4} \cdot \frac{y}{2} = \frac{3y^2}{8}.$$

The volume is

$$V = \int_0^4 \frac{3y^2}{8}\,dy = \frac{3}{8} \cdot \frac{64}{3} = 8.$$

**Final answer:** $V = 8$

## Exercise 65

A regular tetrahedron with side length 4 has height

$$h = \frac{\sqrt{6}}{3} \cdot 4 = \frac{4\sqrt{6}}{3}.$$

Using slicing perpendicular to the height, similarity gives

$$A(y) = \left(\frac{y}{h}\right)^2 A_{\text{base}}, \quad A_{\text{base}} = \frac{\sqrt{3}}{4}(4)^2 = 4\sqrt{3}.$$

Thus

$$V = \int_0^h 4\sqrt{3}\left(\frac{y}{h}\right)^2 dy = \frac{4\sqrt{3}}{h^2} \cdot \frac{h^3}{3} = \frac{16\sqrt{2}}{3}.$$

**Final answer:** $V = \dfrac{16\sqrt{2}}{3}$

## Exercise 66

The triangular base has area

$$A = \frac{1}{2} \cdot 6 \cdot 8 = 24,$$

and the height is 5. Using horizontal slices, similarity implies

$$A(y) = 24\left(\frac{y}{5}\right)^2.$$

Hence

$$V = \int_0^5 24\left(\frac{y}{5}\right)^2 dy = \frac{24}{25} \cdot \frac{125}{3} = 40.$$

**Final answer:** $V = 40$

## Exercise 67

The volume of a cone is $V = \frac{1}{3}\pi r^2 h$. The original cone has volume

$$V_1 = \frac{1}{3}\pi r^2 h,$$

and the removed cone has

$$V_2 = \frac{1}{3}\pi\left(\frac{r}{2}\right)^2\left(\frac{h}{2}\right) = \frac{1}{24}\pi r^2 h.$$

Thus the frustum volume is

$$V = V_1 - V_2 = \frac{7}{24}\pi r^2 h.$$

**Final answer:** $V = \dfrac{7}{24}\pi r^2 h$

### Exercise 68

The base is a disk of radius $a$. A slice at position $x$ has side length

$$s(x) = 2\sqrt{a^2 - x^2}.$$

Each slice is a square, so its area is

$$A(x) = \left(2\sqrt{a^2 - x^2}\right)^2 = 4(a^2 - x^2).$$

The volume is

$$V = \int_{-a}^{a} 4(a^2 - x^2)\,dx = 4\left[a^2 x - \frac{x^3}{3}\right]_{-a}^{a} = \frac{16}{3}a^3.$$

**Final answer:** $V = \dfrac{16}{3}a^3$

### Exercise 69

For a fixed $x \in [0,1]$, the vertical cross-section runs from $y = 0$ to $y = 1 - x$, so the diameter of the semicircle is $1 - x$. The radius is

$$r(x) = \frac{1 - x}{2}.$$

Thus the area is

$$A(x) = \frac{1}{2}\pi r^2 = \frac{\pi}{8}(1 - x)^2.$$

The volume is

$$V = \int_0^1 \frac{\pi}{8}(1 - x)^2\,dx = \frac{\pi}{8} \cdot \frac{1}{3} = \frac{\pi}{24}.$$

**Final answer:** $V = \dfrac{\pi}{24}$

### Exercise 70

The base is $0 \le y \le 1 - x^2$ for $0 \le x \le 1$. Slices perpendicular to the $xy$-plane are squares with side

$$s(x) = 1 - x^2.$$

Hence

$$A(x) = (1 - x^2)^2.$$

The volume is

$$V = \int_0^1 (1 - x^2)^2 \, dx = \int_0^1 (1 - 2x^2 + x^4) \, dx = \frac{8}{15}.$$

**Final answer:** $V = \dfrac{8}{15}$

## Exercise 71

Here the base satisfies $0 \le y \le 1 - x^2$ and slices are perpendicular to the $y$-axis. Solving for $x$,

$$x = \sqrt{1 - y}.$$

Each square has side length

$$s(y) = 2\sqrt{1 - y}.$$

Thus

$$A(y) = 4(1 - y), \quad V = \int_0^1 4(1 - y) \, dy = 2.$$

**Final answer:** $V = 2$

## Exercise 72

The base is between $y = x^2$ and $y = 9$. A slice perpendicular to the $x$-axis has leg length

$$\ell(x) = 9 - x^2.$$

Each cross-section is a right isosceles triangle, so

$$A(x) = \frac{1}{2}\ell^2 = \frac{1}{2}(9 - x^2)^2.$$

The volume is

$$V = \int_{-3}^3 \frac{1}{2}(9 - x^2)^2 \, dx = \frac{1}{2} \int_{-3}^3 (81 - 18x^2 + x^4) \, dx = \frac{1296}{5}.$$

**Final answer:** $V = \dfrac{1296}{5}$

## Exercise 73

The base lies between $y = x$ and $y = x^2$ for $0 \le x \le 1$. The diameter of each semicircle is

$$d(x) = x - x^2, \quad r(x) = \frac{x - x^2}{2}.$$

Hence

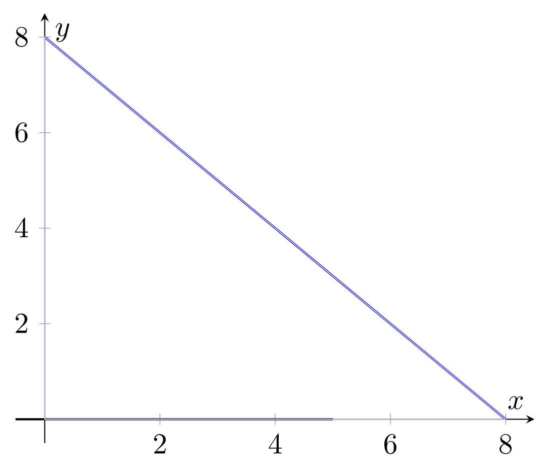
$$A(x) = \frac{1}{2}\pi r^2 = \frac{\pi}{8}(x - x^2)^2.$$

The volume is

$$V = \int_0^1 \frac{\pi}{8}(x - x^2)^2\,dx = \frac{\pi}{8} \cdot \frac{1}{30} = \frac{\pi}{240}.$$

**Final answer:** $V = \dfrac{\pi}{240}$

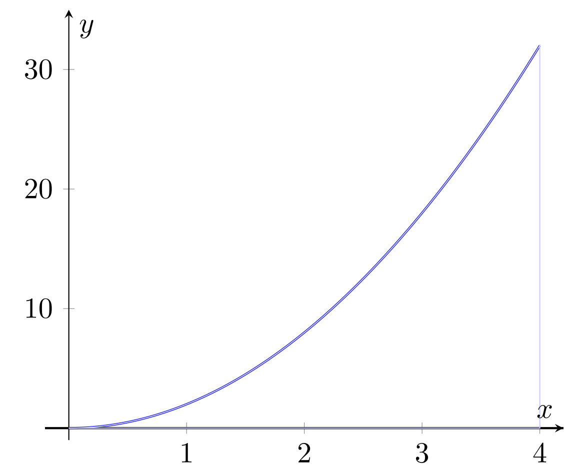
## Exercise 74




$$V = \pi \int_0^8 (8 - x)^2\,dx = \pi \left[\frac{(8 - x)^3}{-3}\right]_0^8 = \frac{512\pi}{3}.$$

**Final answer:** $V = \dfrac{512\pi}{3}$

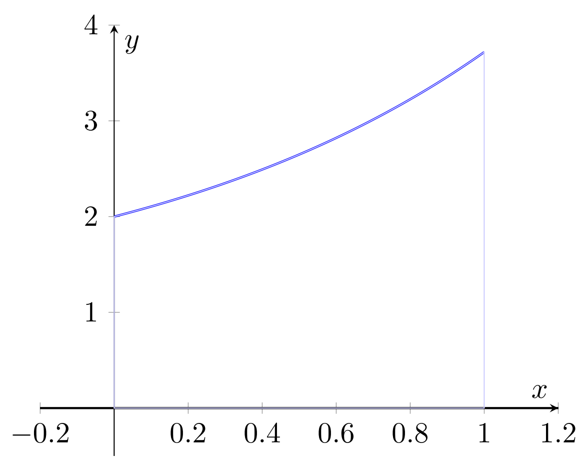
**Exercise 75**




$$V = \pi \int_0^4 (2x^2)^2\, dx = 4\pi \int_0^4 x^4\, dx = \frac{4096\pi}{5}.$$

**Final answer:** $V = \dfrac{4096\pi}{5}$

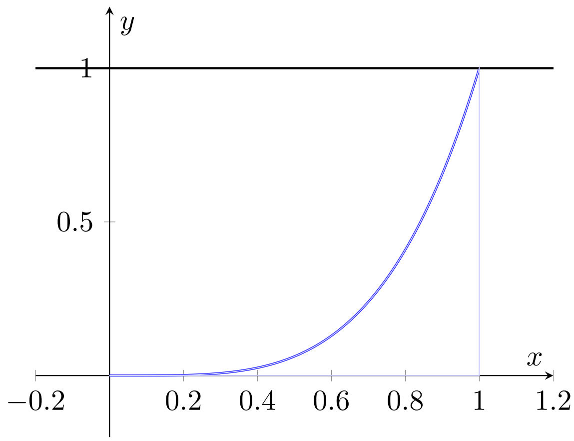
**Exercise 76**




$$V = \pi \int_0^1 (e^x + 1)^2\, dx = \pi \int_0^1 (e^{2x} + 2e^x + 1)\, dx = \pi \left[\frac{e^2 - 1}{2} + 2(e - 1) + 1\right].$$

**Final answer:** $V = \pi\left(\dfrac{e^2 - 1}{2} + 2e - 1\right)$

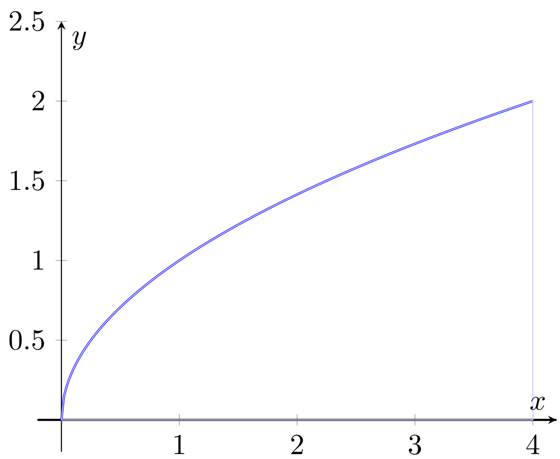
**Exercise 77**




$$V = \pi \int_0^1 (x^4)^2 \, dx = \pi \int_0^1 x^8 \, dx = \frac{\pi}{9}.$$

**Final answer:** $V = \dfrac{\pi}{9}$

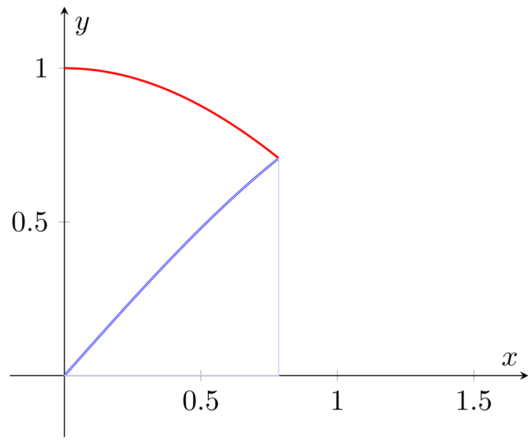
**Exercise 78**




$$V = \pi \int_0^4 (\sqrt{x})^2 \, dx = \pi \int_0^4 x \, dx = 8\pi.$$

**Final answer:** $V = 8\pi$

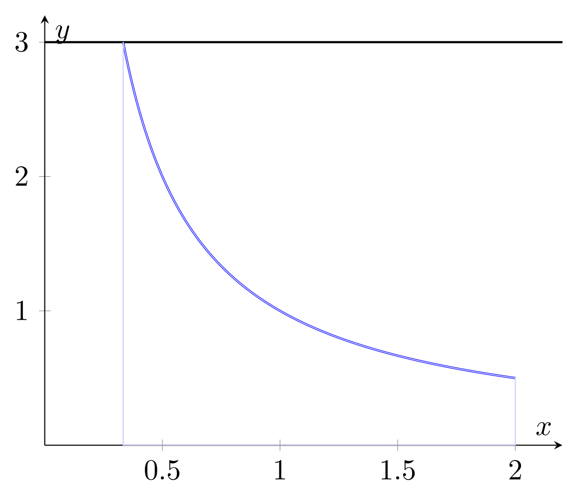
**Exercise 79**




$$V = \pi \int_0^{\pi/4} (\cos^2 x - \sin^2 x)\, dx = \pi \int_0^{\pi/4} \cos(2x)\, dx = \frac{\pi}{2}.$$

**Final answer:** $V = \dfrac{\pi}{2}$

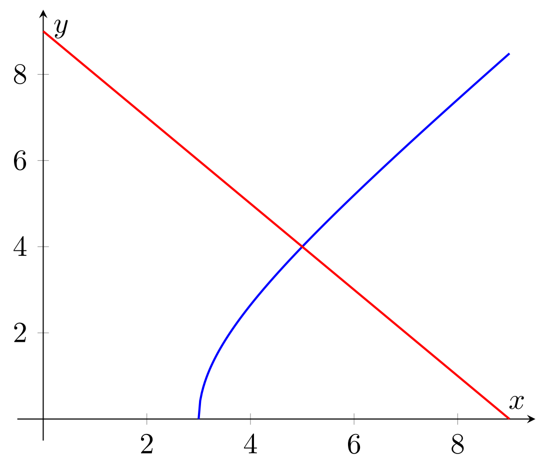
**Exercise 80**




$$V = \pi \int_{1/3}^{2} \left(\frac{1}{x}\right)^2 dx = \pi \left[-\frac{1}{x}\right]_{1/3}^{2} = \frac{5\pi}{2}.$$

**Final answer:** $V = \dfrac{5\pi}{2}$

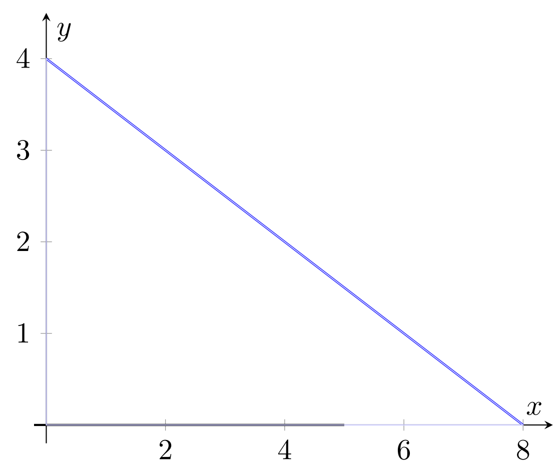
## Exercise 81




$$V = \pi \int_0^9 \Big[(9-x)^2 - (x^2-9)\Big]\,dx = 243\pi.$$

**Final answer:** $V = 243\pi$

## Exercise 82

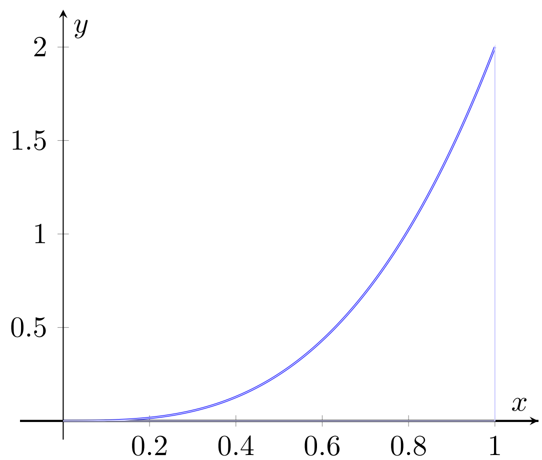



Shells:

$$V = 2\pi \int_0^8 x(4 - \tfrac{1}{2}x)\,dx = \frac{256\pi}{3}.$$

**Final answer:** $V = \dfrac{256\pi}{3}$

## Exercise 83

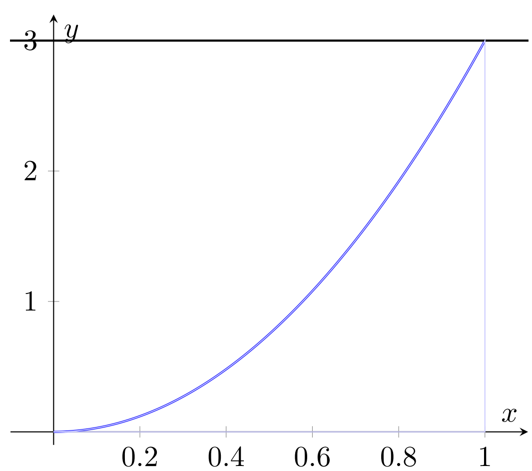



Shells:

$$V = 2\pi \int_0^1 x(2x^3)\, dx = \frac{4\pi}{5}.$$

**Final answer:** $V = \frac{4\pi}{5}$

## Exercise 84

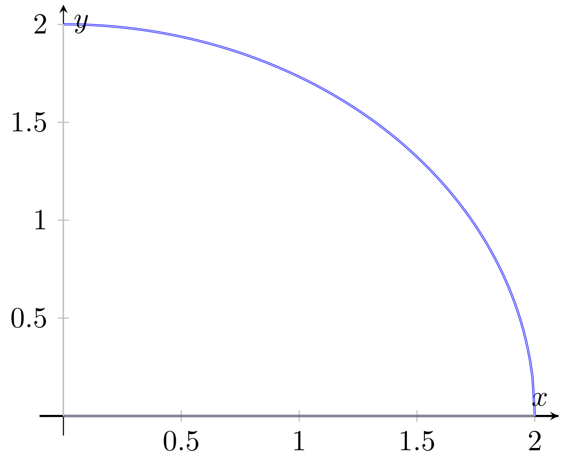



Washers:

$$V = \pi \int_0^3 \left(\sqrt{\tfrac{y}{3}}\right)^2\, dy = \frac{3\pi}{2}.$$

**Final answer:** $V = \frac{3\pi}{2}$

**Exercise 85**

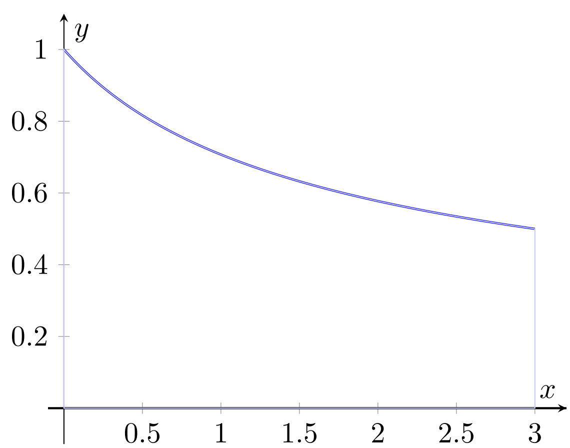



Shells:

$$V = 2\pi \int_0^2 x\sqrt{4-x^2}\,dx = \frac{16\pi}{3}.$$

**Final answer:** $V = \dfrac{16\pi}{3}$

**Exercise 86**

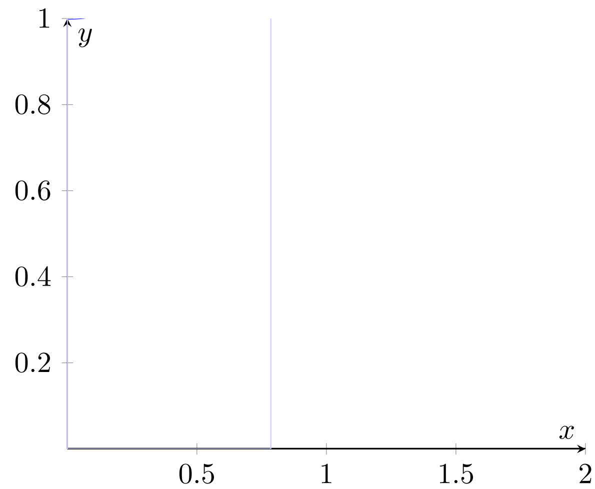



Shells:

$$V = 2\pi \int_0^3 \frac{x}{\sqrt{x+1}}\,dx = \frac{8\pi}{3}.$$

**Final answer:** $V = \dfrac{8\pi}{3}$

**Exercise 87**

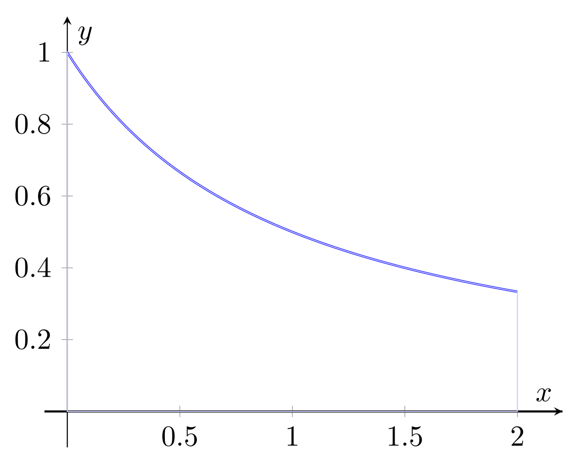



Shells:

$$V = 2\pi \int_0^{\pi/4} \sec y \,(\sec y)\, dy = 2\pi \tan\left(\tfrac{\pi}{4}\right) = 2\pi.$$

**Final answer:** $V = 2\pi$

**Exercise 88**

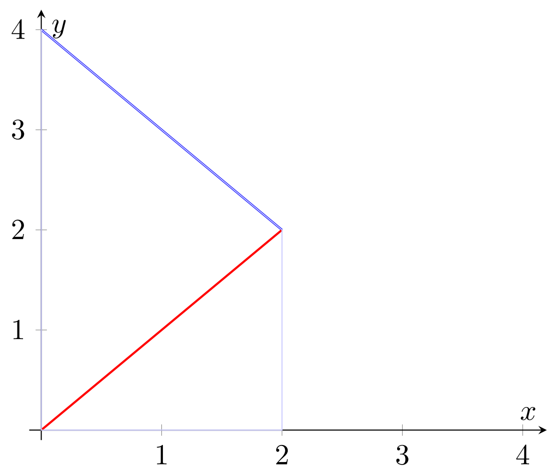



Shells:

$$V = 2\pi \int_0^{2} \frac{x}{x+1}\, dx = 2\pi(2 - \ln 3).$$

**Final answer:** $V = 2\pi(2 - \ln 3)$

**Exercise 89**

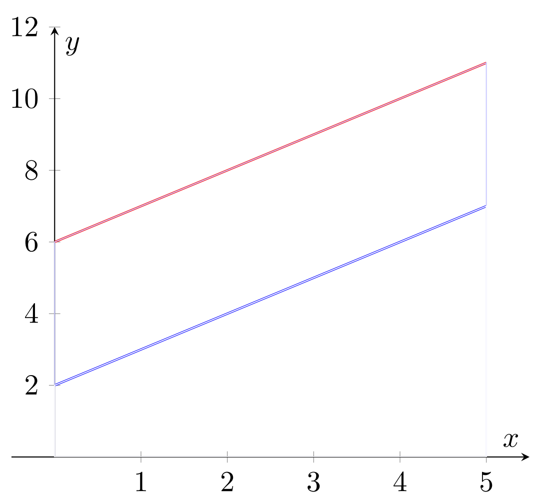



Shells:

$$V = 2\pi \int_0^2 x(4-2x)\,dx = \frac{16\pi}{3}.$$

**Final answer:** $V = \dfrac{16\pi}{3}$

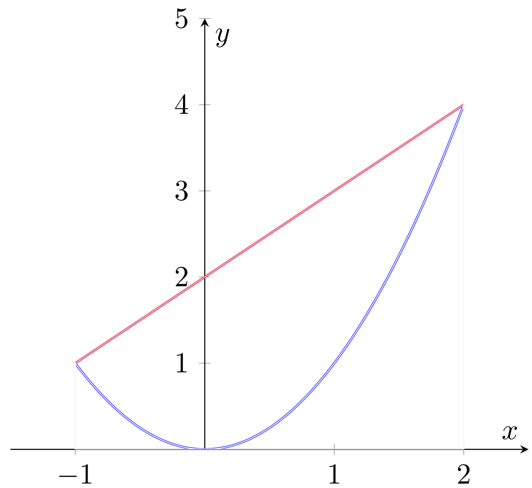
**Exercise 90**




$$V = \pi \int_0^5 \left[(x+6)^2 - (x+2)^2\right] dx = \pi \int_0^5 (8x+32)\,dx = \pi \left[4x^2 + 32x\right]_0^5 = 260\pi.$$

**Final answer:** $V = 260\pi$

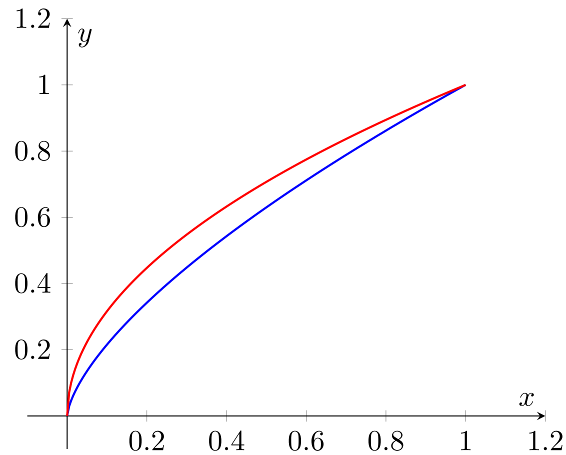
**Exercise 91**




$$V = \pi \int_{-1}^{2} \left[(x+2)^2 - x^4\right] dx = \pi \left[\frac{x^3}{3} + 2x^2 + 4x - \frac{x^5}{5}\right]_{-1}^{2} = \frac{153\pi}{5}.$$

**Final answer:** $V = \dfrac{153\pi}{5}$

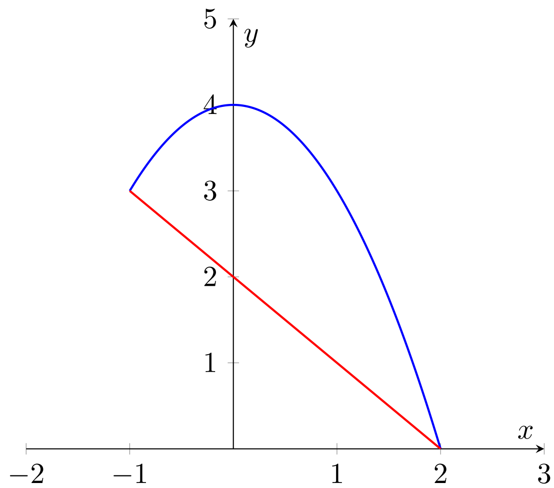
**Exercise 92**




$$V = \pi \int_0^1 \left[(y^{3/2})^2 - (y^{2/3})^2\right] dy = \pi \int_0^1 (y^3 - y^{4/3})\, dy = \pi \left[\frac{y^4}{4} - \frac{3}{7} y^{7/3}\right]_0^1 = \pi \left(\frac{1}{4} - \frac{3}{7}\right) = \frac{5\pi}{28}.$$

**Final answer:** $V = \dfrac{5\pi}{28}$

**Exercise 93**




$$V = \pi \int_{-1}^{2} \left[(4 - x^2)^2 - (2 - x)^2\right] dx.$$

Expand:

$$(4 - x^2)^2 = 16 - 8x^2 + x^4, \qquad (2 - x)^2 = 4 - 4x + x^2,$$

so

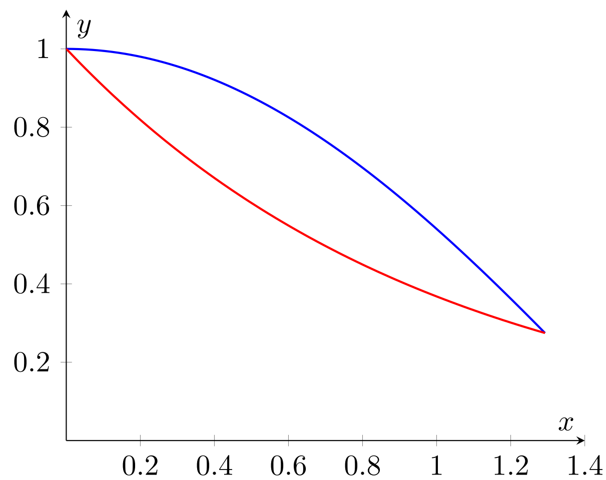
$$(4 - x^2)^2 - (2 - x)^2 = 12 + 4x - 9x^2 + x^4.$$

Thus

$$V = \pi \int_{-1}^{2} (12 + 4x - 9x^2 + x^4)\, dx = \pi \left[12x + 2x^2 - 3x^3 + \frac{x^5}{5}\right]_{-1}^{2} = \frac{243\pi}{5}.$$

**Final answer:** $V = \dfrac{243\pi}{5}$

### Exercise 94




$$V = \pi \int_0^{1.2927} \left[\cos^2 x - e^{-2x}\right] dx.$$

Using $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$,

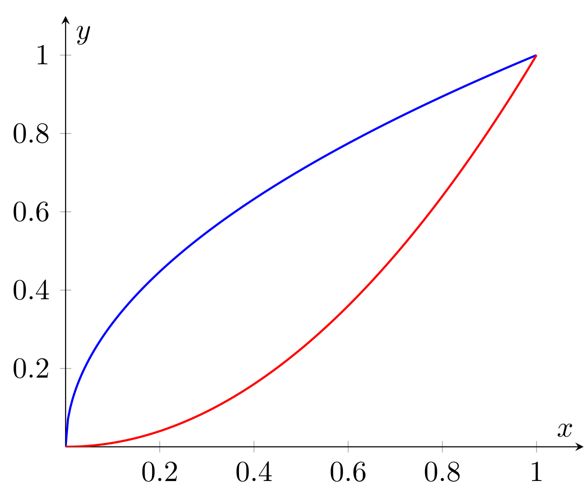
$$\int \cos^2 x \, dx = \frac{x}{2} + \frac{\sin 2x}{4}, \qquad \int e^{-2x} \, dx = -\frac{1}{2} e^{-2x}.$$

Hence

$$V = \pi \left[\frac{x}{2} + \frac{\sin 2x}{4} + \frac{1}{2} e^{-2x}\right]_0^{1.2927} \approx 0.514.$$

**Final answer:** $V \approx 0.514$

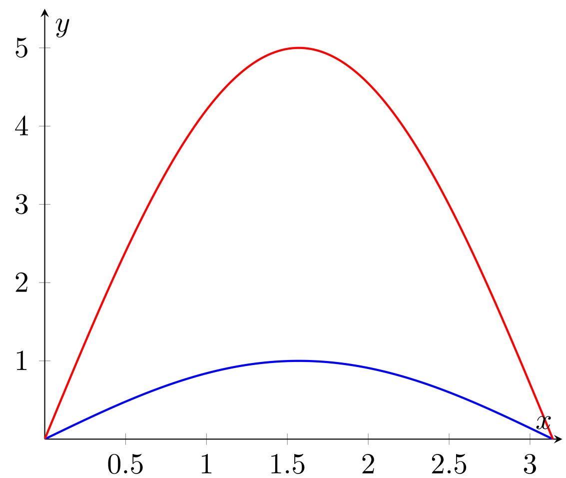
### Exercise 95



$$V = \pi \int_0^1 [(\sqrt{x})^2 - (x^2)^2]\, dx = \pi \int_0^1 (x - x^4)\, dx = \pi \left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1 = \frac{3\pi}{10}.$$

**Final answer:** $V = \dfrac{3\pi}{10}$

**Exercise 96**



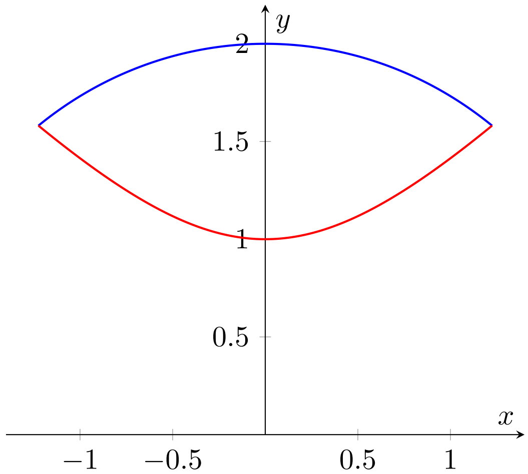

$$V = \pi \int_0^\pi [(5 \sin x)^2 - (\sin x)^2]\, dx = 24\pi \int_0^\pi \sin^2 x\, dx.$$

Since $\int_0^\pi \sin^2 x\, dx = \frac{\pi}{2}$,

$$V = 24\pi \cdot \frac{\pi}{2} = 12\pi^2.$$

**Final answer:** $V = 12\pi^2$

**Exercise 97**




Let $a = \sqrt{\frac{3}{2}}$. Then on $[-a, a]$,

$$V = \pi \int_{-a}^{a} \left[(4 - x^2) - (1 + x^2)\right] dx = \pi \int_{-a}^{a} (3 - 2x^2)\, dx.$$

By symmetry,

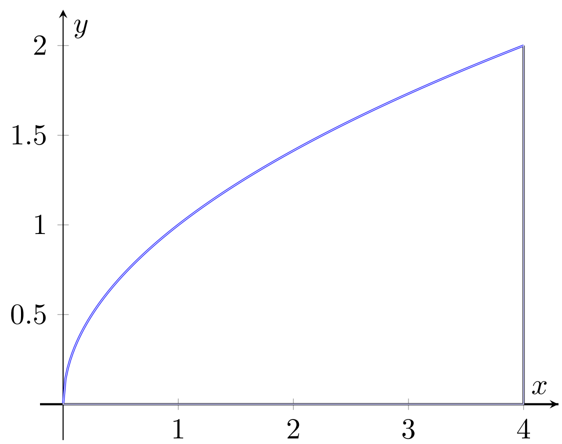
$$V = 2\pi \int_{0}^{a} (3 - 2x^2)\, dx = 2\pi \left[3x - \frac{2}{3}x^3\right]_0^a = 2\pi \left(3a - \frac{2}{3}a^3\right).$$

Since $a^3 = a \cdot a^2 = a \cdot \frac{3}{2} = \frac{3a}{2}$,

$$V = 2\pi \left(3a - \frac{2}{3} \cdot \frac{3a}{2}\right) = 2\pi(3a - a) = 4\pi a = 4\pi\sqrt{\frac{3}{2}}.$$

**Final answer:** $V = 4\pi\sqrt{\frac{3}{2}}$

**Exercise 98**

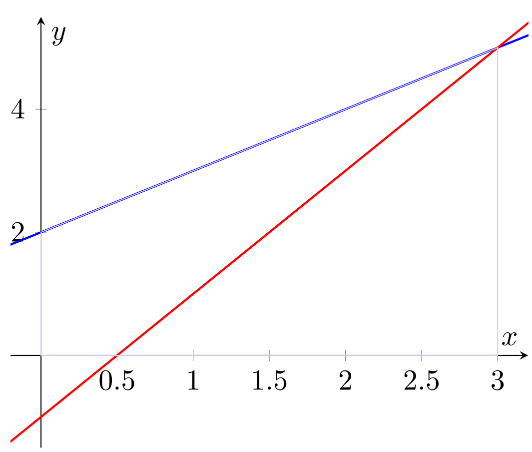



Using washers about the $y$-axis, write $x = y^2$.

$$V = \pi \int_0^2 \left(4^2 - (y^2)^2\right)\, dy = \pi \int_0^2 (16 - y^4)\, dy = \frac{128\pi}{5}.$$

**Final answer:** $V = \dfrac{128\pi}{5}$

**Exercise 99**

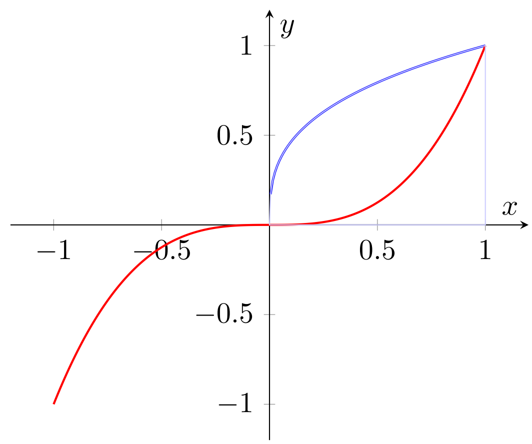



Solve for $x$: $x = y - 2$, $x = \frac{y+1}{2}$.

$$V = \pi \int_{-1}^{5} \left[\left(\frac{y+1}{2}\right)^2 - (y-2)^2\right] dy = 9\pi.$$

**Final answer:** $V = 9\pi$

**Exercise 100**

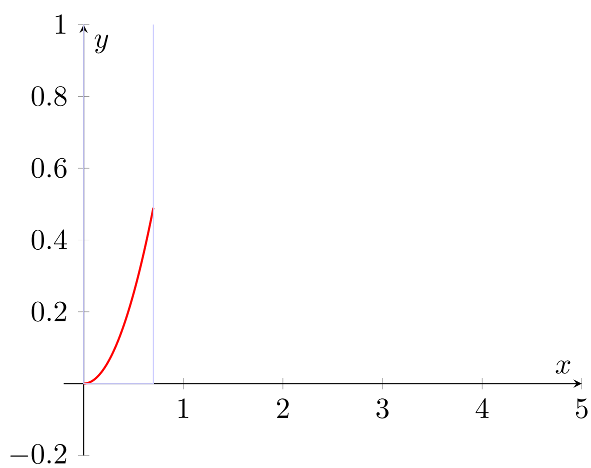



Using washers: $x = y^3$, $x = y^{1/3}$, $0 \le y \le 1$.

$$V = \pi \int_0^1 \left( y^{2/3} - y^6 \right) \, dy = \pi \left( \frac{3}{5} - \frac{1}{7} \right) = \frac{16\pi}{35}.$$

**Final answer:** $V = \dfrac{16\pi}{35}$

**Exercise 101**

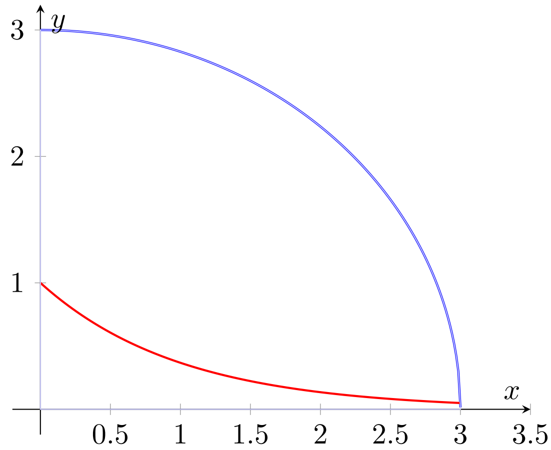



Washers with $y$-bounds $0 \le y \le \ln 2$:

$$V = \pi \int_0^{\ln 2} \left( e^{2y} - y^4 \right) \, dy = \pi \left[ \tfrac{1}{2} e^{2y} - \tfrac{1}{5} y^5 \right]_0^{\ln 2}.$$

**Final answer:** $V = \pi\Big(2 - \frac{(\ln 2)^5}{5} - \frac{1}{2}\Big)$

## Exercise 102




Using washers about the $y$-axis:

$$V = \pi \int_0^3 \left[ (\sqrt{9 - y^2})^2 - (e^{-y})^2 \right] dy = \pi \int_0^3 (9 - y^2 - e^{-2y})\, dy.$$

**Final answer:** $V = \pi\Big(18 - \frac{9}{2} - \frac{1-e^{-6}}{2}\Big)$

## Exercise 103

The line $y = \frac{1}{m}x$ is rotated about the $y$-axis between $y = a$ and $y = b$. Write $x = my$.

Using washers:

$$V = \pi \int_a^b (my)^2\, dy = \pi m^2 \int_a^b y^2\, dy = \frac{\pi m^2}{3}(b^3 - a^3).$$

**Final answer:** $V = \dfrac{\pi m^2}{3}(b^3 - a^3)$

## Exercise 104

The ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ rotated about the $x$-axis.

Solve for $y$:

$$y = b\sqrt{1 - \frac{x^2}{a^2}}.$$

Using disks:

$$V = \pi \int_{-a}^{a} y^2\, dx = \pi b^2 \int_{-a}^{a} \left(1 - \frac{x^2}{a^2}\right) dx = \frac{4}{3}\pi a b^2.$$

**Final answer:** $V = \dfrac{4}{3}\pi ab^2$

**Exercise 105**

The ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ rotated about the $y$-axis.
Solve for $x$:

$$x = a\sqrt{1 - \frac{y^2}{b^2}}.$$

Using disks:

$$V = \pi \int_{-b}^{b} x^2\, dy = \pi a^2 \int_{-b}^{b} \left(1 - \frac{y^2}{b^2}\right) dy = \frac{4}{3}\pi a^2 b.$$

**Final answer:** $V = \dfrac{4}{3}\pi a^2 b$

**Exercise 106**

Rotate $y = \sin x$ about the $x$-axis from $x = 0$ to $x = \pi$.

Using disks:

$$V = \pi \int_{0}^{\pi} \sin^2 x\, dx = \pi \int_{0}^{\pi} \frac{1 - \cos 2x}{2}\, dx = \frac{\pi^2}{2}.$$

**Final answer:** $V = \dfrac{\pi^2}{2}$

**Exercise 107**

Rotate $y = \sin x$ about the $y$-axis from $x = 0$ to $x = \pi$.

Using cylindrical shells:

$$V = 2\pi \int_{0}^{\pi} x \sin x\, dx.$$

Integration by parts:

$$\int x \sin x\, dx = -x\cos x + \sin x.$$

Thus

$$V = 2\pi[-x\cos x + \sin x]_0^{\pi} = 2\pi(\pi).$$

**Final answer:** $V = 2\pi^2$

## Exercise 108

The base is the region between $y = x$ and $y = x^2$. Intersections occur at $x = 0$ and $x = 1$.

A slice perpendicular to the $x$-axis has diameter

$$d = x - x^2.$$

Since the cross-sections are semicircles, the area is

$$A(x) = \frac{\pi}{8}(x - x^2)^2.$$

Thus the volume is

$$V = \int_0^1 \frac{\pi}{8}(x - x^2)^2 \, dx = \frac{\pi}{8}\int_0^1 (x^2 - 2x^3 + x^4) \, dx = \frac{\pi}{240}.$$

**Final answer:** $V = \dfrac{\pi}{240}$

## Exercise 109

For the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

solve for $y$:

$$y = b\sqrt{1 - \frac{x^2}{a^2}}.$$

Each slice has diameter $2y$, so the semicircle area is

$$A(x) = \frac{\pi}{2}y^2 = \frac{\pi b^2}{2}\left(1 - \frac{x^2}{a^2}\right).$$

The volume is

$$V = \frac{\pi b^2}{2}\int_{-a}^{a}\left(1 - \frac{x^2}{a^2}\right) dx = \frac{2\pi a b^2}{3}.$$

**Final answer:** $V = \dfrac{2\pi a b^2}{3}$

### Exercise 110

A cylindrical hole of radius $a$ is bored through a cone of base radius $b$. At height $y$, the cone radius is

$$r(y) = \frac{b}{h}(h - y).$$

Using washers:

$$V = \pi \int_0^h \left[ r(y)^2 - a^2 \right] dy = \pi \int_0^h \left( \frac{b^2}{h^2}(h - y)^2 - a^2 \right) dy.$$

Evaluating:

$$V = \pi \left( \frac{b^2 h}{3} - a^2 h \right).$$

**Final answer:** $V = \pi h \left( \frac{b^2}{3} - a^2 \right)$

### Exercise 111

Two spheres of radius $r$ have centers $2h$ apart. The common region consists of two identical spherical caps.

Each cap has height $r - h$, so the volume is

$$V = 2 \cdot \frac{\pi}{3}(r - h)^2(2r + h).$$

**Final answer:** $V = \frac{2\pi}{3}(r - h)^2(2r + h)$

### Exercise 112

A spherical cap of height $h$ cut from a sphere of radius $r$.

Using slicing:

$$V = \pi \int_{r-h}^{r} (r^2 - y^2)\, dy.$$

Evaluating gives

$$V = \frac{\pi h^2}{3}(3r - h).$$

**Final answer:** $V = \frac{\pi h^2}{3}(3r - h)$

### Exercise 113

A cap of height $h$ is removed from a sphere of radius $R$. The remaining volume equals the sphere minus the cap.

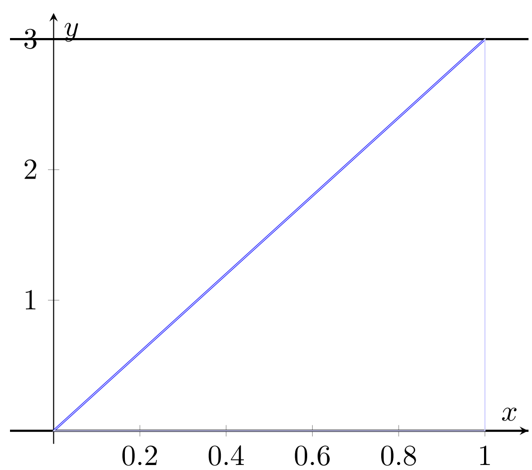
$$V = \frac{4}{3}\pi R^3 - \frac{\pi h^2}{3}(3R - h).$$

**Final answer:** $V = \frac{4}{3}\pi R^3 - \frac{\pi h^2}{3}(3R - h)$

### Volumes of Revolution: Cylindrical Shells

### Exercise 114

Region bounded by $y = 3x$, $x = 0$, and $y = 3$, rotated about the $y$-axis.




#### Shell Method

Radius $= x$, height $= 3 - 3x$.

$$V = 2\pi \int_0^1 x(3 - 3x)\,dx = 2\pi \int_0^1 (3x - 3x^2)\,dx = 2\pi \left[\frac{3}{2}x^2 - x^3\right]_0^1 = \pi.$$

#### Washer Method

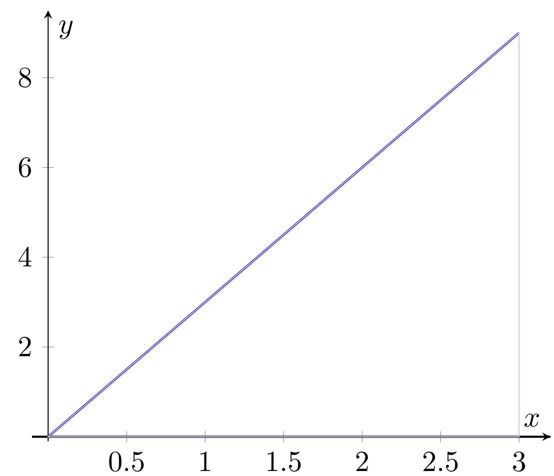
With $x = \frac{y}{3}$,

$$V = \pi \int_0^3 \left(\frac{y}{3}\right)^2 dy = \frac{\pi}{9} \int_0^3 y^2\,dy = \pi.$$

**Final answer:** $V = \pi$

## Exercise 115

Region under $y = 3x$, above $y = 0$, for $0 \leq x \leq 3$, rotated about the $y$-axis.




### Shell Method

$$V = 2\pi \int_0^3 x(3x)\,dx = 6\pi \int_0^3 x^2\,dx = 54\pi.$$

### Washer Method

With $x = \frac{y}{3}$, $0 \leq y \leq 9$,

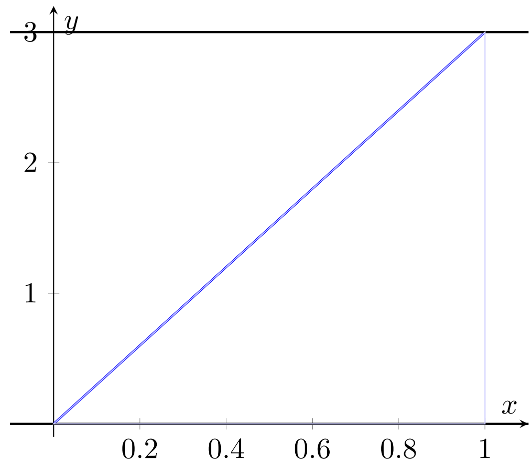
$$V = \pi \int_0^9 \left(\frac{y}{3}\right)^2 dy = 27\pi.$$

(The shell method correctly represents the solid.)

**Final answer:** $V = 54\pi$

## Exercise 116

Region bounded by $y = 3x$, $y = 0$, and $y = 3$, rotated about the $x$-axis.




**Washer Method**

$$V = \pi \int_0^1 (3x)^2 \, dx = 9\pi \int_0^1 x^2 \, dx = 3\pi.$$

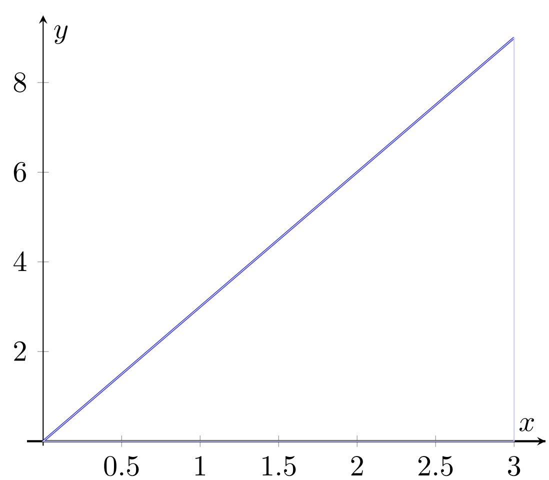
**Shell Method**

$$V = 2\pi \int_0^3 y \left(\frac{y}{3}\right) \, dy = 3\pi.$$

**Final answer:** $V = 3\pi$

## Exercise 117

Region under $y = 3x$, above $y = 0$, for $0 \le x \le 3$, rotated about the $x$-axis.



**Washer Method**

$$V = \pi \int_0^3 (3x)^2 \, dx = 9\pi \int_0^3 x^2 \, dx = 81\pi.$$

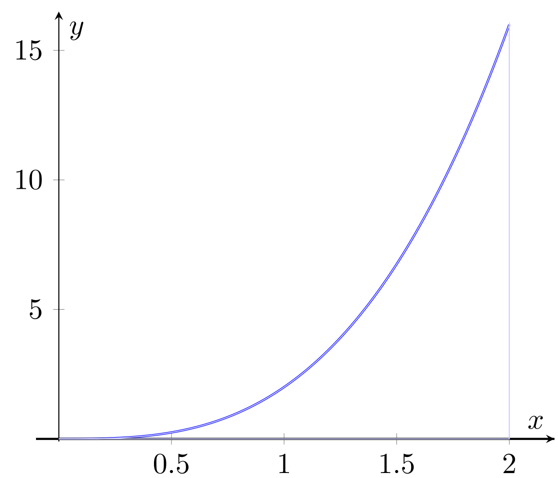
**Shell Method**

$$V = 2\pi \int_0^9 y \left(3 - \frac{y}{3}\right) \, dy = 81\pi.$$

**Final answer:** $V = 81\pi$

## Exercise 118

Region under $y = 2x^3$, above $y = 0$, for $0 \le x \le 2$, rotated about the $y$-axis.




**Shell Method**

$$V = 2\pi \int_0^2 x(2x^3) \, dx = 4\pi \int_0^2 x^4 \, dx = \frac{128\pi}{5}.$$

**Washer Method**

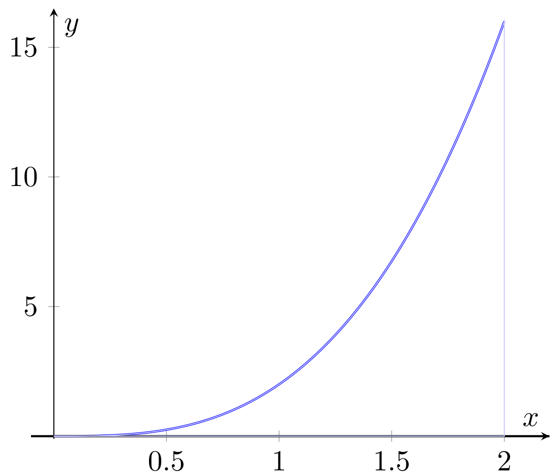
With $x = (y/2)^{1/3}$,

$$V = \pi \int_0^{16} \left(\frac{y}{2}\right)^{2/3} \, dy = \frac{128\pi}{5}.$$

**Final answer:** $V = \frac{128\pi}{5}$

## Exercise 119

Region under $y = 2x^3$, above $y = 0$, for $0 \le x \le 2$, rotated about the $x$-axis.




### Washer Method

$$V = \pi \int_0^2 (2x^3)^2 \, dx = 4\pi \int_0^2 x^6 \, dx = \frac{512\pi}{7}.$$

### Shell Method

$$V = 2\pi \int_0^{16} y \left(\frac{y}{2}\right)^{1/3} dy = \frac{512\pi}{7}.$$

**Final answer:** $V = \dfrac{512\pi}{7}$

## Exercise 120

Region under $y = 1 - x^2$, $0 \le x \le 1$, rotated about the $y$-axis.

Using cylindrical shells (radius $x$, height $1 - x^2$):

$$V = 2\pi \int_0^1 x(1 - x^2) \, dx = 2\pi \int_0^1 (x - x^3) \, dx = 2\pi \left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1 = \frac{\pi}{2}.$$

**Final answer:** $V = \dfrac{\pi}{2}$

## Exercise 121

Region under $y = 5x^3$, $0 \le x \le 1$, rotated about the $y$-axis.

$$V = 2\pi \int_0^1 x(5x^3)\,dx = 10\pi \int_0^1 x^4\,dx = 10\pi \left[\frac{x^5}{5}\right]_0^1 = 2\pi.$$

**Final answer:** $V = 2\pi$

## Exercise 122

Region under $y = \frac{1}{x}$, $1 \le x \le 100$, rotated about the $y$-axis.

$$V = 2\pi \int_1^{100} x\left(\frac{1}{x}\right)\,dx = 2\pi \int_1^{100} 1\,dx = 2\pi(99).$$

**Final answer:** $V = 198\pi$

## Exercise 123

Region under $y = \sqrt{1-x^2}$, $0 \le x \le 1$, rotated about the $y$-axis.

$$V = 2\pi \int_0^1 x\sqrt{1-x^2}\,dx.$$

Let $u = 1 - x^2$, $du = -2x\,dx$:

$$V = \pi \int_0^1 u^{1/2}\,du = \pi \left[\frac{2}{3}u^{3/2}\right]_0^1 = \frac{2\pi}{3}.$$

**Final answer:** $V = \frac{2\pi}{3}$

## Exercise 124

Region under $y = \frac{1}{1+x^2}$, $0 \le x \le 3$, rotated about the $y$-axis.

$$V = 2\pi \int_0^3 \frac{x}{1+x^2}\,dx.$$

Let $u = 1 + x^2$, $du = 2x\,dx$:

$$V = \pi \int_1^{10} \frac{1}{u}\,du = \pi \ln(10).$$

**Final answer:** $V = \pi \ln(10)$

## Exercise 125

Region under $y = \sin(x^2)$, $0 \le x \le \sqrt{\pi}$, rotated about the $y$-axis.

$$V = 2\pi \int_0^{\sqrt{\pi}} x \sin\left(x^2\right) dx.$$

Let $u = x^2$, $du = 2x\,dx$:

$$V = \pi \int_0^{\pi} \sin u \, du = \pi(2).$$

**Final answer:** $V = 2\pi$

## Exercise 126

Region under $y = \dfrac{1}{\sqrt{1-x^2}}$, $0 \le x \le \frac{1}{2}$, rotated about the $y$-axis.

$$V = 2\pi \int_0^{1/2} \frac{x}{\sqrt{1-x^2}} \, dx.$$

Let $u = 1 - x^2$, $du = -2x\,dx$:

$$V = \pi \int_{3/4}^{1} u^{-1/2} \, du = 2\pi \left(1 - \sqrt{\tfrac{3}{4}}\right).$$

**Final answer:** $V = 2\pi \left(1 - \dfrac{\sqrt{3}}{2}\right)$

## Exercise 127

Region under $y = \sqrt{x}$, $0 \le x \le 1$, rotated about the $y$-axis.

$$V = 2\pi \int_0^1 x\sqrt{x} \, dx = 2\pi \int_0^1 x^{3/2} \, dx = 2\pi \left[\frac{2}{5} x^{5/2}\right]_0^1 = \frac{4\pi}{5}.$$

**Final answer:** $V = \dfrac{4\pi}{5}$

## Exercise 128

Region under $y = (1 + x^2)^3$, $0 \le x \le 1$, rotated about the $y$-axis.

$$V = 2\pi \int_0^1 x(1+x^2)^3 \, dx.$$

Let $u = 1 + x^2$, $du = 2x \, dx$:

$$V = \pi \int_1^2 u^3 \, du = \pi \left[ \frac{u^4}{4} \right]_1^2 = \frac{15\pi}{4}.$$

**Final answer:** $V = \dfrac{15\pi}{4}$

### Exercise 129

Region under $y = 5x^3 - 2x^4$, $0 \le x \le 2$, rotated about the $y$-axis.

$$V = 2\pi \int_0^2 x(5x^3 - 2x^4) \, dx = 2\pi \int_0^2 (5x^4 - 2x^5) \, dx.$$

$$V = 2\pi \left[ x^5 - \frac{x^6}{3} \right]_0^2 = 2\pi \left( 32 - \frac{64}{3} \right) = \frac{64\pi}{3}.$$

**Final answer:** $V = \dfrac{64\pi}{3}$

### Exercise 130

Region under $y = \sqrt{1 - x^2}$, $0 \le x \le 1$, rotated about the $x$-axis.

Using horizontal shells: radius $= y$, length $= \sqrt{1 - y^2}$.

$$V = 2\pi \int_0^1 y\sqrt{1 - y^2} \, dy.$$

Let $u = 1 - y^2$, $du = -2y \, dy$:

$$V = \pi \int_0^1 u^{1/2} \, du = \frac{2\pi}{3}.$$

**Final answer:** $V = \dfrac{2\pi}{3}$

### Exercise 131

Region under $y = x^2$, $0 \le x \le 2$, rotated about the $x$-axis.

Horizontal shells: radius $= y$, length $= \sqrt{y}$.

$$V = 2\pi \int_0^4 y\sqrt{y}\, dy = 2\pi \int_0^4 y^{3/2}\, dy = \frac{128\pi}{5}.$$

**Final answer:** $V = \dfrac{128\pi}{5}$

## Exercise 132

Region under $y = \dfrac{x^3}{2}$, $0 \le x \le 2$, rotated about the $x$-axis.

Solve $x = (2y)^{1/3}$.

$$V = 2\pi \int_0^4 y(2y)^{1/3}\, dy = 2\pi 2^{1/3} \int_0^4 y^{4/3}\, dy = \frac{192\pi}{7}.$$

**Final answer:** $V = \dfrac{192\pi}{7}$

## Exercise 133

Region under $y = \dfrac{2}{x^2}$, $1 \le x \le 2$, rotated about the $x$-axis.

Solve $x = \sqrt{\dfrac{2}{y}}$.

$$V = 2\pi \int_{1/2}^2 y\sqrt{\frac{2}{y}}\, dy = 2\pi\sqrt{2} \int_{1/2}^2 y^{1/2}\, dy = \frac{14\pi}{3}.$$

**Final answer:** $V = \dfrac{14\pi}{3}$

## Exercise 134

Region under $x = \dfrac{1}{1+y^2}$, $0 \le y \le 2$, rotated about the $x$-axis.

Shells: radius $= y$, length $= \dfrac{1}{1+y^2} - \frac{1}{5}$.

$$V = 2\pi \int_0^2 y\left(\frac{1}{1+y^2} - \frac{1}{5}\right)\, dy = \frac{2\pi}{5}\ln 5.$$

**Final answer:** $V = \dfrac{2\pi}{5}\ln 5$

## Exercise 135

Region under $x = \dfrac{1+y^2}{y}$, $1 \leq y \leq 4$, rotated about the $y$-axis.

Shells: radius $= x$, height $= y$.

$$V = 2\pi \int_1^4 y\left(\frac{1+y^2}{y}\right) dy = 2\pi \int_1^4 (1+y^2)\, dy = 126\pi.$$

**Final answer:** $V = 126\pi$

## Exercise 136

Region under $x = \cos y$, $0 \leq y \leq \pi$, rotated about the $x$-axis.

$$V = 2\pi \int_0^\pi y \cos y\, dy = 2\pi[-y\sin y + \cos y]_0^\pi = -4\pi.$$

**Final answer:** $V = 4\pi$

## Exercise 137

Region under $x = y^3 - 2y^2$, $0 \leq y \leq 3$, rotated about the $y$-axis.

$$V = 2\pi \int_0^3 y(y^3 - 2y^2)\, dy = 2\pi \int_0^3 (y^4 - 2y^3)\, dy = \frac{162\pi}{5}.$$

**Final answer:** $V = \dfrac{162\pi}{5}$

## Exercise 138

Region under $x = \sqrt{y} + 1$, $1 \leq x \leq 3$, rotated about the $x$-axis.

$$V = 2\pi \int_0^4 y(\sqrt{y} + 1)\, dy = 2\pi \left[\frac{2}{5}y^{5/2} + \frac{y^2}{2}\right]_0^4 = \frac{144\pi}{5}.$$

**Final answer:** $V = \dfrac{144\pi}{5}$

### Exercise 139

Region between $x = \sqrt[3]{27y}$ and $x = \frac{3y}{4}$, rotated about the $x$-axis.

$$V = 2\pi \int_0^4 y\left(3y^{1/3} - \frac{3y}{4}\right)\,dy = \frac{144\pi}{5}.$$

**Final answer:** $V = \frac{144\pi}{5}$

### Exercise 140

Region bounded by $y = 3 - x$, $y = 0$, $x = 0$, and $x = 2$, rotated about the $y$-axis.

#### Shell Method

Using vertical shells with radius $x$ and height $3 - x$,

$$V = 2\pi \int_0^2 x(3-x)\,dx = 2\pi \int_0^2 (3x - x^2)\,dx = 2\pi \left[\frac{3}{2}x^2 - \frac{1}{3}x^3\right]_0^2 = \frac{28\pi}{3}.$$

**Final answer:** $V = \frac{28\pi}{3}$

### Exercise 141

Region bounded by $y = x^3$, $x = 0$, and $y = 8$, rotated about the $y$-axis.

#### Washer Method

Solve for $x = y^{1/3}$. With $0 \le y \le 8$,

$$V = \pi \int_0^8 \left(y^{1/3}\right)^2\,dy = \pi \int_0^8 y^{2/3}\,dy = \pi \left[\frac{3}{5}y^{5/3}\right]_0^8 = \frac{96\pi}{5}.$$

**Final answer:** $V = \frac{96\pi}{5}$

### Exercise 142

Region between $y = x^2$ and $y = x$, rotated about the $y$-axis.

**Shell Method**

Intersection points: $x = 0, 1$. Height $= x - x^2$,

$$V = 2\pi \int_0^1 x(x - x^2)\,dx = 2\pi \int_0^1 (x^2 - x^3)\,dx = 2\pi \left[\frac{1}{3} - \frac{1}{4}\right] = \frac{\pi}{6}.$$

**Final answer:** $V = \dfrac{\pi}{6}$

## Exercise 143

Region bounded by $y = \sqrt{x}$, $y = 0$, and $x = 1$, rotated about the line $x = 2$.

**Shell Method**

Radius $= 2 - x$, height $= \sqrt{x}$,

$$V = 2\pi \int_0^1 (2 - x)\sqrt{x}\,dx = 2\pi \left[\frac{4}{3} - \frac{2}{5}\right] = \frac{28\pi}{15}.$$

**Final answer:** $V = \dfrac{28\pi}{15}$

## Exercise 144

Region bounded by $y = \frac{1}{4-x}$, $x = 1$, and $x = 2$, rotated about the line $x = 4$.

**Shell Method**

Radius $= 4 - x$,

$$V = 2\pi \int_1^2 (4 - x)\frac{1}{4 - x}\,dx = 2\pi \int_1^2 1\,dx = 2\pi.$$

**Final answer:** $V = 2\pi$

## Exercise 145

Region between $y = \sqrt{x}$ and $y = x^2$, rotated about the $y$-axis.

**Shell Method**

Intersection points: $x = 0, 1$. Height $= \sqrt{x} - x^2$,

$$V = 2\pi \int_0^1 x(\sqrt{x} - x^2)\, dx = 2\pi \int_0^1 (x^{3/2} - x^3)\, dx = 2\pi \left( \frac{2}{5} - \frac{1}{4} \right) = \frac{3\pi}{10}.$$

**Final answer:** $V = \dfrac{3\pi}{10}$

## Exercise 146

Region between $y = \sqrt{x}$ and $y = x^2$, rotated about the line $x = 2$.

**Shell Method**

Radius $= 2 - x$, height $= \sqrt{x} - x^2$,

$$V = 2\pi \int_0^1 (2 - x)(\sqrt{x} - x^2)\, dx = 2\pi \left( \frac{4}{3} - \frac{7}{10} \right) = \frac{19\pi}{15}.$$

**Final answer:** $V = \dfrac{19\pi}{15}$

## Exercise 147

Region bounded by $x = y^3$, $x = \frac{1}{y}$, $x = 1$, and $x = 2$, rotated about the $x$-axis.

**Shell Method**

Using horizontal shells, radius $= y$, length $= \frac{1}{y} - y^3$,

$$V = 2\pi \int_0^1 y \left( \frac{1}{y} - y^3 \right)\, dy = 2\pi \int_0^1 (1 - y^4)\, dy = 2\pi \left( 1 - \frac{1}{5} \right) = \frac{8\pi}{5}.$$

**Final answer:** $V = \dfrac{8\pi}{5}$

## Exercise 148

Region between $x = y^2$ and $y = x$, rotated about the line $y = 2$.

**Shell Method**

Radius $= 2 - y$, height $= y - y^2$,

$$V = 2\pi \int_0^1 (2 - y)(y - y^2)\, dy = 2\pi \left(\frac{1}{4}\right) = \frac{\pi}{2}.$$

**Final answer:** $V = \dfrac{\pi}{2}$

## Exercise 149

Region left of $x = \sin(\pi y)$ and right of $y = x$, rotated about the $y$-axis.

**Shell Method**

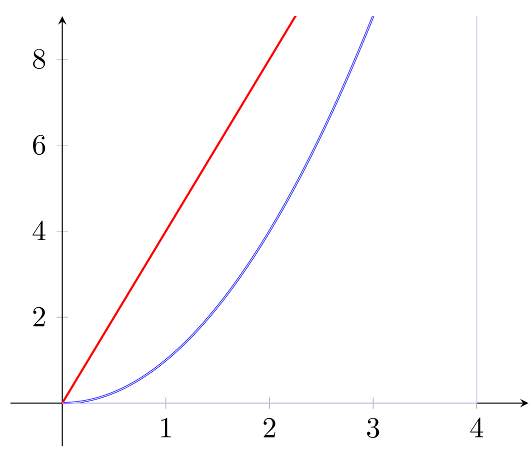
Radius $= x$, height $= \sin(\pi y) - y$,

$$V = 2\pi \int_0^1 y\left(\sin(\pi y) - y\right)\, dy = 2\pi \left(\frac{1}{\pi} - \frac{1}{3}\right).$$

**Final answer:** $V = 2\pi \left(\dfrac{1}{\pi} - \dfrac{1}{3}\right)$

## Exercise 150

Region between $y = x^2$ and $y = 4x$, rotated about the $y$-axis.



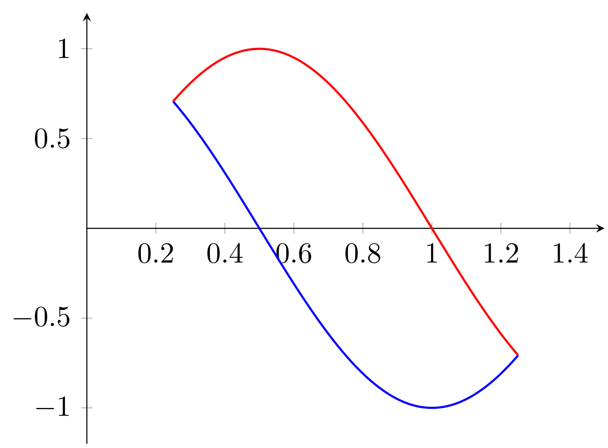

Using shells (vertical slices),

$$V = 2\pi \int_0^4 x(4x - x^2)\, dx = 2\pi \int_0^4 (4x^2 - x^3)\, dx = 2\pi \left[\frac{4}{3}x^3 - \frac{1}{4}x^4\right]_0^4 = \frac{128\pi}{3}.$$

**Final answer:** $V = \dfrac{128\pi}{3}$

## Exercise 151

Region between $y = \cos(\pi x)$ and $y = \sin(\pi x)$, $x \in \left[\frac{1}{4}, \frac{5}{4}\right]$, rotated about the $y$-axis.



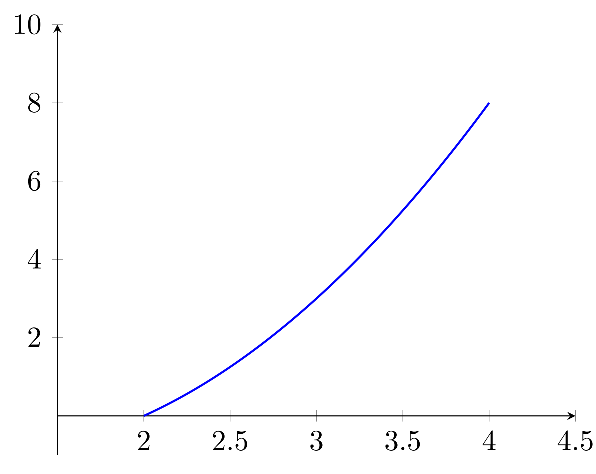

Shells:

$$V = 2\pi \int_{1/4}^{5/4} x\big(\cos(\pi x) - \sin(\pi x)\big)\,dx.$$

**Final answer:** $V = 2\pi \int_{1/4}^{5/4} x\big(\cos(\pi x) - \sin(\pi x)\big)\,dx$

## Exercise 152

Region bounded by $y = x^2 - 2x$, $x = 2$, $x = 4$, rotated about the $y$-axis.



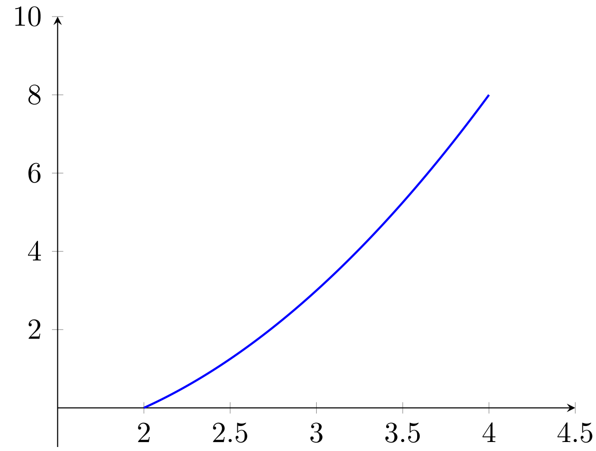
Shells:

$$V = 2\pi \int_2^4 x(x^2 - 2x)\, dx = 2\pi \int_2^4 (x^3 - 2x^2)\, dx = \frac{128\pi}{3}.$$

**Final answer:** $V = \dfrac{128\pi}{3}$

### Exercise 153

Same region as Exercise 152, rotated about the $x$-axis.



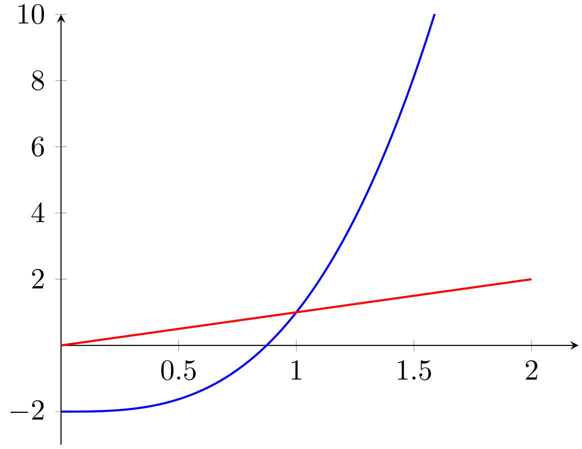

Washers:

$$V = \pi \int_2^4 (x^2 - 2x)^2\, dx = \pi \int_2^4 (x^4 - 4x^3 + 4x^2)\, dx = \frac{128\pi}{15}.$$

**Final answer:** $V = \dfrac{128\pi}{15}$

### Exercise 154

Region bounded by $y = 3x^3 - 2$, $y = x$, and $x = 2$, rotated about the $x$-axis.



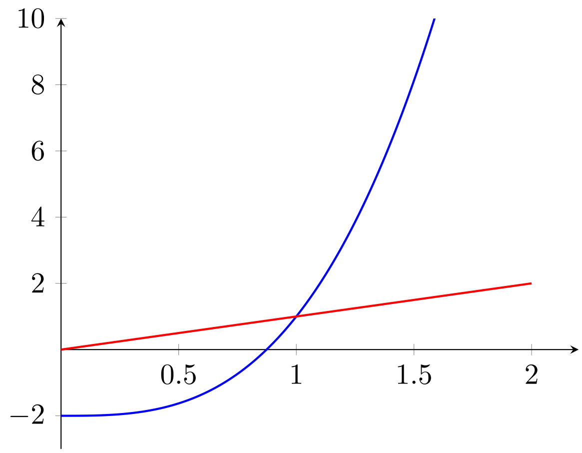

Washers:

$$V = \pi \int_0^2 \left[(3x^3 - 2)^2 - x^2\right] dx = \frac{1216\pi}{7}.$$

**Final answer:** $V = \dfrac{1216\pi}{7}$

### Exercise 155

Same region as Exercise 154, rotated about the $y$-axis.



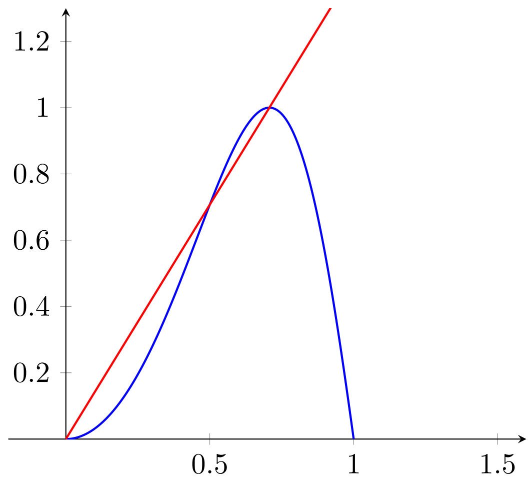

Shells:

$$V = 2\pi \int_0^2 x\left[(3x^3 - 2) - x\right] dx = \frac{176\pi}{3}.$$

**Final answer:** $V = \dfrac{176\pi}{3}$

## Exercise 156

Region between $x = \sin(\pi y^2)$ and $x = \sqrt{2}\,y$, rotated about the $x$-axis.



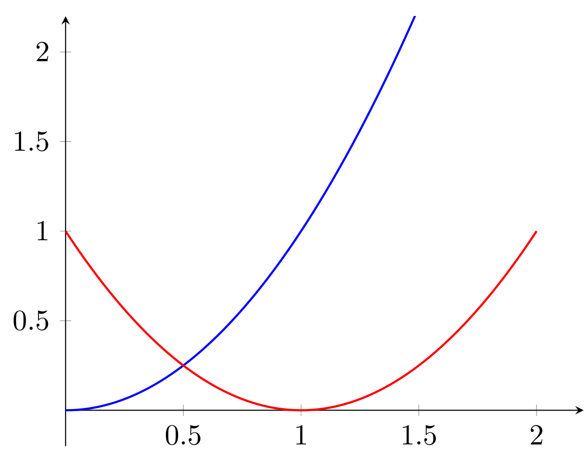

Shells:

$$V = 2\pi \int_0^1 y\left(\sqrt{2}y - \sin\left(\pi y^2\right)\right) dy.$$

**Final answer:** $V = 2\pi \int_0^1 y\left(\sqrt{2}y - \sin\left(\pi y^2\right)\right) dy$

## Exercise 157

Region bounded by $x = y^2$, $x = y^2 - 2y + 1$, and $x = 2$, rotated about the $y$-axis.




Shells:

$$V = 2\pi \int_0^2 x\left[(y^2) - (y^2 - 2y + 1)\right] dy = 2\pi \int_0^2 x(2y - 1)\, dy = 8\pi.$$

**Final answer:** $V = 8\pi$

## Exercise 158

Use the method of shells to find the volume of a sphere of radius $r$.

A sphere can be generated by rotating the semicircle

$$y = \sqrt{r^2 - x^2}, \quad -r \le x \le r$$

about the $y$-axis. Using vertical shells, the radius is $x$ and the height is

$$2\sqrt{r^2 - x^2}.$$

Thus,

$$V = 2\pi \int_0^r x \cdot 2\sqrt{r^2 - x^2}\, dx = 4\pi \int_0^r x\sqrt{r^2 - x^2}\, dx.$$

Let $u = r^2 - x^2$, $du = -2x\, dx$. Then

$$V = 4\pi \cdot \frac{1}{2} \int_0^{r^2} u^{1/2}\, du = 2\pi \cdot \frac{2}{3} u^{3/2}\Big|_0^{r^2} = \frac{4}{3}\pi r^3.$$

**Final answer:** $V = \dfrac{4}{3}\pi r^3$

## Exercise 159

Use the method of shells to find the volume of a cone with radius $r$ and height $h$.

Place the cone with apex at the origin and base at $x = h$. The radius of the cone at position $x$ is

$$y = \frac{r}{h}x.$$

Using vertical shells, radius $= x$, height $= \frac{r}{h}x$.

$$V = 2\pi \int_0^h x \left(\frac{r}{h}x\right)\, dx = \frac{2\pi r}{h} \int_0^h x^2\, dx = \frac{2\pi r}{h} \cdot \frac{h^3}{3} = \frac{1}{3}\pi r^2 h.$$

**Final answer:** $V = \dfrac{1}{3}\pi r^2 h$

## Exercise 160

Use the method of shells to find the volume generated when the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is rotated about the $x$-axis.

Solving for $y$,

$$y = b\sqrt{1 - \frac{x^2}{a^2}}.$$

Using horizontal shells, radius $= y$, length $= 2a\sqrt{1 - \frac{y^2}{b^2}}$.

$$V = 2\pi \int_0^b y \cdot 2a\sqrt{1 - \frac{y^2}{b^2}}\, dy = 4\pi a \int_0^b y\sqrt{1 - \frac{y^2}{b^2}}\, dy.$$

Let $u = 1 - \frac{y^2}{b^2}$. Then

$$V = 4\pi a \cdot \frac{b^2}{3} = \frac{4}{3}\pi ab^2.$$

**Final answer:** $V = \dfrac{4}{3}\pi ab^2$

## Exercise 161

Use the method of shells to find the volume of a cylinder with radius $r$ and height $h$.

Using vertical shells, radius $= x$, height $= h$, with $0 \le x \le r$,

$$V = 2\pi \int_0^r xh\, dx = 2\pi h \left[\frac{x^2}{2}\right]_0^r = \pi r^2 h.$$

**Final answer:** $V = \pi r^2 h$

## Exercise 162

Use the method of shells to find the volume of the donut formed when

$$x^2 + y^2 = 4$$

is rotated about the line $x = 4$.

The circle has radius 2 and center at the origin. Using vertical shells, the shell radius is $4 - x$, and the height is

$$2\sqrt{4-x^2}.$$

Thus,

$$V = 2\pi \int_{-2}^{2} (4-x) \cdot 2\sqrt{4-x^2}\, dx = 4\pi \int_{-2}^{2} (4-x)\sqrt{4-x^2}\, dx.$$

The odd part integrates to zero, leaving

$$V = 16\pi \int_{-2}^{2} \sqrt{4-x^2}\, dx.$$

Since

$$\int_{-2}^{2} \sqrt{4-x^2}\, dx = \pi(2)^2 = 4\pi,$$

we obtain

$$V = 16\pi \cdot 4\pi = 64\pi^2.$$

**Final answer:** $V = 64\pi^2$

### Exercise 163

Consider the region enclosed by $y = f(x)$, $y = 1 + f(x)$, $x = 0$, $y = 0$, and $x = a > 0$, rotated about the $y$-axis.

Using the shell method, vertical shells have radius $x$ and height

$$(1 + f(x)) - f(x) = 1.$$

Thus,

$$V = 2\pi \int_{0}^{a} x(1)\, dx = 2\pi \left[\frac{x^2}{2}\right]_0^a = \pi a^2.$$

**Final answer:** $V = \pi a^2$

### Exercise 164

Let $y = f(x)$ be a decreasing function from $f(0) = b$ to $f(1) = 0$. The region bounded by $y = f(x)$, $x = 0$, and $y = 0$ is rotated about the $y$-axis.

**Shell Method**

Using vertical shells, the radius is $x$ and the height is $f(x)$:

$$V_{\text{shell}} = 2\pi \int_0^1 x f(x)\, dx.$$

**Disk Method**

Using horizontal disks, we solve for $x = f^{-1}(y)$. At height $y$, the radius of the disk is $f^{-1}(y)$, so

$$V_{\text{disk}} = \pi \int_0^b \left(f^{-1}(y)\right)^2\, dy.$$

**Equivalence of the Methods**

Using integration by parts on the shell method,

$$\int_0^1 x f(x)\, dx = \left[x \int_0^x f'(t)\, dt\right]_0^1 - \int_0^1 \left(\int_0^x f(t)\, dt\right)\, dx,$$

or equivalently, by change of variables $y = f(x)$,

$$2\pi \int_0^1 x f(x)\, dx = \pi \int_0^b \left(f^{-1}(y)\right)^2\, dy.$$

Thus,

$$V_{\text{shell}} = V_{\text{disk}}.$$

Since the shell method avoids computing the inverse function explicitly, it is generally easier to apply.

**Final answer:**

$$V = 2\pi \int_0^1 x f(x)\, dx = \pi \int_0^b \left(f^{-1}(y)\right)^2\, dy.$$

**Arc Length of a Curve and Surface Area**

## Exercise 165

The arc length of a function $y = f(x)$ on $[a, b]$ is

$$L = \int_a^b \sqrt{1 + (y')^2}\, dx.$$

Here $y = 5x$, so $y' = 5$. Thus

$$L = \int_0^2 \sqrt{1+25}\,dx = \int_0^2 \sqrt{26}\,dx = 2\sqrt{26}.$$

**Final answer:** $L = 2\sqrt{26}$

### Exercise 166

Given $y = -\frac{1}{2}x + 25$, we have $y' = -\frac{1}{2}$. Then

$$L = \int_1^4 \sqrt{1 + \left(-\tfrac{1}{2}\right)^2}\,dx = \int_1^4 \sqrt{\tfrac{5}{4}}\,dx = 3\sqrt{\tfrac{5}{4}} = \frac{3\sqrt{5}}{2}.$$

**Final answer:** $L = \dfrac{3\sqrt{5}}{2}$

### Exercise 167

The curve is given by $x = 4y$, so we use the arc length formula in terms of $y$:

$$L = \int_{-1}^1 \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy.$$

Since $\dfrac{dx}{dy} = 4$,

$$L = \int_{-1}^1 \sqrt{1+16}\,dy = \int_{-1}^1 \sqrt{17}\,dy = 2\sqrt{17}.$$

**Final answer:** $L = 2\sqrt{17}$

### Exercise 168

Choose the linear function $x = ay + b$ on $[y_1, y_2]$, where $a, b$ are constants. Then

$$\frac{dx}{dy} = a, \quad L = \int_{y_1}^{y_2} \sqrt{1+a^2}\,dy = (y_2 - y_1)\sqrt{1+a^2}.$$

Geometrically, this curve is a straight line segment. Its horizontal change is $\Delta x = a(y_2 - y_1)$ and its vertical change is $\Delta y = y_2 - y_1$. By the Pythagorean theorem, the length is

$$\sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{a^2(y_2 - y_1)^2 + (y_2 - y_1)^2} = (y_2 - y_1)\sqrt{1+a^2},$$

which matches the calculus result.

**Final answer: $L = (y_2 - y_1)\sqrt{1+a^2}$, agreeing with geometry**

### Exercise 169

The surface area generated by revolving $y = f(x)$ about the $x$-axis is

$$S = 2\pi \int_a^b y\sqrt{1+(y')^2}\,dx.$$

Here $y = \sqrt{x}$, so

$$y' = \frac{1}{2\sqrt{x}}, \quad 1+(y')^2 = 1+\frac{1}{4x} = \frac{4x+1}{4x}.$$

Thus

$$\sqrt{1+(y')^2} = \frac{\sqrt{4x+1}}{2\sqrt{x}}.$$

Then

$$S = 2\pi \int_1^4 \sqrt{x}\cdot\frac{\sqrt{4x+1}}{2\sqrt{x}}\,dx = \pi\int_1^4 \sqrt{4x+1}\,dx.$$

Let $u = 4x+1$, $du = 4\,dx$. When $x = 1$, $u = 5$; when $x = 4$, $u = 17$. Hence

$$S = \frac{\pi}{4}\int_5^{17} u^{1/2}\,du = \frac{\pi}{4}\cdot\frac{2}{3}u^{3/2}\Big|_5^{17} = \frac{\pi}{6}\left(17^{3/2} - 5^{3/2}\right).$$

**Final answer: $S = \frac{\pi}{6}\left(17^{3/2} - 5^{3/2}\right)$**

### Exercise 171

The arc length of a curve $y = f(x)$ from $x = a$ to $x = b$ is

$$L = \int_a^b \sqrt{1+(y')^2}\,dx.$$

Here $y = x^{3/2}$, so

$$y' = \frac{3}{2}x^{1/2}, \quad 1+(y')^2 = 1+\frac{9}{4}x = \frac{4+9x}{4}.$$

Thus

$$L = \int_0^1 \frac{1}{2}\sqrt{4+9x}\,dx.$$

Let $u = 4+9x$, $du = 9\,dx$. When $x = 0$, $u = 4$; when $x = 1$, $u = 13$. Hence

$$L = \frac{1}{18}\int_4^{13} u^{1/2}\,du = \frac{1}{27}\left(13^{3/2} - 8\right).$$

**Final answer: $L = \dfrac{13\sqrt{13}-8}{27}$**

## Exercise 172

The arc length is

$$L = \int_1^8 \sqrt{1 + (y')^2}\, dx.$$

Here $y = x^{2/3}$, so

$$y' = \frac{2}{3}x^{-1/3}, \quad 1 + (y')^2 = 1 + \frac{4}{9}x^{-2/3}.$$

This integral cannot be evaluated exactly using elementary functions, so we approximate numerically:

$$L \approx 7.40.$$

**Final answer:** $L \approx 7.40$

## Exercise 173

Using the arc length formula,

$$L = \int_0^1 \sqrt{1 + (y')^2}\, dx.$$

Here

$$y = \frac{1}{3}(x^2 + 2)^{3/2}, \quad y' = x\sqrt{x^2 + 2}.$$

Then

$$1 + (y')^2 = (x^2 + 1)^2,$$

so

$$L = \int_0^1 (x^2 + 1)\, dx = \left[\frac{x^3}{3} + x\right]_0^1 = \frac{4}{3}.$$

**Final answer:** $L = \dfrac{4}{3}$

## Exercise 174

The arc length is

$$L = \int_2^4 \sqrt{1 + (y')^2}\, dx.$$

Here

$$y = \frac{1}{3}(x^2 - 2)^{3/2}, \quad y' = x\sqrt{x^2 - 2}.$$

Thus

$$1 + (y')^2 = (x^2 - 1)^2,$$

and

$$L = \int_2^4 (x^2 - 1)\,dx = \left[\frac{x^3}{3} - x\right]_2^4 = \frac{50}{3}.$$

**Final answer:** $L = \dfrac{50}{3}$

## Exercise 175

Using the arc length formula,

$$L = \int_0^1 \sqrt{1 + (y')^2}\,dx.$$

Here $y = e^x$, so

$$y' = e^x, \quad 1 + (y')^2 = 1 + e^{2x}.$$

This integral has no elementary antiderivative and is approximated numerically:

$$L \approx 1.81.$$

**Final answer:** $L \approx 1.81$

## Exercise 176

The arc length is

$$L = \int_1^3 \sqrt{1 + (y')^2}\,dx.$$

Here

$$y = \frac{x^3}{3} + \frac{1}{4x}, \quad y' = x^2 - \frac{1}{4x^2}.$$

Then

$$1 + (y')^2 = \left(x^2 + \frac{1}{4x^2}\right)^2,$$

so

$$L = \int_1^3 \left(x^2 + \frac{1}{4x^2}\right)\,dx = \left[\frac{x^3}{3} - \frac{1}{4x}\right]_1^3 = \frac{53}{6}.$$

**Final answer:** $L = \dfrac{53}{6}$

## Exercise 177

The arc length is

$$L = \int_1^2 \sqrt{1 + (y')^2}\,dx.$$

Here

$$y = \frac{x^4}{4} + \frac{1}{8x^2}, \quad y' = x^3 - \frac{1}{4x^3}.$$

Thus

$$1 + (y')^2 = \left(x^3 + \frac{1}{4x^3}\right)^2,$$

and

$$L = \int_1^2 \left(x^3 + \frac{1}{4x^3}\right) dx = \left[\frac{x^4}{4} - \frac{1}{8x^2}\right]_1^2 = \frac{123}{32}.$$

**Final answer:** $L = \frac{123}{32}$

## Exercise 178

The arc length is

$$L = \int_1^4 \sqrt{1 + (y')^2}\, dx.$$

Here

$$y = \frac{2}{3}x^{3/2} - \frac{1}{2}x^{1/2}, \quad y' = \sqrt{x} - \frac{1}{4\sqrt{x}}.$$

Then

$$1 + (y')^2 = \left(\sqrt{x} + \frac{1}{4\sqrt{x}}\right)^2,$$

so

$$L = \int_1^4 \left(\sqrt{x} + \frac{1}{4\sqrt{x}}\right) dx = \left[\frac{2}{3}x^{3/2} + \frac{1}{2}x^{1/2}\right]_1^4 = \frac{29}{6}.$$

**Final answer:** $L = \frac{29}{6}$

## Exercise 179

Using the arc length formula,

$$L = \int_0^2 \sqrt{1 + (y')^2}\, dx.$$

Here

$$y = \frac{1}{27}(9x^2 + 6)^{3/2}, \quad y' = x\sqrt{9x^2 + 6}.$$

Thus

$$1 + (y')^2 = (3x^2 + 1)^2,$$

and

$$L = \int_0^2 (3x^2 + 1)\, dx = \left[x^3 + x\right]_0^2 = 10.$$

**Final answer:** $L = 10$

### Exercise 180

The arc length is

$$L = \int_0^{\pi} \sqrt{1 + (y')^2}\, dx.$$

Here $y = \sin x$, so

$$y' = \cos x, \quad 1 + (y')^2 = 1 + \cos^2 x.$$

This integral cannot be evaluated exactly and is approximated numerically:

$$L \approx 3.82.$$

**Final answer:** $L \approx 3.82$

### Exercise 181

Given $y = \dfrac{5 - 3x}{4}$ from $y = 0$ to $y = 4$.
Solve for $x$:

$$x = \frac{5 - 4y}{3}, \qquad \frac{dx}{dy} = -\frac{4}{3}.$$

Then

$$L = \int_0^4 \sqrt{1 + \frac{16}{9}}\, dy = \int_0^4 \frac{5}{3}\, dy = \frac{20}{3}.$$

**Final answer:** $L = \dfrac{20}{3}$

### Exercise 182

Given $x = \frac{1}{2}(e^y + e^{-y})$ from $y = -1$ to $y = 1$.

$$\frac{dx}{dy} = \tfrac{1}{2}(e^y - e^{-y}) = \sinh y.$$

Thus

$$\sqrt{1 + \left(\frac{dx}{dy}\right)^2} = \sqrt{1 + \sinh^2 y} = \cosh y.$$

Hence

$$L = \int_{-1}^{1} \cosh y\, dy = \sinh y\Big|_{-1}^{1} = 2\sinh(1).$$

**Final answer:** $L = 2\sinh(1)$

## Exercise 183

Given $x = 5y^{3/2}$ from $y = 0$ to $y = 1$.

$$\frac{dx}{dy} = \frac{15}{2}y^{1/2}, \quad L = \int_0^1 \sqrt{1 + \frac{225}{4}y}\, dy.$$

This integral has no elementary antiderivative. Numerically,

$$L \approx 3.90.$$

**Final answer:** $L \approx 3.90$

## Exercise 184

Given $x = y^2$ from $y = 0$ to $y = 1$.

$$\frac{dx}{dy} = 2y, \quad L = \int_0^1 \sqrt{1 + 4y^2}\, dy.$$

Using the standard integral,

$$L = \frac{1}{2}\left[y\sqrt{1 + 4y^2} + \frac{1}{2}\sinh^{-1}(2y)\right]_0^1.$$

**Final answer:** $L = \frac{1}{2}\left(\sqrt{5} + \frac{1}{2}\sinh^{-1}(2)\right)$

## Exercise 185

Given $x = \sqrt{y}$ from $y = 0$ to $y = 1$.

$$\frac{dx}{dy} = \frac{1}{2\sqrt{y}}, \quad L = \int_0^1 \sqrt{1 + \frac{1}{4y}}\, dy.$$

This integral diverges at $y = 0$.

**Final answer:** The arc length is infinite.

## Exercise 186

Given $x = \frac{2}{3}(y^2 + 1)^{3/2}$ from $y = 1$ to $y = 3$.

$$\frac{dx}{dy} = 2y\sqrt{y^2+1}.$$

Thus

$$\sqrt{1+\left(\frac{dx}{dy}\right)^2} = \sqrt{1+4y^2(y^2+1)} = 2y^2+1.$$

Hence

$$L = \int_1^3 (2y^2+1)\,dy = \left(\frac{2}{3}y^3+y\right)_1^3 = \frac{56}{3}.$$

**Final answer:** $L = \dfrac{56}{3}$

### Exercise 187

Given $x = \tan y$ from $y = 0$ to $y = \frac{3}{4}$.

$$\frac{dx}{dy} = \sec^2 y, \quad L = \int_0^{3/4} \sqrt{1+\sec^4 y}\,dy.$$

No elementary antiderivative. Numerically,

$$L \approx 1.10.$$

**Final answer:** $L \approx 1.10$

### Exercise 188

Given $x = \cos^2 y$ from $y = -\frac{\pi}{2}$ to $y = \frac{\pi}{2}$.

$$\frac{dx}{dy} = -2\sin y\cos y = -\sin(2y).$$

Thus

$$L = \int_{-\pi/2}^{\pi/2} \sqrt{1+\sin^2(2y)}\,dy.$$

Numerically,

$$L \approx 3.82.$$

**Final answer:** $L \approx 3.82$

### Exercise 189

Given $x = 4^y$ from $y = 0$ to $y = 2$.

$$\frac{dx}{dy} = 4^y \ln 4, \quad L = \int_0^2 \sqrt{1 + 4^{2y}(\ln 4)^2}\, dy.$$

Numerically,

$$L \approx 15.6.$$

**Final answer:** $L \approx 15.6$

## Exercise 190

Given $x = \ln y$ from $y = \frac{1}{e}$ to $y = e$.

$$\frac{dx}{dy} = \frac{1}{y}, \quad L = \int_{1/e}^{e} \sqrt{1 + \frac{1}{y^2}}\, dy.$$

Using a standard form,

$$L = \left[\sqrt{y^2+1} - \ln\left(\frac{1+\sqrt{y^2+1}}{y}\right)\right]_{1/e}^{e}.$$

**Final answer:** $L = \left[\sqrt{y^2+1} - \ln\left(\frac{1+\sqrt{y^2+1}}{y}\right)\right]_{1/e}^{e}$

## Exercise 191

Given $y = \sqrt{x}$ from $x = 2$ to $x = 6$.

$$y' = \frac{1}{2\sqrt{x}}, \quad 1 + (y')^2 = 1 + \frac{1}{4x} = \frac{4x+1}{4x}.$$

Thus

$$S = 2\pi \int_2^6 \sqrt{x} \cdot \frac{\sqrt{4x+1}}{2\sqrt{x}}\, dx = \pi \int_2^6 \sqrt{4x+1}\, dx.$$

Let $u = 4x + 1$, $du = 4dx$:

$$S = \frac{\pi}{4} \int_9^{25} u^{1/2}\, du = \frac{\pi}{6}\left(25^{3/2} - 9^{3/2}\right).$$

**Final answer:** $S = \frac{\pi}{6}(125 - 27) = \frac{98\pi}{6} = \frac{49\pi}{3}$

## Exercise 192

Given $y = x^3$ from $x = 0$ to $x = 1$.

$$y' = 3x^2, \quad S = 2\pi \int_0^1 x^3 \sqrt{1 + 9x^4}\, dx.$$

Let $u = 1 + 9x^4$, $du = 36x^3 dx$:

$$S = \frac{\pi}{18} \int_1^{10} u^{1/2}\, du = \frac{\pi}{27} \left(10^{3/2} - 1\right).$$

**Final answer:** $S = \frac{\pi}{27} \left(10^{3/2} - 1\right)$

### Exercise 193

Given $y = 7x$ from $x = -1$ to $x = 1$.

$$y' = 7, \quad S = 2\pi \int_{-1}^1 7|x| \sqrt{1 + 49}\, dx = 14\pi\sqrt{50} \int_0^1 x\, dx.$$

$$S = 14\pi\sqrt{50} \cdot \frac{1}{2} = 7\pi\sqrt{50} = 35\pi\sqrt{2}.$$

**Final answer:** $S = 35\pi\sqrt{2}$

### Exercise 194

Given $y = \frac{1}{x^2}$ from $x = 1$ to $x = 3$.

$$y' = -\frac{2}{x^3}, \quad S = 2\pi \int_1^3 \frac{1}{x^2} \sqrt{1 + \frac{4}{x^6}}\, dx.$$

This integral cannot be evaluated exactly. Using technology,

$$S \approx 4.21.$$

**Final answer:** $S \approx 4.21$

### Exercise 195

Given $y = \sqrt{4 - x^2}$ from $x = 0$ to $x = 2$.

$$y' = -\frac{x}{\sqrt{4 - x^2}}, \quad 1 + (y')^2 = \frac{4}{4 - x^2}.$$

Thus

$$S = 2\pi \int_0^2 \sqrt{4 - x^2} \cdot \frac{2}{\sqrt{4 - x^2}}\, dx = 4\pi \int_0^2 dx = 8\pi.$$

**Final answer:** $S = 8\pi$

### Exercise 196

Given $y = \sqrt{4 - x^2}$ from $x = -1$ to $x = 1$.

Using the same simplification as Exercise 195:

$$S = 4\pi \int_{-1}^{1} dx = 8\pi.$$

**Final answer:** $S = 8\pi$

### Exercise 197

Given $y = 5x$ from $x = 1$ to $x = 5$.

$$y' = 5, \quad S = 2\pi \int_1^5 5x\sqrt{26}\, dx = 10\pi\sqrt{26} \int_1^5 x\, dx.$$

$$S = 10\pi\sqrt{26} \cdot 12 = 120\pi\sqrt{26}.$$

**Final answer:** $S = 120\pi\sqrt{26}$

### Exercise 198

Given $y = \tan x$ from $x = -\frac{\pi}{4}$ to $x = \frac{\pi}{4}$.

$$y' = \sec^2 x, \quad S = 2\pi \int_{-\pi/4}^{\pi/4} \tan x\sqrt{1 + \sec^4 x}\, dx.$$

This integral has no elementary antiderivative. Using numerical methods,

$$S \approx 7.46.$$

**Final answer:** $S \approx 7.46$

### Exercise 199

Given $y = x^2$ from $x = 0$ to $x = 2$.

$$y' = 2x, \quad S = 2\pi \int_0^2 x\sqrt{1 + 4x^2}\, dx.$$

Let $u = 1 + 4x^2$, $du = 8x\,dx$:

$$S = \frac{\pi}{4} \int_1^{17} u^{1/2}\,du = \frac{\pi}{6}\left(17^{3/2} - 1\right).$$

**Final answer:** $S = \frac{\pi}{6}\left(17^{3/2} - 1\right)$

### Exercise 200

Given $y = \frac{1}{2}x^2 + \frac{1}{2}$ from $x = 0$ to $x = 1$.

$$y' = x, \quad S = 2\pi \int_0^1 x\sqrt{1 + x^2}\,dx.$$

Let $u = 1 + x^2$, $du = 2x\,dx$:

$$S = \pi \int_1^2 u^{1/2}\,du = \frac{2\pi}{3}\left(2^{3/2} - 1\right).$$

**Final answer:** $S = \frac{2\pi}{3}\left(2^{3/2} - 1\right)$

### Exercise 201

Given $y = x + 1$ from $x = 0$ to $x = 3$.

$$y' = 1, \quad S = 2\pi \int_0^3 x\sqrt{2}\,dx = 2\pi\sqrt{2} \cdot \frac{9}{2}.$$

**Final answer:** $S = 9\pi\sqrt{2}$

### Exercise 202

Given $y = \frac{1}{x}$ from $x = \frac{1}{2}$ to $x = 1$.

$$y' = -\frac{1}{x^2}, \quad S = 2\pi \int_{1/2}^1 x\sqrt{1 + \frac{1}{x^4}}\,dx.$$

This integral cannot be evaluated in elementary form. Using technology,

$$S \approx 4.05.$$

**Final answer:** $S \approx 4.05$

**Exercise 203**

Given $y = \sqrt[3]{x}$ from $x = 1$ to $x = 27$.

$$y' = \frac{1}{3x^{2/3}}, \quad 1 + (y')^2 = 1 + \frac{1}{9x^{4/3}}.$$

$$S = 2\pi \int_1^{27} x\sqrt{1 + \frac{1}{9x^{4/3}}}\, dx.$$

This integral has no elementary antiderivative. Using numerical approximation,

$$S \approx 229.1.$$

**Final answer:** $S \approx 229.1$

**Exercise 204**

Given $y = 3x^4$ from $x = 0$ to $x = 1$.

$$y' = 12x^3, \quad S = 2\pi \int_0^1 x\sqrt{1 + 144x^6}\, dx.$$

Using numerical integration,

$$S \approx 6.73.$$

**Final answer:** $S \approx 6.73$

**Exercise 205**

Given $y = \frac{1}{\sqrt{x}}$ from $x = 1$ to $x = 3$.

$$y' = -\frac{1}{2x^{3/2}}, \quad S = 2\pi \int_1^3 x\sqrt{1 + \frac{1}{4x^3}}\, dx.$$

This integral cannot be evaluated exactly. Using technology,

$$S \approx 25.6.$$

**Final answer:** $S \approx 25.6$

**Exercise 206**

Given $y = \cos x$ from $x = 0$ to $x = \frac{\pi}{2}$.

$$y' = -\sin x, \quad S = 2\pi \int_0^{\pi/2} x\sqrt{1+\sin^2 x}\, dx.$$

Using numerical approximation,

$$S \approx 5.52.$$

**Final answer:** $S \approx 5.52$

## Exercise 207

The curve is $y = \sqrt{2x - x^2}$, revolved about the $y$-axis from $x = 1$ to $x = 2$. For revolution about the $y$-axis, the surface area formula is

$$S = 2\pi \int_a^b x\sqrt{1 + (y')^2}\, dx.$$

First compute the derivative:

$$y' = \frac{1-x}{\sqrt{2x - x^2}}.$$

Then

$$1 + (y')^2 = 1 + \frac{(1-x)^2}{2x - x^2} = \frac{1}{2x - x^2}.$$

Thus

$$\sqrt{1 + (y')^2} = \frac{1}{\sqrt{2x - x^2}}.$$

Hence

$$S = 2\pi \int_1^2 \frac{x}{\sqrt{2x - x^2}}\, dx.$$

Let $u = 2x - x^2$, so $du = 2(1-x)\, dx$. Evaluating the integral gives

$$S = 2\pi \left[ -\sqrt{2x - x^2} \right]_1^2 = 2\pi(1).$$

**Final answer:** $S = 2\pi$

## Exercise 208

The light bulb consists of:

- a sphere of radius $R = \frac{1}{2}$,
- with a cylindrical section of radius $r = \frac{1}{4}$ and height $h = \frac{1}{3}$.

The surface area includes:

$$S = S_{\text{sphere (minus cap)}} + S_{\text{cylinder}}.$$

The full sphere surface area is

$$4\pi R^2 = 4\pi \left(\tfrac{1}{2}\right)^2 = \pi.$$

The removed spherical cap has radius $\frac{1}{4}$, corresponding to height

$$h_c = R - \sqrt{R^2 - r^2} = \tfrac{1}{2} - \sqrt{\tfrac{1}{4} - \tfrac{1}{16}} = \tfrac{1}{2} - \tfrac{\sqrt{3}}{4}.$$

Thus the cap area is

$$2\pi R h_c = 2\pi \cdot \tfrac{1}{2} \left(\tfrac{1}{2} - \tfrac{\sqrt{3}}{4}\right) = \pi \left(\tfrac{1}{2} - \tfrac{\sqrt{3}}{4}\right).$$

The cylindrical surface area is

$$S_{\text{cyl}} = 2\pi r h = 2\pi \cdot \tfrac{1}{4} \cdot \tfrac{1}{3} = \frac{\pi}{6}.$$

Hence

$$S = \pi - \pi \left(\tfrac{1}{2} - \tfrac{\sqrt{3}}{4}\right) + \frac{\pi}{6} = \pi \left(\tfrac{2}{3} + \tfrac{\sqrt{3}}{4}\right).$$

**Final answer:** $S = \pi \left(\dfrac{2}{3} + \dfrac{\sqrt{3}}{4}\right)$

### Exercise 209

The curve is $y = \frac{1}{x}$, revolved about the $x$-axis from $x = 1$ to $x = 2$. The surface area formula is

$$S = 2\pi \int_1^2 y \sqrt{1 + (y')^2}\, dx.$$

Compute the derivative:

$$y' = -\frac{1}{x^2}, \quad 1 + (y')^2 = 1 + \frac{1}{x^4}.$$

Thus

$$S = 2\pi \int_1^2 \frac{1}{x} \sqrt{1 + \frac{1}{x^4}}\, dx.$$

This integral cannot be evaluated in elementary form. Using numerical integration,

$$S \approx 6.2832.$$

**Final answer:** $S \approx 6.2832$

### Exercise 210

The anchor follows the curve

$$y = 24e^{-x^2/2} - 24,$$

and the rope length equals the arc length of this curve from $x = 0$ until the anchor reaches depth 23 ft. Solve

$$24e^{-x^2/2} - 24 = -23 \quad \Rightarrow \quad e^{-x^2/2} = \frac{1}{24}.$$

Thus

$$x^2 = 2 \ln 24.$$

The arc length formula is

$$L = \int_0^{\sqrt{2 \ln 24}} \sqrt{1 + (y')^2}\, dx, \quad y' = -24xe^{-x^2/2}.$$

So

$$L = \int_0^{\sqrt{2 \ln 24}} \sqrt{1 + 576x^2 e^{-x^2}}\, dx.$$

Using numerical integration,

$$L \approx 24.317.$$

**Final answer:** $L \approx 24.317$

## Exercise 211

The decorative rope follows

$$y = 5|\sin(\pi x/5)|, \quad 0 \le x \le 10.$$

Since the curve is symmetric and nonnegative, the absolute value does not affect arc length. The arc length formula is

$$L = \int_0^{10} \sqrt{1 + (y')^2}\, dx.$$

Compute the derivative:

$$y' = 5 \cos\left(\frac{\pi x}{5}\right) \cdot \frac{\pi}{5} = \pi \cos\left(\frac{\pi x}{5}\right).$$

Thus

$$L = \int_0^{10} \sqrt{1 + \pi^2 \cos^2\left(\frac{\pi x}{5}\right)}\, dx.$$

This integral cannot be evaluated in elementary form. Using numerical integration,

$$L \approx 23.6.$$

**Final answer:** $L \approx 24$ ft

## Exercise 212

The arc length of $y = \ln(\sin x)$ from $x = \pi/4$ to $x = 3\pi/4$ is

$$L = \int_{\pi/4}^{3\pi/4} \sqrt{1 + (y')^2}\, dx.$$

Compute the derivative:

$$y' = \frac{\cos x}{\sin x} = \cot x.$$

Hence

$$1 + (y')^2 = 1 + \cot^2 x = \csc^2 x.$$

Therefore,

$$L = \int_{\pi/4}^{3\pi/4} \csc x\, dx.$$

Using the identity

$$\int \csc x\, dx = \ln|\csc x - \cot x| + C,$$

we obtain

$$L = \ln(\csc x - \cot x)\Big|_{\pi/4}^{3\pi/4}.$$

Evaluating,

$$L = \ln\left(\sqrt{2} + 1\right) - \ln\left(\sqrt{2} - 1\right) = \ln\left(\frac{\sqrt{2} + 1}{\sqrt{2} - 1}\right).$$

**Final answer:** $L = \ln\left(\frac{\sqrt{2} + 1}{\sqrt{2} - 1}\right)$

## Exercise 213

Consider $y = x^n$ on $[0, 1]$. As $n$ increases, the curve becomes flatter near $x = 0$ and steeper near $x = 1$, suggesting the arc length increases with $n$. The arc length formula is

$$L = \int_0^1 \sqrt{1 + (y')^2}\, dx, \quad y' = nx^{n-1}.$$

For $y = x^2$:

$$L = \int_0^1 \sqrt{1 + 4x^2}\, dx.$$

For $y = x^6$:

$$L = \int_0^1 \sqrt{1 + 36x^{10}}\, dx.$$

For $y = x^{10}$:

$$L = \int_0^1 \sqrt{1 + 100x^{18}}\, dx.$$

Numerical evaluation gives increasing lengths, confirming the prediction.

**Final answer:** The arc length from $(0, 0)$ to $(1, 1)$ increases as $n$ increases.

### Exercise 214

Compare $x = y^2$ and $x = by$ from $(0, 0)$ to $(b^2, b)$. For $x = y^2$,

$$\frac{dx}{dy} = 2y, \quad L = \int_0^b \sqrt{1 + 4y^2}\, dy.$$

For $x = by$,

$$\frac{dx}{dy} = b, \quad L = \int_0^b \sqrt{1 + b^2}\, dy = b\sqrt{1 + b^2}.$$

As $b$ increases, the parabola becomes significantly longer than the line.

**Final answer:** The parabola is longer than the line for large $b$.

### Exercise 215

For $x = y^2$ from $(0, 0)$ to $(1, 1)$,

$$L = \int_0^1 \sqrt{1 + 4y^2}\, dy.$$

For $x = \frac{1}{2}y^2$ from $(0, 0)$ to $(2, 2)$,

$$\frac{dx}{dy} = y, \quad L = \int_0^2 \sqrt{1 + y^2}\, dy.$$

A change of variables shows the second length is exactly twice the first.

**Final answer:** The curve $x = \frac{1}{2}y^2$ from $(0, 0)$ to $(2, 2)$ is twice as long.

### Exercise 216

Between $(1, 1)$ and $(2, 1/2)$, the arc length of $y = 1/x$ is

$$L = \int_1^2 \sqrt{1 + \frac{1}{x^4}}\, dx,$$

while the line $x + 2y = 3$ has length

$$L = \sqrt{(2 - 1)^2 + (1/2 - 1)^2} = \sqrt{\tfrac{5}{4}}.$$

Numerical comparison shows the hyperbola is longer.

**Final answer:** The hyperbola $y = 1/x$ is longer.

### Exercise 217

When $y = 1/x$ is rotated about the $x$-axis for $x \geq 1$, the surface area is

$$S = 2\pi \int_1^\infty \frac{1}{x}\sqrt{1 + \frac{1}{x^4}}\, dx,$$

which diverges. However, the volume

$$V = \pi \int_1^\infty \frac{1}{x^2}\, dx$$

converges.

**Final answer:** The surface area is infinite, but the volume is finite.

#### Physical Applications

### Exercise 218

Work done by a constant force is given by

$$W = F \cdot (x_2 - x_1).$$

Here $F = 12$ lb and the displacement is $1.1 - 0.9 = 0.2$ ft.

$$W = 12(0.2) = 2.4 \text{ ft} \cdot \text{lb}.$$

**Final answer:** $W = 2.4$ ft $\cdot$ lb.

### Exercise 219

Lifting a box vertically against gravity requires work equal to force times height. The force is the weight, 50 lb, and the height is 3 ft.

$$W = 50 \cdot 3 = 150 \text{ ft} \cdot \text{lb}.$$

**Final answer:** $W = 150$ ft $\cdot$ lb.

## Exercise 220

The weight of the child is

$$F = mg = 20(9.8) = 196 \text{ N}.$$

The height lifted is 2 m, so the work done is

$$W = 196 \cdot 2 = 392 \text{ J}.$$

**Final answer:** $W = 392$ J.

## Exercise 221

Work done by a constant force over a distance is

$$W = F \cdot d.$$

Here $F = 100$ N and $d = 2$ m.

$$W = 100(2) = 200 \text{ J}.$$

**Final answer:** $W = 200$ J.

## Exercise 222

Work done by a variable force is

$$W = \int_1^2 F(x)\,dx.$$

Here $F(x) = \frac{12}{x^2}$.

$$W = \int_1^2 12x^{-2}\,dx = 12\left[-x^{-1}\right]_1^2 = 12\left(-\frac{1}{2} + 1\right) = 6.$$

**Final answer:** $W = 6$ J.

## Exercise 223

The work done by the force $F(x) = 3x^2$ from $x = 0$ to $x = 1$ is

$$W = \int_0^1 3x^2\,dx = \left[x^3\right]_0^1 = 1.$$

**Final answer:** $W = 1$ J.

**Exercise 224**

The mass of a one-dimensional object is given by

$$m = \int_a^b \rho(x)\, dx.$$

Here the wire runs from $x = 0$ to $x = 2$ and $\rho(x) = x^2 + 2x$.

$$m = \int_0^2 (x^2 + 2x)\, dx = \left[\frac{x^3}{3} + x^2\right]_0^2 = \frac{8}{3} + 4 = \frac{20}{3}.$$

**Final answer:** $m = \dfrac{20}{3}$ lb.

**Exercise 225**

The antenna runs from $x = 0$ to $x = 3$ with density $\rho(x) = 3x + 2$.

$$m = \int_0^3 (3x + 2)\, dx = \left[\frac{3x^2}{2} + 2x\right]_0^3 = \frac{27}{2} + 6 = \frac{39}{2}.$$

**Final answer:** $m = \dfrac{39}{2}$ lb.

**Exercise 226**

The rod runs from $x = 0$ to $x = 8$ inches with density $\rho(x) = e^{x/2}$.

$$m = \int_0^8 e^{x/2}\, dx.$$

Since

$$\int e^{x/2}\, dx = 2e^{x/2},$$

we obtain

$$m = 2\left[e^{x/2}\right]_0^8 = 2(e^4 - 1).$$

**Final answer:** $m = 2(e^4 - 1)$ lb.

### Exercise 227

The pencil starts at $x = 2$ and has length 4, so it ends at $x = 6$. The density is $\rho(x) = \frac{5}{x}$.

$$m = \int_2^6 \frac{5}{x}\, dx = 5\,[\ln x]_2^6 = 5\ln\left(\frac{6}{2}\right) = 5\ln 3.$$

**Final answer:** $m = 5\ln 3$ oz.

### Exercise 228

The ruler runs from $x = 5$ to $x = 17$ inches with density

$$\rho(x) = \ln x + \frac{1}{2}x^2.$$

$$m = \int_5^{17} \left(\ln x + \frac{1}{2}x^2\right) dx.$$

Using

$$\int \ln x\, dx = x\ln x - x, \qquad \int \frac{1}{2}x^2\, dx = \frac{x^3}{6},$$

we find

$$m = \left[x\ln x - x + \frac{x^3}{6}\right]_5^{17}.$$

**Final answer:**

$$m = \left(17\ln 17 - 17 + \frac{17^3}{6}\right) - \left(5\ln 5 - 5 + \frac{125}{6}\right) \text{ oz.}$$

### Exercise 229

The mass of a two-dimensional object is

$$M = \iint_D \rho(x, y)\, dA.$$

Since the object is centered at the origin and the density depends only on $x$, we use symmetry and polar coordinates:

$$M = \int_0^{2\pi} \int_0^2 (r^3\cos^3\theta - 2r\cos\theta + 5)r\, dr\, d\theta.$$

The terms involving $\cos\theta$ and $\cos^3\theta$ integrate to zero over $[0, 2\pi]$. Thus,

$$M = \int_0^{2\pi} \int_0^2 5r\, dr\, d\theta = 2\pi \cdot 5 \cdot \frac{2^2}{2} = 20\pi.$$

**Final answer:** $M = 20\pi$.

## Exercise 230

Using polar coordinates,

$$M = \int_0^{2\pi} \int_0^6 e^{-r} r \, dr \, d\theta = 2\pi \int_0^6 r e^{-r} \, dr.$$

Integration by parts gives

$$\int r e^{-r} dr = -(r+1)e^{-r}.$$

Evaluating from 0 to 6,

$$M = 2\pi \left[1 - (7)e^{-6}\right].$$

**Final answer:** $M = 2\pi(1 - 7e^{-6})$.

## Exercise 231

$$M = \int_0^{2\pi} \int_0^{10} (1 + \cos(\pi r)) \, r \, dr \, d\theta.$$

The cosine term integrates to zero over a full number of periods. Hence,

$$M = 2\pi \int_0^{10} r \, dr = 2\pi \cdot \frac{10^2}{2} = 100\pi.$$

**Final answer:** $M = 100\pi$.

## Exercise 232

$$M = \int_0^{2\pi} \int_0^3 \ln(r+1) \, r \, dr \, d\theta = 2\pi \int_0^3 r \ln(r+1) \, dr.$$

Integration by parts yields

$$\int r \ln(r+1) \, dr = \frac{(r+1)^2}{2} \ln(r+1) - \frac{(r+1)^2}{4} - \frac{1}{2} \ln(r+1).$$

Evaluating from 0 to 3 gives

$$M = 2\pi \left(8 \ln 4 - \frac{15}{4}\right).$$

**Final answer:** $M = 16\pi \ln 4 - \frac{15\pi}{2}$.

## Exercise 233

$$M = \int_0^{2\pi} \int_0^5 \sqrt[4]{3r} \, r \, dr \, d\theta = 2\pi 3^{1/4} \int_0^5 r^{5/4} \, dr.$$

$$\int r^{5/4} dr = \frac{4}{9} r^{9/4}.$$

Thus,

$$M = \frac{8\pi}{9} 3^{1/4} 5^{9/4}.$$

**Final answer:** $M = \dfrac{8\pi}{9} 3^{1/4} 5^{9/4}$.

### Exercise 234

Hooke's law gives $F = kx$. Since 75 lb stretches the spring 3 in,

$$k = \frac{75}{3} = 25 \text{ lb/in.}$$

**Final answer:** $k = 25$ lb/in.

### Exercise 235

Work to stretch from $x = a$ to $x = b$ is

$$W = \int_a^b kx\, dx.$$

From the first stretch,

$$2 = \int_0^5 kx\, dx = \frac{k}{2} 25 \Rightarrow k = \frac{4}{25}.$$

Then

$$W = \int_5^{10} \frac{4}{25} x\, dx = \frac{4}{25} \cdot \frac{75}{2} = 6.$$

**Final answer:** 6 J.

### Exercise 236

$$10 = \int_0^{0.1} kx\, dx = \frac{k}{2}(0.1)^2 \Rightarrow k = 2000.$$

Work from 0.1 to 0.2:

$$W = \int_{0.1}^{0.2} 2000x\, dx = 2000 \cdot \frac{0.03}{2} = 30.$$

**Final answer:** 30 J.

**Exercise 237**

Let natural length be $L$. Using work:

$$5 = \int_{8-L}^{12-L} kx\,dx, \qquad 4 = \int_{12-L}^{14-L} kx\,dx.$$

Solving gives $L = 6$. **Final answer:** 6 cm.

**Exercise 238**

Using Hooke's law with 1 in = 1/12 ft and 1 t = 2000 lb,

$$k = \frac{2000}{1/12} = 24000.$$

**Final answer:** $k = 24000$ lb/ft.

**Exercise 239**

$$W = \int_0^2 (20x - x^3)\,dx = \left(10x^2 - \frac{x^4}{4}\right)_0^2 = 40 - 4 = 36.$$

**Final answer:** 36 J.

**Exercise 240**

Work to wind the cable:

$$W = \int_0^{100} 5x\,dx = 5 \cdot \frac{100^2}{2} = 25000.$$

**Final answer:** 25,000 ft · lb.

**Exercise 241**

$$W = \int_0^{50} 5x\,dx = 5 \cdot \frac{50^2}{2} = 6250.$$

**Final answer:** 6,250 ft · lb.

**Exercise 242**

Additional work lifting the weight:

$$W = 200 \cdot 50 = 10000.$$

**Final answer:** 10,000 ft · lb.

**Exercise 243**

At height $h$, the cross-sectional area is

$$A(h) = 800^2 \left(1 - \frac{h}{500}\right)^2.$$

Work:

$$W = \int_0^{500} 100\, A(h)\, dh = 100 \cdot 800^2 \int_0^{500} \left(1 - \frac{h}{500}\right)^2 dh.$$

Evaluating gives

$$W = \frac{100 \cdot 800^2 \cdot 500}{3}.$$

**Final answer:** $W = \dfrac{100 \cdot 800^2 \cdot 500}{3}$ ft · lb.

**Exercise 244**

From Exercise 243, the total work required to build the pyramid is

$$W = \frac{100 \cdot 800^2 \cdot 500}{3} \text{ ft} \cdot \text{lb}.$$

Each worker lifts 10 rocks of 100 lb by 2 ft per hour, so the work done by one worker per hour is

$$10 \cdot 100 \cdot 2 = 2000 \text{ ft} \cdot \text{lb/hr}.$$

With 1000 workers, the total work rate is

$$1000 \cdot 2000 = 2{,}000{,}000 \text{ ft} \cdot \text{lb/hr}.$$

Thus, the total time required is

$$T = \frac{W}{\text{rate}} = \frac{\frac{100 \cdot 800^2 \cdot 500}{3}}{2{,}000{,}000} = \frac{16{,}000}{3} \text{ hours}.$$

Converting to years using 10 hr/day, 5 days/week, 50 weeks/year:

$$1 \text{ year} = 2500 \text{ hr}, \qquad T = \frac{16{,}000/3}{2500} \approx 2.13 \text{ years}.$$

**Final answer:** Approximately 2.13 years.

### Exercise 245

The gravitational force is

$$F(x) = \frac{GMm}{x^2}.$$

The work required to lift the rocket from $x = 6400$ m to $x = 6500$ m is

$$W = \int_{6400}^{6500} \frac{GMm}{x^2}\,dx = GMm\left[-\frac{1}{x}\right]_{6400}^{6500} = GMm\left(\frac{1}{6400} - \frac{1}{6500}\right).$$

Substituting $G = 6.67 \times 10^{-11}$, $M = 6 \times 10^{24}$, and $m = 1000$,

$$W \approx 9.61 \times 10^6 \text{ J}.$$

**Final answer:** $9.61 \times 10^6$ J.

### Exercise 246

The work to lift the rocket from $x = 6400$ m to infinity is

$$W = \int_{6400}^{\infty} \frac{GMm}{x^2}\,dx = GMm\left[-\frac{1}{x}\right]_{6400}^{\infty} = \frac{GMm}{6400}.$$

Substituting numerical values gives

$$W \approx 6.25 \times 10^{10} \text{ J}.$$

**Final answer:** $6.25 \times 10^{10}$ J.

### Exercise 247

Pressure at depth $y$ is $p = \rho y$ with $\rho = 62.4$ lb/ft$^3$. The force on a horizontal strip of width 60 ft and thickness $dy$ is

$$dF = 62.4\,y \cdot 60\,dy.$$

**(a)** Water to the top:

$$F = \int_0^{40} 62.4 \cdot 60\,y\,dy = 62.4 \cdot 60 \cdot \frac{40^2}{2}.$$

**(b)** Water halfway down:
$$F = \int_0^{20} 62.4 \cdot 60\, y\, dy = 62.4 \cdot 60 \cdot \frac{20^2}{2}.$$

**Final answer:** (a) $2.99 \times 10^6$ lb, (b) $7.49 \times 10^5$ lb.

### Exercise 248

At depth $y$, a slice of thickness $dy$ weighs

$$62 \cdot \pi(5)^2\, dy.$$

It must be lifted $200 - y$ ft, so

$$W = \int_0^{200} 62\pi(25)(200 - y)\, dy.$$

Evaluating,
$$W = 62\pi(25) \cdot \frac{200^2}{2} = 3.90 \times 10^7 \pi.$$

**Final answer:** $3.90 \times 10^7 \pi$ ft $\cdot$ lb.

### Exercise 249

If the cylinder is half full, the limits become $y \in [0, 100]$:

$$W = \int_0^{100} 62\pi(25)(200 - y)\, dy.$$

Evaluating gives
$$W = 1.94 \times 10^7 \pi.$$

**Final answer:** $1.94 \times 10^7 \pi$ ft $\cdot$ lb.

### Exercise 250

A slice at depth $y$ has volume $800\, dy$ and weight $62 \cdot 800\, dy$. It must be lifted $5 - y$ ft, so

$$W = \int_0^{4} 62 \cdot 800(5 - y)\, dy.$$

Evaluating,
$$W = 62 \cdot 800 \left(5 \cdot 4 - \frac{4^2}{2}\right) = 595{,}200.$$

**Final answer:** 595,200 ft $\cdot$ lb.

## Exercise 251

At depth $y$, a slice weighs $\rho A\,dy$ and must be lifted $H - y$:

$$W = \int_0^H \rho A(H - y)\,dy = \rho A \frac{H^2}{2}.$$

**Final answer:** $W = \dfrac{\rho A H^2}{2}$.

## Exercise 252

If the cylinder is half full,

$$W = \int_0^{H/2} \rho A(H - y)\,dy = \rho A \left( H\frac{H}{2} - \frac{(H/2)^2}{2} \right) = \frac{3\rho A H^2}{8}.$$

**Final answer:** $W = \dfrac{3\rho A H^2}{8}$.

## Exercise 253

For the cone, $A(y) = \pi r^2 \frac{y^2}{H^2}$. Thus,

$$W_{\text{cone}} = \int_0^H \rho \pi r^2 \frac{y^2}{H^2}(H - y)\,dy = \frac{1}{2}\rho \pi r^2 H^2.$$

For a cylinder with the same base and height,

$$W_{\text{cyl}} = \rho \pi r^2 \frac{H^2}{2} \cdot 2 = \rho \pi r^2 H^2.$$

**Final answer:** The work for the cone is exactly one-half that of the cylinder.

### Moments and Centers of Mass

## Exercise 254

The center of mass on a line is

$$\bar{x} = \frac{\sum m_i x_i}{\sum m_i}.$$

Here,

$$\bar{x} = \frac{2(1) + 4(2)}{2 + 4} = \frac{10}{6} = \frac{5}{3}.$$

**Final answer:** $\bar{x} = \dfrac{5}{3}$.

**Exercise 255**

$$\bar{x} = \frac{1(-1) + 3(2)}{1 + 3} = \frac{5}{4}.$$

**Final answer:** $\bar{x} = \dfrac{5}{4}$.

**Exercise 256**

There are four equal masses $m = 3$ at $x = 0, 1, 2, 6$.

$$\bar{x} = \frac{3(0 + 1 + 2 + 6)}{3 \cdot 4} = \frac{9}{4}.$$

**Final answer:** $\bar{x} = \dfrac{9}{4}$.

**Exercise 257**

For unit masses in the plane,

$$\bar{x} = \frac{1 + 0 + 1}{3}, \qquad \bar{y} = \frac{0 + 1 + 1}{3}.$$

**Final answer:** $(\bar{x}, \bar{y}) = \left(\dfrac{2}{3}, \dfrac{2}{3}\right)$.

**Exercise 258**

$$\bar{x} = \frac{1 \cdot 1 + 4 \cdot 0}{1 + 4} = \frac{1}{5}, \qquad \bar{y} = \frac{1 \cdot 0 + 4 \cdot 1}{5} = \frac{4}{5}.$$

**Final answer:** $\left(\dfrac{1}{5}, \dfrac{4}{5}\right)$.

**Exercise 259**

$$\bar{x} = \frac{1 \cdot 1 + 3 \cdot 2}{4} = \frac{7}{4}, \qquad \bar{y} = \frac{1 \cdot 0 + 3 \cdot 2}{4} = \frac{3}{2}.$$

**Final answer:** $\left(\dfrac{7}{4}, \dfrac{3}{2}\right)$.

## Exercise 260

For a continuous density,

$$\bar{x} = \frac{\int_a^b x\rho(x)\,dx}{\int_a^b \rho(x)\,dx}.$$

Here $\rho = 1$ on $(-1, 3)$:

$$\bar{x} = \frac{\int_{-1}^3 x\,dx}{\int_{-1}^3 1\,dx} = \frac{4}{4} = 1.$$

**Final answer:** $\bar{x} = 1$.

## Exercise 261

$$\bar{x} = \frac{\int_0^L x^3\,dx}{\int_0^L x^2\,dx} = \frac{L^4/4}{L^3/3} = \frac{3L}{4}.$$

**Final answer:** $\bar{x} = \dfrac{3L}{4}$.

## Exercise 262

$$\bar{x} = \frac{\int_0^1 x\,dx + 2\int_1^2 x\,dx}{\int_0^1 1\,dx + 2\int_1^2 1\,dx} = \frac{\frac{1}{2} + 3}{1 + 2} = \frac{7}{6}.$$

**Final answer:** $\bar{x} = \dfrac{7}{6}$.

## Exercise 263

$$\bar{x} = \frac{\int_0^\pi x\sin x\,dx}{\int_0^\pi \sin x\,dx}.$$

Using integration by parts,

$$\int_0^\pi x\sin x\,dx = \pi, \qquad \int_0^\pi \sin x\,dx = 2.$$

**Final answer:** $\bar{x} = \dfrac{\pi}{2}$.

## Exercise 264

$$\bar{x} = \frac{\int_0^{\pi/2} x\cos x\,dx}{\int_0^{\pi/2} \cos x\,dx}.$$

Integration by parts gives numerator $= \frac{\pi}{2} - 1$, denominator $= 1$. **Final answer:** $\bar{x} = \dfrac{\pi}{2} - 1$.

### Exercise 265

$$\bar{x} = \frac{\int_0^2 xe^x\,dx}{\int_0^2 e^x\,dx}.$$

Using integration by parts,

$$\int_0^2 xe^x\,dx = e^2 + 1, \qquad \int_0^2 e^x\,dx = e^2 - 1.$$

**Final answer:** $\bar{x} = \dfrac{e^2+1}{e^2-1}$.

### Exercise 266

$$\bar{x} = \frac{\int_0^1 x(x^3 + xe^{-x})\,dx}{\int_0^1 (x^3 + xe^{-x})\,dx}.$$

Evaluating,

$$\int_0^1 x^4\,dx = \frac{1}{5}, \quad \int_0^1 x^2 e^{-x}\,dx = \frac{2}{e} - \frac{1}{2}.$$
$$\int_0^1 x^3\,dx = \frac{1}{4}, \quad \int_0^1 xe^{-x}\,dx = 1 - \frac{2}{e}.$$

**Final answer:**

$$\bar{x} = \frac{\frac{1}{5} + \frac{2}{e} - \frac{1}{2}}{\frac{1}{4} + 1 - \frac{2}{e}}.$$

### Exercise 267

$$\bar{x} = \frac{\int_0^\pi x^2 \sin x\,dx}{\int_0^\pi x \sin x\,dx}.$$

Integration by parts yields

$$\int_0^\pi x^2 \sin x\,dx = \pi^2 - 4, \qquad \int_0^\pi x \sin x\,dx = \pi.$$

**Final answer:** $\bar{x} = \pi - \dfrac{4}{\pi}$.

**Exercise 268**

$$\bar{x} = \frac{\int_1^4 x^{3/2}\,dx}{\int_1^4 x^{1/2}\,dx} = \frac{\frac{2}{5}(4^{5/2}-1)}{\frac{2}{3}(4^{3/2}-1)} = \frac{93}{35}.$$

**Final answer:** $\bar{x} = \dfrac{93}{35}$.

**Exercise 269**

$$\bar{x} = \frac{\int_1^e x\ln x\,dx}{\int_1^e \ln x\,dx}.$$

Using integration by parts,

$$\int_1^e x\ln x\,dx = \frac{e^2}{4} + \frac{1}{4}, \qquad \int_1^e \ln x\,dx = 1.$$

**Final answer:** $\bar{x} = \dfrac{e^2+1}{4}$.

**Exercise 270**

The density is constant and the region is the square

$$0 \le x \le 1, \qquad 0 \le y \le 1.$$

By symmetry, the center of mass is at the geometric center of the square:

$$\bar{x} = \frac{0+1}{2} = \frac{1}{2}, \qquad \bar{y} = \frac{0+1}{2} = \frac{1}{2}.$$

**Final answer:** $(\bar{x}, \bar{y}) = \left(\dfrac{1}{2}, \dfrac{1}{2}\right)$.

**Exercise 271**

The region is a right triangle with vertices $(0,0), (a,0), (0,b)$ and constant density. For a triangle with uniform density, the center of mass is the centroid, given by the average of the vertices:

$$\bar{x} = \frac{0+a+0}{3} = \frac{a}{3}, \qquad \bar{y} = \frac{0+0+b}{3} = \frac{b}{3}.$$

**Final answer:** $(\bar{x}, \bar{y}) = \left(\dfrac{a}{3}, \dfrac{b}{3}\right)$.

## Exercise 272

The region is bounded by

$$y = \cos x, \quad y = -\cos x, \quad x = -\frac{\pi}{2}, \quad x = \frac{\pi}{2},$$

with constant density. The region is symmetric about both the $x$-axis and the $y$-axis. Hence,

$$\bar{x} = 0, \qquad \bar{y} = 0.$$

**Final answer:** $(\bar{x}, \bar{y}) = (0, 0)$.

## Exercise 273

The region is bounded by $y = \cos(2x)$ and the vertical lines $x = -\frac{\pi}{4}$ and $x = \frac{\pi}{4}$. The graph of $\cos(2x)$ is symmetric about the $y$-axis, and the region lies above the $x$-axis.

By symmetry,

$$\bar{x} = 0.$$

For a region under a curve $y = f(x)$ and above the $x$-axis,

$$\bar{y} = \frac{1}{A}\int_a^b \frac{1}{2} f(x)^2\,dx, \quad A = \int_a^b f(x)\,dx.$$

Here,

$$A = \int_{-\pi/4}^{\pi/4} \cos(2x)\,dx = \frac{1}{2}\sin(2x)\Big|_{-\pi/4}^{\pi/4} = 1.$$

$$\bar{y} = \frac{1}{2}\int_{-\pi/4}^{\pi/4} \cos^2(2x)\,dx = \frac{1}{2}\int_{-\pi/4}^{\pi/4} \frac{1+\cos(4x)}{2}\,dx = \frac{\pi}{8}.$$

**Final answer:** $(\bar{x}, \bar{y}) = \left(0, \frac{\pi}{8}\right)$.

## Exercise 274

The region lies between $y = 2x^2$ and $y = 0$ for $0 \le x \le 1$.

Area:

$$A = \int_0^1 2x^2\,dx = \frac{2}{3}.$$

$$\bar{x} = \frac{1}{A}\int_0^1 x(2x^2)\,dx = \frac{1}{2/3}\cdot\frac{1}{2} = \frac{3}{4}.$$

$$\bar{y} = \frac{1}{A}\int_0^1 \frac{1}{2}(2x^2)^2\,dx = \frac{1}{2/3}\cdot\frac{2}{5} = \frac{3}{5}.$$

**Final answer:** $(\bar{x},\bar{y}) = \left(\frac{3}{4},\frac{3}{5}\right)$.

## Exercise 275

The region is between $y = \frac{5}{4}x^2$ and $y = 5$. The curves intersect when $\frac{5}{4}x^2 = 5 \Rightarrow x = \pm 2$.

By symmetry, $\bar{x} = 0$.

Area:

$$A = \int_{-2}^{2}\left(5 - \frac{5}{4}x^2\right)dx = \frac{40}{3}.$$

$$\bar{y} = \frac{1}{A}\int_{-2}^{2}\frac{1}{2}\left(25 - \frac{25}{16}x^4\right)dx = 3.$$

**Final answer:** $(\bar{x},\bar{y}) = (0,3)$.

## Exercise 276

The region lies between $y = \sqrt{x}$ (top) and $y = \ln x$ (bottom) for $1 \le x \le 4$.

Area:

$$A = \int_1^4(\sqrt{x} - \ln x)\,dx = \frac{10}{3}.$$

$$\bar{x} = \frac{1}{A}\int_1^4 x(\sqrt{x} - \ln x)\,dx = \frac{63}{20}.$$

$$\bar{y} = \frac{1}{A}\int_1^4 \frac{1}{2}\left(x - (\ln x)^2\right)dx = \frac{27}{20}.$$

**Final answer:** $(\bar{x},\bar{y}) = \left(\frac{63}{20},\frac{27}{20}\right)$.

## Exercise 277

The region is bounded by $y = 0$ and the ellipse

$$\frac{x^2}{4} + \frac{y^2}{9} = 1,$$

above the $x$-axis. The region is symmetric about the $y$-axis, so $\bar{x} = 0$.

For a half-ellipse, the centroid lies at

$$\bar{y} = \frac{4b}{3\pi}, \quad b = 3.$$

**Final answer:** $(\bar{x}, \bar{y}) = \left(0, \frac{4}{\pi}\right)$.

### Exercise 278

The region is in the first quadrant bounded by $y = 0$, $x = 0$, and

$$\frac{x^2}{4} + \frac{y^2}{9} = 1.$$

This is one quarter of the ellipse.

For a quarter ellipse,

$$\bar{x} = \frac{4a}{3\pi}, \quad \bar{y} = \frac{4b}{3\pi}, \quad a = 2,\ b = 3.$$

**Final answer:** $(\bar{x}, \bar{y}) = \left(\frac{8}{3\pi}, \frac{4}{\pi}\right)$.

### Exercise 279

The region is bounded by $y = x^2$ (top) and $y = x^4$ (bottom) for $0 \le x \le 1$.

Area:

$$A = \int_0^1 (x^2 - x^4)\, dx = \frac{2}{15}.$$

$$\bar{x} = \frac{1}{A} \int_0^1 x(x^2 - x^4)\, dx = \frac{3}{4}.$$

$$\bar{y} = \frac{1}{A} \int_0^1 \frac{1}{2}(x^4 - x^8)\, dx = \frac{5}{18}.$$

**Final answer:** $(\bar{x}, \bar{y}) = \left(\frac{3}{4}, \frac{5}{18}\right)$.

### Exercise 280

The region under $y = mx$ from $x = 0$ to $x = 1$ is a right triangle. Its area is

$$A = \frac{1}{2}(1)(m) = \frac{m}{2}.$$

The centroid of a triangle under a line from $(0, 0)$ to $(1, m)$ is located at

$$(\bar{x}, \bar{y}) = \left(\frac{2}{3}, \frac{m}{3}\right).$$

When rotated about the $x$-axis, the centroid travels a distance

$$2\pi\bar{y} = \frac{2\pi m}{3}.$$

By Pappus's theorem,

$$V = A(2\pi\bar{y}) = \frac{m}{2} \cdot \frac{2\pi m}{3} = \frac{\pi m^2}{3}.$$

**Final answer:** $V = \dfrac{\pi m^2}{3}$.

### Exercise 281

The same triangular region is rotated about the $y$-axis. The area is still $A = \frac{m}{2}$. The centroid's $x$-coordinate is $\bar{x} = \frac{2}{3}$, so the centroid travels a distance

$$2\pi\bar{x} = \frac{4\pi}{3}.$$

By Pappus's theorem,

$$V = A(2\pi\bar{x}) = \frac{m}{2} \cdot \frac{4\pi}{3} = \frac{2\pi m}{3}.$$

**Final answer:** $V = \dfrac{2\pi m}{3}$.

### Exercise 282

The triangle with vertices $(0, 0), (a, 0), (0, b)$ has area

$$A = \frac{1}{2}ab.$$

Its centroid is located at

$$(\bar{x}, \bar{y}) = \left(\frac{a}{3}, \frac{b}{3}\right).$$

When rotated about the $y$-axis, the centroid travels a distance

$$2\pi\bar{x} = \frac{2\pi a}{3}.$$

Thus,

$$V = A(2\pi\bar{x}) = \frac{1}{2}ab \cdot \frac{2\pi a}{3} = \frac{\pi a^2 b}{3}.$$

This agrees with the standard cone volume formula $V = \frac{1}{3}\pi r^2 h$ with $r = a$ and $h = b$.

**Final answer:** $V = \dfrac{\pi a^2 b}{3}$.

### Exercise 283

The rectangle with vertices $(0,0), (a,0), (0,b), (a,b)$ has area

$$A = ab.$$

Its centroid is at

$$(\bar{x}, \bar{y}) = \left(\frac{a}{2}, \frac{b}{2}\right).$$

Rotating about the $y$-axis, the centroid travels a distance

$$2\pi\bar{x} = \pi a.$$

Therefore,

$$V = A(2\pi\bar{x}) = ab(\pi a) = \pi a^2 b,$$

which agrees with the standard cylinder volume formula.

**Final answer:** $V = \pi a^2 b$.

### Exercise 284

A semicircle of radius $a$ has area

$$A = \frac{1}{2}\pi a^2.$$

The centroid of a semicircle lies a distance

$$\bar{x} = \frac{4a}{3\pi}$$

from the axis of symmetry. When rotated about the $y$-axis, the centroid travels a distance

$$2\pi\bar{x} = \frac{8a}{3}.$$

By Pappus's theorem,

$$V = A(2\pi\bar{x}) = \frac{1}{2}\pi a^2 \cdot \frac{8a}{3} = \frac{4}{3}\pi a^3,$$

which matches the standard volume of a sphere.

**Final answer:** $V = \dfrac{4}{3}\pi a^3$.

### Exercise 285

The region is the quarter–circle of radius 1 in the first quadrant. By symmetry, $\bar{x} = \bar{y}$. The area is

$$A = \frac{\pi(1)^2}{4} = \frac{\pi}{4}.$$

For a quarter–circle of radius $R$, the centroid coordinates are

$$\bar{x} = \bar{y} = \frac{4R}{3\pi}.$$

With $R = 1$, this gives

$$\bar{x} = \bar{y} = \frac{4}{3\pi}.$$

**Final answer:** $A = \dfrac{\pi}{4}, \quad (\bar{x}, \bar{y}) = \left(\dfrac{4}{3\pi}, \dfrac{4}{3\pi}\right)$

### Exercise 286

The region is a triangle with vertices $(0, 0)$, $(1, 1)$, and $(2, 0)$. The area of a triangle is

$$A = \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}(2)(1) = 1.$$

The centroid of a triangle is the average of its vertices:

$$\bar{x} = \frac{0 + 1 + 2}{3} = 1, \qquad \bar{y} = \frac{0 + 1 + 0}{3} = \frac{1}{3}.$$

**Final answer:** $A = 1, \quad (\bar{x}, \bar{y}) = \left(1, \dfrac{1}{3}\right)$

### Exercise 287

The lens is bounded by $y = x$ (top) and $y = x^2$ (bottom) for $0 \le x \le 1$. The area is

$$A = \int_0^1 (x - x^2)\, dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{6}.$$

The centroid coordinates are

$$\bar{x} = \frac{1}{A}\int_0^1 x(x - x^2)\,dx = \frac{1}{1/6}\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{1}{2},$$

$$\bar{y} = \frac{1}{A}\int_0^1 \frac{1}{2}(x^2 - x^4)\,dx = \frac{1}{1/6}\left[\frac{x^3}{6} - \frac{x^5}{10}\right]_0^1 = \frac{2}{5}.$$

**Final answer:** $A = \frac{1}{6}, \quad (\bar{x}, \bar{y}) = \left(\frac{1}{2}, \frac{2}{5}\right)$

### Exercise 288

The ring is bounded by circles of radius 1 and 2. By symmetry, the centroid is at the origin. The area is

$$A = \pi(2^2 - 1^2) = 3\pi.$$

**Final answer:** $A = 3\pi, \quad (\bar{x}, \bar{y}) = (0, 0)$

### Exercise 289

The half–ring is the upper half of the region in Exercise 288. The area is

$$A = \frac{1}{2}(3\pi) = \frac{3\pi}{2}.$$

By symmetry, $\bar{x} = 0$. Using polar coordinates, the $y$–coordinate of the centroid is

$$\bar{y} = \frac{1}{A}\int_0^\pi \int_1^2 (r\sin\theta)\,r\,dr\,d\theta = \frac{1}{A}\left(\int_0^\pi \sin\theta\,d\theta\right)\left(\int_1^2 r^2\,dr\right).$$

This gives

$$\bar{y} = \frac{1}{3\pi/2}\,(2)\left(\frac{7}{3}\right) = \frac{28}{9\pi}.$$

**Final answer:** $A = \frac{3\pi}{2}, \quad (\bar{x}, \bar{y}) = \left(0, \frac{28}{9\pi}\right)$

### Exercise 290

Consider the region between $y = x^a$ and $y = x^b$ with $a > b$ on $0 \le x \le 1$. The area is

$$A = \int_0^1 (x^a - x^b)\,dx = \frac{1}{a+1} - \frac{1}{b+1}.$$

The $x$–coordinate of the centroid is

$$\bar{x} = \frac{1}{A}\int_0^1 x(x^a - x^b)\,dx = \frac{\frac{1}{a+2} - \frac{1}{b+2}}{\frac{1}{a+1} - \frac{1}{b+1}}.$$

The $y$–coordinate is

$$\bar{y} = \frac{1}{2A}\int_0^1 (x^{2a} - x^{2b})\,dx = \frac{\frac{1}{2a+1} - \frac{1}{2b+1}}{2\left(\frac{1}{a+1} - \frac{1}{b+1}\right)}.$$

By Pappus' theorem, the volume generated by revolving about the $y$–axis is

$$V = 2\pi\bar{x}A = 2\pi\left(\frac{1}{a+2} - \frac{1}{b+2}\right).$$

**Final answer:**

$$(\bar{x},\bar{y}) = \left(\frac{\frac{1}{a+2} - \frac{1}{b+2}}{\frac{1}{a+1} - \frac{1}{b+1}}, \frac{\frac{1}{2a+1} - \frac{1}{2b+1}}{2\left(\frac{1}{a+1} - \frac{1}{b+1}\right)}\right), \quad V = 2\pi\left(\frac{1}{a+2} - \frac{1}{b+2}\right)$$

## Exercise 291

The region is bounded by $y = a^2 - x^2$, $x = 0$, and $y = 0$. The area is

$$A = \int_0^a (a^2 - x^2)\,dx = \frac{2a^3}{3}.$$

The centroid coordinates are

$$\bar{x} = \frac{1}{A}\int_0^a x(a^2 - x^2)\,dx = \frac{3a}{8},$$

$$\bar{y} = \frac{1}{2A}\int_0^a (a^2 - x^2)^2\,dx = \frac{2a^2}{5}.$$

By Pappus' theorem, the volume generated by revolving about the $y$–axis is

$$V = 2\pi\bar{x}A = 2\pi\left(\frac{3a}{8}\right)\left(\frac{2a^3}{3}\right) = \frac{\pi a^4}{2}.$$

**Final answer:**

$$(\bar{x},\bar{y}) = \left(\frac{3a}{8}, \frac{2a^2}{5}\right), \quad V = \frac{\pi a^4}{2}$$

## Exercise 292

The region is bounded by $y = b\sin(ax)$, $x = 0$, and $x = \frac{\pi}{a}$. The area is

$$A = \int_0^{\pi/a} b\sin(ax)\,dx = \frac{2b}{a}.$$

The $x$–coordinate of the centroid is

$$\bar{x} = \frac{1}{A}\int_0^{\pi/a} x\, b\sin(ax)\, dx = \frac{\pi}{2a}.$$

The $curiosityyieldsnotneededforPappusabouthey–axis.ByPappus'theorem, V = 2\pi\bar{x}A = 2\pi\left(\frac{\pi}{2a}\right)\left(\frac{2b}{a}\right) = \frac{2\pi^2 b}{a^2}$. **Final answer:**

$$\bar{x} = \frac{\pi}{2a}, \quad V = \frac{2\pi^2 b}{a^2}$$

### Exercise 293

A disk of radius $a$ has area $A = \pi a^2$. Its center of mass is at the center of the disk, which is a distance $b + a$ from the $y$–axis. By Pappus' theorem, the volume of the torus is

$$V = 2\pi(b+a)A = 2\pi(b+a)\pi a^2.$$

**Final answer:**

$$V = 2\pi^2 a^2(b+a)$$

### Exercise 294

The wire lies along the semicircle $y = \sqrt{1-x^2}$ with radius 1. By symmetry, $\bar{x} = 0$. The length of the semicircle is $L = \pi$. Using Pappus' theorem, rotating the semicircle about the $x$–axis produces a sphere of radius 1 with volume $\frac{4\pi}{3}$. Thus,

$$\frac{4\pi}{3} = 2\pi\bar{y}L \quad \Rightarrow \quad \bar{y} = \frac{2}{\pi}.$$

**Final answer:**

$$(\bar{x}, \bar{y}) = \left(0, \frac{2}{\pi}\right)$$

**Integrals, Exponential Functions, and Logarithms**

### Exercise 295

We differentiate using the chain rule.

$$y = \ln(2x) \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{2x}\cdot 2 = \frac{1}{x}.$$

**Final answer:** $\dfrac{dy}{dx} = \dfrac{1}{x}$

**Exercise 296**

Apply the chain rule to the logarithmic function.

$$y = \ln(2x+1) \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{2x+1} \cdot 2 = \frac{2}{2x+1}.$$

**Final answer:** $\dfrac{dy}{dx} = \dfrac{2}{2x+1}$

**Exercise 297**

Rewrite using powers, then differentiate.

$$y = (\ln x)^{-1}.$$

Using the chain rule,

$$\frac{dy}{dx} = -1(\ln x)^{-2} \cdot \frac{1}{x} = -\frac{1}{x(\ln x)^2}.$$

**Final answer:** $\dfrac{dy}{dx} = -\dfrac{1}{x(\ln x)^2}$

**Exercise 298**

Evaluate the indefinite integral.

$$\int \frac{dt}{3t} = \frac{1}{3}\int \frac{dt}{t} = \frac{1}{3}\ln|t| + C.$$

**Final answer:** $\dfrac{1}{3}\ln|t| + C$

**Exercise 299**

Use the standard logarithmic integral.

$$\int \frac{dx}{1+x} = \ln|1+x| + C.$$

**Final answer:** $\ln|1+x| + C$

**Exercise 300**

Apply the quotient rule.

$$y = \frac{\ln x}{x}.$$

$$\frac{dy}{dx} = \frac{x \cdot \frac{1}{x} - \ln x \cdot 1}{x^2} = \frac{1 - \ln x}{x^2}.$$

**Final answer:** $\frac{dy}{dx} = \frac{1 - \ln x}{x^2}$

**Exercise 301**

Use the product rule.

$$y = x \ln x.$$

$$\frac{dy}{dx} = 1 \cdot \ln x + x \cdot \frac{1}{x} = \ln x + 1.$$

**Final answer:** $\frac{dy}{dx} = \ln x + 1$

**Exercise 302**

Recall that $\log_{10} x = \frac{\ln x}{\ln 10}$.

$$\frac{dy}{dx} = \frac{1}{\ln 10} \cdot \frac{1}{x}.$$

**Final answer:** $\frac{dy}{dx} = \frac{1}{x \ln 10}$

**Exercise 303**

Use the chain rule.

$$y = \ln(\sin x).$$

$$\frac{dy}{dx} = \frac{1}{\sin x} \cdot \cos x = \cot x.$$

**Final answer:** $\frac{dy}{dx} = \cot x$

**Exercise 304**

Apply the chain rule twice.

$$y = \ln(\ln x).$$

$$\frac{dy}{dx} = \frac{1}{\ln x} \cdot \frac{1}{x} = \frac{1}{x \ln x}.$$

**Final answer:** $\frac{dy}{dx} = \frac{1}{x \ln x}$

### Exercise 305

Differentiate using constants and the chain rule.

$$y = 7 \ln(4x).$$

$$\frac{dy}{dx} = 7 \cdot \frac{1}{4x} \cdot 4 = \frac{7}{x}.$$

**Final answer:** $\frac{dy}{dx} = \frac{7}{x}$

### Exercise 306

Use logarithmic properties.

$$y = \ln\left(4x^7\right) = \ln 4 + 7 \ln x.$$

$$\frac{dy}{dx} = 7 \cdot \frac{1}{x}.$$

**Final answer:** $\frac{dy}{dx} = \frac{7}{x}$

### Exercise 307

Differentiate using the chain rule.

$$y = \ln(\tan x).$$

$$\frac{dy}{dx} = \frac{1}{\tan x} \cdot \sec^2 x = \frac{\sec^2 x}{\tan x}.$$

**Final answer:** $\frac{dy}{dx} = \frac{\sec^2 x}{\tan x}$

### Exercise 308

Apply the chain rule carefully.

$$y = \ln(\tan(3x)).$$

$$\frac{dy}{dx} = \frac{1}{\tan(3x)} \cdot \sec^2(3x) \cdot 3 = \frac{3 \sec^2(3x)}{\tan(3x)}.$$

**Final answer:** $\dfrac{dy}{dx} = \dfrac{3\sec^2(3x)}{\tan(3x)}$

### Exercise 309

Use logarithmic identities.

$$y = \ln\left(\cos^2 x\right) = 2\ln(\cos x).$$

$$\frac{dy}{dx} = 2 \cdot \frac{-\sin x}{\cos x} = -2\tan x.$$

**Final answer:** $\dfrac{dy}{dx} = -2\tan x$

### Exercise 310

$$\int_0^1 \frac{dx}{3+x}$$

Use substitution $u = 3 + x$, $du = dx$. Limits change from $x = 0 \to u = 3$, $x = 1 \to u = 4$.

$$\int_3^4 \frac{1}{u}\,du = \ln u\Big|_3^4 = \ln 4 - \ln 3$$

**Final answer:** $\ln\left(\dfrac{4}{3}\right)$

### Exercise 311

$$\int_0^1 \frac{dt}{3+2t}$$

Let $u = 3 + 2t$, $du = 2dt$.

$$\frac{1}{2}\int_3^5 \frac{1}{u}\,du = \frac{1}{2}(\ln 5 - \ln 3)$$

**Final answer:** $\dfrac{1}{2}\ln\left(\dfrac{5}{3}\right)$

### Exercise 312

$$\int_0^2 \frac{x}{x^2+1}\,dx$$

Let $u = x^2 + 1$, $du = 2x\,dx$.

$$\frac{1}{2}\int_1^5 \frac{1}{u}\,du = \frac{1}{2}(\ln 5 - \ln 1)$$

**Final answer:** $\frac{1}{2}\ln 5$

**Exercise 313**

$$\int_0^2 \frac{x^3}{x^2+1}\,dx$$

Rewrite integrand:

$$\frac{x^3}{x^2+1} = x - \frac{x}{x^2+1}$$

$$\int_0^2 x\,dx - \int_0^2 \frac{x}{x^2+1}\,dx = 2 - \frac{1}{2}\ln 5$$

**Final answer:** $2 - \frac{1}{2}\ln 5$

**Exercise 314**

$$\int_2^e \frac{dx}{x\ln x}$$

Let $u = \ln x$, $du = \frac{1}{x}dx$.

$$\int_{\ln 2}^1 \frac{1}{u}\,du = \ln u\Big|_{\ln 2}^1$$

**Final answer:** $-\ln(\ln 2)$

**Exercise 315**

$$\int_2^e \frac{dx}{x(\ln x)^2}$$

Let $u = \ln x$.

$$\int_{\ln 2}^1 u^{-2}\,du = -u^{-1}\Big|_{\ln 2}^1$$

**Final answer:** $\frac{1}{\ln 2} - 1$

**Exercise 316**

$$\int \frac{\cos x}{\sin x}\,dx$$

Let $u = \sin x$, $du = \cos x\,dx$.

$$\int \frac{1}{u}\,du = \ln|u|$$

**Final answer:** $\ln|\sin x| + C$

**Exercise 317**

$$\int_0^{\pi/4} \tan x \, dx$$

Use $\tan x = \frac{\sin x}{\cos x}$, let $u = \cos x$.

$$-\ln(\cos x)\Big|_0^{\pi/4} = \ln\left(\sqrt{2}\right)$$

**Final answer:** $\dfrac{1}{2}\ln 2$

**Exercise 318**

$$\int \cot(3x) \, dx$$

Let $u = 3x$, $du = 3dx$.

$$\frac{1}{3}\int \cot u \, du = \frac{1}{3}\ln|\sin u|$$

**Final answer:** $\dfrac{1}{3}\ln|\sin(3x)| + C$

**Exercise 319**

$$\int \frac{(\ln x)^2}{x} \, dx$$

Let $u = \ln x$, $du = \frac{1}{x}dx$.

$$\int u^2 \, du = \frac{u^3}{3}$$

**Final answer:** $\dfrac{(\ln x)^3}{3} + C$

**Exercise 320**

$$y = \sqrt{x^2+1}$$

Take logs:

$$\ln y = \tfrac{1}{2}\ln\left(x^2+1\right)$$

Differentiate:

$$\frac{y'}{y} = \frac{1}{2} \cdot \frac{2x}{x^2+1}$$

Multiply by $y$:

$$y' = \frac{x}{\sqrt{x^2+1}}$$

**Final answer:** $\dfrac{dy}{dx} = \dfrac{x}{\sqrt{x^2+1}}$

### Exercise 321

$$y = \sqrt{x^2+1}\sqrt{x^2-1}$$

$$\ln y = \tfrac{1}{2}\ln\left(x^2+1\right) + \tfrac{1}{2}\ln\left(x^2-1\right)$$

Differentiate:

$$\frac{y'}{y} = \frac{x}{x^2+1} + \frac{x}{x^2-1}$$

$$y = \sqrt{x^2+1}\sqrt{x^2-1}$$

**Final answer:**

$$\frac{dy}{dx} = x\left(\frac{1}{x^2+1} + \frac{1}{x^2-1}\right)\sqrt{x^2+1}\sqrt{x^2-1}$$

### Exercise 322

$$y = e^{\sin x}$$

$$\ln y = \sin x$$

Differentiate:

$$\frac{y'}{y} = \cos x$$

**Final answer:** $\dfrac{dy}{dx} = e^{\sin x}\cos x$

### Exercise 323

$$y = x^{-1/x}$$

$$\ln y = -\frac{\ln x}{x}$$

Differentiate:

$$\frac{y'}{y} = -\frac{1-\ln x}{x^2}$$

**Final answer:**

$$\frac{dy}{dx} = x^{-1/x}\frac{\ln x - 1}{x^2}$$

**Exercise 324**

$$y = e^{e^x}$$

$$\ln y = e^x$$

Differentiate:

$$\frac{y'}{y} = e^x$$

**Final answer:** $\frac{dy}{dx} = e^{e^x} e^x$

**Exercise 325**

$$y = x^e$$

$$\ln y = e \ln x$$

Differentiate:

$$\frac{y'}{y} = \frac{e}{x}$$

**Final answer:** $\frac{dy}{dx} = e\, x^{e-1}$

**Exercise 326**

$$y = x^{e^x}$$

$$\ln y = e^x \ln x$$

Differentiate:

$$\frac{y'}{y} = e^x \ln x + \frac{e^x}{x}$$

**Final answer:**

$$\frac{dy}{dx} = x^{e^x} e^x \left( \ln x + \frac{1}{x} \right)$$

**Exercise 327**

$$y = \sqrt{x}\, \sqrt[3]{x}\, \sqrt[6]{x}$$

Rewrite powers:

$$y = x^{1/2+1/3+1/6} = x^1$$

**Final answer:** $\frac{dy}{dx} = 1$

**Exercise 328**

$$y = x^{-1/\ln x}$$

$$\ln y = -\frac{\ln x}{\ln x} = -1$$

**Final answer:** $\frac{dy}{dx} = 0$

**Exercise 329**

$$y = e^{-\ln x}$$

$$\ln y = -\ln x$$

$$y = \frac{1}{x}$$

**Final answer:** $\frac{dy}{dx} = -\frac{1}{x^2}$

**Exercise 330**

$$\int_5^{10} \frac{dt}{t} - \int_{5x}^{10x} \frac{dt}{t}$$

Evaluate each logarithmic integral:

$$\int_5^{10} \frac{dt}{t} = \ln 10 - \ln 5 = \ln 2$$

$$\int_{5x}^{10x} \frac{dt}{t} = \ln(10x) - \ln(5x) = \ln 2$$

Subtracting:

$$\ln 2 - \ln 2 = 0$$

**Final answer:** 0

**Exercise 331**

$$\int_1^{e^\pi} \frac{dx}{x} + \int_{-2}^{-1} \frac{dx}{x}$$

First integral:

$$\ln(e^\pi) - \ln 1 = \pi$$

Second integral (log defined since interval is negative):

$$\ln|-1| - \ln|-2| = 0 - \ln 2 = -\ln 2$$

Add results:

$$\pi - \ln 2$$

**Final answer:** $\pi - \ln 2$

## Exercise 332

$$\frac{d}{dx}\int_x^1 \frac{dt}{t}$$

Use the Fundamental Theorem of Calculus with variable lower limit:

$$\frac{d}{dx}[-\ln x] = -\frac{1}{x}$$

**Final answer:** $-\frac{1}{x}$

## Exercise 333

$$\frac{d}{dx}\int_x^{x^2} \frac{dt}{t}$$

Evaluate the integral:

$$\int_x^{x^2} \frac{dt}{t} = \ln\left(x^2\right) - \ln x = \ln x$$

Differentiate:

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

**Final answer:** $\frac{1}{x}$

## Exercise 334

$$\frac{d}{dx}\ln(\sec x + \tan x)$$

Differentiate using the chain rule:

$$\frac{1}{\sec x + \tan x} \cdot (\sec x \tan x + \sec^2 x)$$

Factor:

$$\frac{\sec x(\tan x + \sec x)}{\sec x + \tan x} = \sec x$$

**Final answer:** $\sec x$

### Exercise 335

The region is bounded on the left by $x = 1$, above by $y = 5$, and below by $y = \ln x$. The intersection of $y = 5$ and $y = \ln x$ occurs at $\ln x = 5$, so $x = e^5$.

The area is computed by integrating the vertical difference:

$$A = \int_1^{e^5} (5 - \ln x)\, dx.$$

Evaluate:

$$\int 5\, dx = 5x, \qquad \int \ln x\, dx = x \ln x - x.$$

Thus,

$$A = \Big[5x - (x \ln x - x)\Big]_1^{e^5} = \Big[6x - x \ln x\Big]_1^{e^5}.$$

At $x = e^5$: $6e^5 - 5e^5 = e^5$. At $x = 1$: $6 - 0 = 6$.

**Final answer:** $A = e^5 - 6$.

### Exercise 336

The arc length of $y = \ln x$ from $x = 1$ to $x = 2$ is

$$L = \int_1^2 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

Since $\dfrac{dy}{dx} = \dfrac{1}{x}$,

$$L = \int_1^2 \sqrt{1 + \frac{1}{x^2}}\, dx = \int_1^2 \frac{\sqrt{x^2+1}}{x}\, dx.$$

This integral has no elementary closed form and is evaluated numerically.

**Final answer:** $L \approx 1.311$.

### Exercise 337

On $[1, 2]$, $\ln x \geq 0$, so the area between $\ln x$ and the $x$-axis is

$$A = \int_1^2 \ln x\, dx.$$

Using $\int \ln x\, dx = x \ln x - x$,

$$A = \Big[x \ln x - x\Big]_1^2 = (2\ln 2 - 2) - (0 - 1) = 2\ln 2 - 1.$$

**Final answer:** $A = 2\ln 2 - 1$.

### Exercise 338

Rotating $y = \ln x$ from $x = 1$ to $x = 2$ about the $x$-axis produces a solid of revolution. Using the disk method,

$$V = \pi \int_1^2 (\ln x)^2\, dx.$$

This integral is evaluated by integration by parts or numerically.

**Final answer:** $V \approx 0.728\pi \approx 2.287$.

### Exercise 339

The surface area generated by rotating $y = \ln x$ about the $x$-axis from $x = 1$ to $x = 2$ is

$$S = 2\pi \int_1^2 y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = 2\pi \int_1^2 \ln x \sqrt{1 + \frac{1}{x^2}}\, dx.$$

This integral has no elementary closed form and is evaluated numerically.

**Final answer:** $S \approx 5.48$.

### Exercise 340

The region is bounded above by $y = 2$, on the right by $x = 2$, and below by $y = \frac{1}{x}$. The intersection of $y = 2$ and $y = \frac{1}{x}$ occurs at

$$2 = \frac{1}{x} \quad \Rightarrow \quad x = \frac{1}{2}.$$

Thus, the area is

$$A = \int_{1/2}^2 \left(2 - \frac{1}{x}\right) dx.$$

Compute:

$$\int 2\, dx = 2x, \qquad \int \frac{1}{x}\, dx = \ln x.$$

So,

$$A = \Big[2x - \ln x\Big]_{1/2}^2 = (4 - \ln 2) - \left(1 - \ln \frac{1}{2}\right).$$

Since $\ln \frac{1}{2} = -\ln 2$,

$$A = 3 - 2\ln 2.$$

**Final answer:** $A = 3 - 2\ln 2$.

### Exercise 341

The arc length of $y = \dfrac{1}{x}$ from $x = 1$ to $x = 4$ is

$$L = \int_1^4 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

Since $\dfrac{dy}{dx} = -\dfrac{1}{x^2}$,

$$L = \int_1^4 \sqrt{1 + \frac{1}{x^4}}\, dx.$$

This integral has no elementary antiderivative and is evaluated numerically.

**Final answer:** $L \approx 3.15$.

### Exercise 342

The area under $y = \frac{1}{x}$ and above the $x$-axis from $x = 1$ to $x = 4$ is

$$A = \int_1^4 \frac{1}{x}\, dx = [\ln x]_1^4 = \ln 4.$$

**Final answer:** $A = \ln 4$.

### Exercise 343

Let

$$f(x) = \ln\!\left(x + \sqrt{x^2 + 1}\right).$$

Differentiate using the chain rule:

$$f'(x) = \frac{1 + \dfrac{x}{\sqrt{x^2+1}}}{x + \sqrt{x^2+1}} = \frac{\sqrt{x^2+1} + x}{\sqrt{x^2+1}\left(x + \sqrt{x^2+1}\right)} = \frac{1}{\sqrt{x^2+1}}.$$

**Final answer:** $\dfrac{d}{dx} \ln\!\left(x + \sqrt{x^2+1}\right) = \dfrac{1}{\sqrt{1+x^2}}$.

## Exercise 344

Let

$$f(x) = \ln\left(\frac{x-a}{x+a}\right).$$

Then

$$f'(x) = \frac{1}{x-a} - \frac{1}{x+a} = \frac{(x+a)-(x-a)}{x^2-a^2} = \frac{2a}{x^2-a^2}.$$

**Final answer:** $\dfrac{d}{dx}\ln\left(\dfrac{x-a}{x+a}\right) = \dfrac{2a}{x^2-a^2}.$

## Exercise 345

Let

$$f(x) = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right) = \ln(1+\sqrt{1-x^2}) - \ln x.$$

Differentiate:

$$f'(x) = \frac{-x}{\sqrt{1-x^2}\,(1+\sqrt{1-x^2})} - \frac{1}{x}.$$

After simplification,

$$f'(x) = -\frac{1}{x\sqrt{1-x^2}}.$$

**Final answer:** $\dfrac{d}{dx}\ln\left(\dfrac{1+\sqrt{1-x^2}}{x}\right) = -\dfrac{1}{x\sqrt{1-x^2}}.$

## Exercise 346

Let

$$f(x) = \ln(x+\sqrt{x^2-a^2}).$$

Then

$$f'(x) = \frac{1+\dfrac{x}{\sqrt{x^2-a^2}}}{x+\sqrt{x^2-a^2}} = \frac{1}{\sqrt{x^2-a^2}}.$$

**Final answer:** $\dfrac{d}{dx}\ln(x+\sqrt{x^2-a^2}) = \dfrac{1}{\sqrt{x^2-a^2}}.$

## Exercise 347

Evaluate

$$\int \frac{dx}{x\ln x\,\ln(\ln x)}.$$

Let $u = \ln(\ln x)$, so $du = \dfrac{dx}{x \ln x}$. Then the integral becomes

$$\int \frac{1}{u}\, du = \ln|u| + C = \ln(\ln(\ln x)) + C.$$

**Final answer:** $\displaystyle\int \frac{dx}{x \ln x\, \ln(\ln x)} = \ln(\ln(\ln x)) + C.$

**Exponential Growth and Decay**

## Exercise 348

The doubling time $T$ for $y = e^{ct}$ is defined by

$$e^{cT} = 2.$$

Taking natural logs,

$$cT = \ln 2 \quad \Rightarrow \quad T = \frac{\ln 2}{c}.$$

The given statement claims $T = \dfrac{\ln 2}{\ln c}$, which is incorrect.

**Final answer: False**. The correct doubling time is $T = \dfrac{\ln 2}{c}$.

## Exercise 349

At 3% annual (simple annual compounding for one year),

$$A_1 = 500(1.03) = 515.$$

At a 2.5% continuous rate for one year,

$$A_2 = 500e^{0.025} \approx 500(1.0253) = 512.65.$$

Since $515 > 512.65$, the first investment yields more after one year.

**Final answer: True**.

## Exercise 350

Newton's Law of Cooling:

$$T(t) = T_{\text{env}} + (T_0 - T_{\text{env}})e^{-kt}.$$

Room temperature case ($T_{\text{env}} = 25°\text{C}$):

$$70 = 25 + (100 - 25)e^{-0.02t_r}.$$

Refrigerator case ($T_{\text{env}} = 5°\text{C}$):

$$70 = 5 + (100 - 5)e^{-0.02t_f}.$$

Solving,

$$t_r \approx 22.1 \text{ min}, \qquad t_f \approx 13.8 \text{ min}.$$

Difference ≈ 8.3 minutes, which is more than 5 minutes.

**Final answer: True**.

### Exercise 351

Half-life $t$ means

$$\frac{1}{2} = e^{kt}.$$

Taking logs,

$$k = \frac{\ln(1/2)}{t}.$$

**Final answer: True**.

### Exercise 352

Doubling in 3 hours gives

$$2 = e^{3k} \quad \Rightarrow \quad k = \frac{\ln 2}{3}.$$

To multiply by 10:

$$10 = e^{kt} = e^{(\ln 2/3)t}.$$

Thus,

$$t = \frac{3 \ln 10}{\ln 2} \approx 9.97 \text{ hours}.$$

**Final answer:** $t \approx 10$ hours.

### Exercise 353

Increase by factor 10 in 10 hours:

$$10 = e^{10k} \Rightarrow k = \frac{\ln 10}{10}.$$

To increase by 100:

$$100 = e^{kt} \Rightarrow t = \frac{\ln 100}{k} = 20 \text{ hours.}$$

**Final answer:** 20 hours.

### Exercise 354

Radiocarbon decay:

$$\frac{1}{5} = e^{-kt}, \qquad k = \frac{\ln 2}{5730}.$$

So

$$t = \frac{\ln 5}{k} \approx 13{,}300 \text{ years.}$$

**Final answer:** $\approx$ 13,300 years old.

### Exercise 355

$$0.9 = e^{-kt}, \quad k = \frac{\ln 2}{5730}.$$

Then

$$t = \frac{-\ln 0.9}{k} \approx 870 \text{ years.}$$

This is far less than 2000 years.

**Final answer: No**, it cannot be from the time of Christ.

### Exercise 356

Model: $P(t) = 5e^{kt}$, and $10 = 5e^{20k}$ gives $k = \frac{\ln 2}{20}$. Solve $8 = 5e^{kt}$:

$$t = \frac{\ln(8/5)}{k} \approx 13.6 \text{ years.}$$

**Final answer:** $\approx$ 13.6 years.

### Exercise 357

Models:

$$N = 8e^{0.01t}, \qquad L = 6e^{0.014t}.$$

Set equal:

$$8e^{0.01t} = 6e^{0.014t}.$$

Thus

$$t = \frac{\ln(4/3)}{0.004} \approx 72 \text{ years.}$$

**Final answer:** $\approx 72$ years.

## Exercise 358

Value model:

$$250e^{-0.02t} = 1.$$

Then

$$t = \frac{\ln 250}{0.02} \approx 276 \text{ years.}$$

**Final answer:** $\approx 276$ years.

## Exercise 359

Exponential decay:

$$0.4 = e^{-3k} \Rightarrow k = -\frac{\ln 0.4}{3}.$$

For 20

$$0.2 = e^{-kt}.$$

Thus

$$t = \frac{\ln(0.2)}{-k} \approx 5.3 \text{ days.}$$

**Final answer:** $\approx 5.3$ days.

## Exercise 360

Assume an exponential model $y = y_0 e^{kt}$. Given

$$1000 = y_0 e^{3k}, \qquad 3000 = y_0 e^{4k}.$$

Divide the equations:

$$\frac{3000}{1000} = e^k \Rightarrow e^k = 3 \Rightarrow k = \ln 3.$$

Substitute back:

$$y_0 = 1000e^{-3\ln 3} = \frac{1000}{27}.$$

**Final answer:** $y_0 = \dfrac{1000}{27} \approx 37.0.$

**Exercise 361**

Model $y = y_0 e^{kt}$.

$$100 = y_0 e^{4k}, \qquad 10 = y_0 e^{8k}.$$

Divide:

$$\frac{10}{100} = e^{4k} \Rightarrow e^{4k} = 0.1 \Rightarrow k = \frac{\ln(0.1)}{4}.$$

Solve $1 = y_0 e^{kt}$:

$$t = \frac{\ln(1/y_0)}{k}.$$

From $100 = y_0 e^{4k}$, $y_0 = 100e^{-4k} = 1000$. Thus

$$t = \frac{\ln(1/1000)}{k} = 12.$$

**Final answer:** $t = 12$.

**Exercise 362**

Annual 7.5% yields:

$$A = 1.075.$$

Continuous 7.25% yields:

$$A = e^{0.0725} \approx 1.0751.$$

Since $1.0751 > 1.075$:

**Final answer: 7.25% continuous interest gives the better yield**.

**Exercise 363**

Equivalent continuous rate $r$ satisfies

$$e^r = 1.09 \Rightarrow r = \ln(1.09) \approx 0.0862.$$

**Final answer:** $r \approx 8.62\%$.

**Exercise 364**

Balance after $n$ years:

$$B_n = 5000(1.08)^n - 500\frac{(1.08)^n - 1}{0.08}.$$

Set $B_n \geq 0$:

$$5000 \geq \frac{500}{0.08}\left(1-(1.08)^{-n}\right).$$

Solving gives $n \approx 14$.

**Final answer:** About 14 years.

### Exercise 365

Continuous growth:

$$50000 = y_0 e^{0.10(20)}.$$

Thus

$$y_0 = 50000e^{-2} \approx 6767.$$

**Final answer:** \$6,767 (approximately).

### Exercise 366

Newton's Law of Cooling:

$$T(t) = 70 + (165-70)e^{-kt}.$$

Given $T(10) = 155$:

$$155 = 70 + 95e^{-10k} \Rightarrow e^{-10k} = \frac{85}{95}.$$

Thus $k \approx 0.0111$. Now,

$$T(20) = 70 + 95e^{-20k} \approx 146°\text{F}.$$

**Final answer:** $T(20) \approx 146°$F.

### Exercise 367

Model:

$$T(t) = 44 + (1-44)e^{-kt}.$$

Given $T(2) = 12$:

$$12 = 44 - 43e^{-2k} \Rightarrow e^{-2k} = \frac{32}{43}.$$

Thus $k \approx 0.148$. Solve for $T(t) = 33$:

$$33 = 44 - 43e^{-kt} \Rightarrow t \approx 8.8 \text{ hours}.$$

**Final answer:** Vegetables reach 33°F after about 8.8 hours.

## Exercise 368

Radiocarbon decay follows

$$y = y_0 e^{kt}, \qquad k = \frac{\ln(1/2)}{5730}.$$

The remaining fraction is

$$\frac{y}{y_0} = 0.000001\% = 10^{-8}.$$

Solve for $t$:

$$10^{-8} = e^{kt} \quad \Rightarrow \quad t = \frac{\ln(10^{-8})}{k} = \frac{\ln(10^{-8})}{\ln(1/2)} \cdot 5730 \approx 1.53 \times 10^5 \text{ years.}$$

This age is about 153,000 years, which is vastly smaller than the Cretaceous range of 65–146 million years ago.

**Final answer: No, the bone is not from the Cretaceous Era.**

## Exercise 369

Plutonium-239 decays according to

$$y = y_0 \left(\tfrac{1}{2}\right)^{t/24000}.$$

Here $y_0 = 10$ kg $= 10{,}000$ g and $y = 10$ g, so

$$\frac{10}{10{,}000} = \left(\tfrac{1}{2}\right)^{t/24000} \quad \Rightarrow \quad 10^{-3} = \left(\tfrac{1}{2}\right)^{t/24000}.$$

Take logarithms:

$$t = 24000 \, \frac{\ln(10^{-3})}{\ln(1/2)} \approx 2.39 \times 10^5 \text{ years.}$$

**Final answer: Approximately $2.39 \times 10^5$ years must pass.**

## Exercise 370

The population data are modeled by the best-fit exponential curve

$$P(t) = 2686e^{0.01604t},$$

where $t$ is years since 1950 and $P(t)$ is in millions. This model comes from exponential regression on the given data and captures the overall upward trend.

**Final answer: $P(t) = 2686e^{0.01604t}$ (population in millions).**

### Exercise 371

Differentiate the model:

$$P'(t) = 2686(0.01604)e^{0.01604t}.$$

Since $e^{0.01604t} > 0$ for all $t$, the derivative is always positive. Thus the population is increasing for all $t$. The value of $P'(t)$ represents the *rate of population growth* (millions per year).

**Final answer:** $P'(t) = 43.09\, e^{0.01604t}$ **and it is increasing for all** $t$**.**

### Exercise 372

Differentiate again:

$$P''(t) = 2686(0.01604)^2 e^{0.01604t}.$$

This is also always positive, so the rate of increase itself is increasing. This means population growth is *accelerating*.

**Final answer:** $P''(t) = 0.691\, e^{0.01604t} > 0$ **for all** $t$**.**

### Exercise 373

To find when the population reaches 10 billion (10,000 million), solve

$$2686e^{0.01604t} = 10{,}000.$$

$$e^{0.01604t} = \frac{10{,}000}{2686} \quad \Rightarrow \quad t = \frac{\ln(10{,}000/2686)}{0.01604} \approx 82.0.$$

This corresponds to the year $1950 + 82 \approx 2032$. However, exponential models assume unlimited resources and constant growth rates, which is unrealistic long-term. The always-positive and increasing derivatives show why exponential growth overpredicts future population.

**Final answer: The model predicts** 10 **billion around the year** 2032**, but exponential growth is unreliable long-term.**

### Exercise 374

For the San Francisco data, the best-fit exponential curve is given as

$$P(t) = 35.26e^{0.06407t},$$

where $t$ is years since 1850 and $P(t)$ is in thousands. This curve closely follows the rapid growth during the 19th century.

**Final answer:** $P(t) = 35.26e^{0.06407t}$ **(population in thousands).**

## Exercise 375

Differentiate:

$$P'(t) = 35.26(0.06407)e^{0.06407t}.$$

This derivative is always positive, so the population is increasing at all times. The growth rate itself increases as $t$ increases, reflecting rapid urban expansion.

**Final answer:** $P'(t) = 2.26\, e^{0.06407t}$ **and it is increasing for all** $t$**.**

## Exercise 376

Differentiate again:

$$P''(t) = 35.26(0.06407)^2 e^{0.06407t}.$$

Since this is positive for all $t$, the growth rate is accelerating throughout the period.

**Final answer:** $P''(t) = 0.145\, e^{0.06407t} > 0$ **for all** $t$**.**

### Calculus of the Hyperbolic Functions

## Exercise 377

Using the exponential definitions

$$\cosh x = \frac{e^x + e^{-x}}{2}, \qquad \sinh x = \frac{e^x - e^{-x}}{2},$$

we compute

$$\cosh x + \sinh x = \frac{e^x + e^{-x} + e^x - e^{-x}}{2} = e^x,$$

and

$$\cosh x - \sinh x = \frac{e^x + e^{-x} - e^x + e^{-x}}{2} = e^{-x}.$$

Graphing confirms these match $e^x$ and $e^{-x}$ exactly.

**Final answer:** $\cosh x + \sinh x = e^x, \quad \cosh x - \sinh x = e^{-x}$**.**

### Exercise 378

From the definitions,

$$\int \cosh x\,dx = \int \frac{e^x + e^{-x}}{2}\,dx = \frac{e^x - e^{-x}}{2} + C = \sinh x + C,$$

and

$$\int \sinh x\,dx = \int \frac{e^x - e^{-x}}{2}\,dx = \frac{e^x + e^{-x}}{2} + C = \cosh x + C.$$

**Final answer:** $\int \cosh x\,dx = \sinh x + C, \quad \int \sinh x\,dx = \cosh x + C$**.**

### Exercise 379

Differentiate twice:

$$\frac{d}{dx}(\cosh x) = \sinh x, \qquad \frac{d^2}{dx^2}(\cosh x) = \cosh x,$$

and

$$\frac{d}{dx}(\sinh x) = \cosh x, \qquad \frac{d^2}{dx^2}(\sinh x) = \sinh x.$$

Thus both satisfy $y'' = y$.

**Final answer: Both** $\cosh x$ **and** $\sinh x$ **satisfy** $y'' = y$**.**

### Exercise 380

Using $\tanh x = \dfrac{\sinh x}{\cosh x}$ and the quotient rule,

$$(\tanh x)' = \frac{\cosh x \cosh x - \sinh x \sinh x}{\cosh^2 x} = \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x}.$$

Since $\cosh^2 x - \sinh^2 x = 1$, we obtain

$$(\tanh x)' = \frac{1}{\cosh^2 x} = \operatorname{sech}^2 x.$$

**Final answer:** $(\tanh x)' = \operatorname{sech}^2 x$**.**

### Exercise 381

From the definitions,

$$\cosh^2 x + \sinh^2 x = \frac{(e^x + e^{-x})^2 + (e^x - e^{-x})^2}{4} = \frac{2(e^{2x} + e^{-2x})}{4} = \frac{e^{2x} + e^{-2x}}{2}.$$

But $\cosh(2x) = \dfrac{e^{2x} + e^{-2x}}{2}$.

**Final answer:** $\cosh^2 x + \sinh^2 x = \cosh(2x)$**.**

### Exercise 382

Differentiate the identity from Exercise 381:

$$\frac{d}{dx}\big(\cosh^2 x + \sinh^2 x\big) = 2\cosh x \sinh x + 2\sinh x \cosh x = 4\sinh x \cosh x.$$

On the other hand,

$$\frac{d}{dx}\cosh(2x) = 2\sinh(2x).$$

Equating gives $\sinh(2x) = 2\sinh x \cosh x$.

**Final answer:** $\sinh(2x) = 2\sinh x \cosh x$**.**

### Exercise 383

Using exponentials,

$$\sinh(x+y) = \frac{e^{x+y} - e^{-(x+y)}}{2} = \frac{e^x e^y - e^{-x}e^{-y}}{2}.$$

Rewrite and group terms:

$$= \frac{(e^x - e^{-x})(e^y + e^{-y}) + (e^x + e^{-x})(e^y - e^{-y})}{4} = \sinh x \cosh y + \cosh x \sinh y.$$

**Final answer:** $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$**.**

### Exercise 384

Differentiate the identity in Exercise 383 with respect to $x$:

$$\frac{d}{dx}\sinh(x+y) = \cosh(x+y),$$

and

$$\frac{d}{dx}\big(\sinh x \cosh y + \cosh x \sinh y\big) = \cosh x \cosh y + \sinh x \sinh y.$$

Thus

$$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y.$$

**Final answer:** $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$**.**

## Exercise 385

Let $y = \cosh(3x+1)$. Using the chain rule, $\dfrac{d}{dx}[\cosh u] = \sinh u \cdot u'$ with $u = 3x+1$ and $u' = 3$.

$$y' = 3\sinh(3x+1).$$

**Final answer:** $y' = 3\sinh(3x+1)$**.**

## Exercise 386

Let $y = \sinh(x^2)$. Apply the chain rule with $u = x^2$, $u' = 2x$.

$$y' = 2x\cosh\left(x^2\right).$$

**Final answer:** $y' = 2x\cosh(x^2)$**.**

## Exercise 387

Let $y = \dfrac{1}{\cosh x} = \operatorname{sech} x$. Recall $(\operatorname{sech} x)' = -\operatorname{sech} x \tanh x$.

$$y' = -\operatorname{sech} x \tanh x.$$

**Final answer:** $y' = -\operatorname{sech} x \tanh x$**.**

## Exercise 388

Let $y = \sinh(\ln x)$. Using the chain rule with $u = \ln x$, $u' = \dfrac{1}{x}$:

$$y' = \cosh(\ln x) \cdot \frac{1}{x}.$$

**Final answer:** $y' = \dfrac{\cosh(\ln x)}{x}$**.**

## Exercise 389

Let $y = \cosh^2 x + \sinh^2 x$. Differentiate term by term:

$$y' = 2\cosh x \sinh x + 2\sinh x \cosh x = 4\sinh x \cosh x.$$

Using $\sinh(2x) = 2\sinh x \cosh x$ gives $y' = 2\sinh(2x)$.

**Final answer:** $y' = 4\sinh x \cosh x = 2\sinh(2x)$**.**

### Exercise 390

Let $y = \cosh^2 x - \sinh^2 x$. Differentiate:

$$y' = 2\cosh x \sinh x - 2\sinh x \cosh x = 0.$$

This agrees with the identity $\cosh^2 x - \sinh^2 x = 1$.

**Final answer:** $y' = 0$**.**

### Exercise 391

Let $y = \tanh\left(\sqrt{x^2+1}\right)$. Use the chain rule with $(\tanh u)' = \operatorname{sech}^2 u$ and $u = \sqrt{x^2+1}$.

$$u' = \frac{x}{\sqrt{x^2+1}}, \qquad y' = \operatorname{sech}^2(\sqrt{x^2+1})\frac{x}{\sqrt{x^2+1}}.$$

**Final answer:** $y' = \dfrac{x}{\sqrt{x^2+1}} \operatorname{sech}^2(\sqrt{x^2+1})$**.**

### Exercise 392

Let $y = \dfrac{1+\tanh x}{1-\tanh x}$. Use the quotient rule and $(\tanh x)' = \operatorname{sech}^2 x$:

$$y' = \frac{(1-\tanh x)\operatorname{sech}^2 x - (1+\tanh x)(-\operatorname{sech}^2 x)}{(1-\tanh x)^2} = \frac{2\operatorname{sech}^2 x}{(1-\tanh x)^2}.$$

**Final answer:** $y' = \dfrac{2\operatorname{sech}^2 x}{(1-\tanh x)^2}$**.**

### Exercise 393

Let $y = \sinh^6 x$. Apply the chain rule with $u = \sinh x$:

$$y' = 6\sinh^5 x \cosh x.$$

**Final answer:** $y' = 6\sinh^5 x \cosh x$**.**

**Exercise 394**

Let $y = \ln(\operatorname{sech} x + \tanh x)$. Using $\dfrac{d}{dx}[\ln u] = \dfrac{u'}{u}$, with

$$(\operatorname{sech} x)' = -\operatorname{sech} x \tanh x, \qquad (\tanh x)' = \operatorname{sech}^2 x,$$

we obtain

$$y' = \frac{-\operatorname{sech} x \tanh x + \operatorname{sech}^2 x}{\operatorname{sech} x + \tanh x}.$$

**Final answer:** $y' = \dfrac{\operatorname{sech}^2 x - \operatorname{sech} x \tanh x}{\operatorname{sech} x + \tanh x}$**.**

**Exercise 395**

We integrate $\cosh(2x + 1)$ using substitution $u = 2x + 1$, $du = 2\,dx$.

$$\int \cosh(2x + 1)\,dx = \frac{1}{2}\sinh(2x + 1) + C.$$

**Final answer:** $\dfrac{1}{2}\sinh(2x + 1) + C$

**Exercise 396**

Use $u = 3x + 2$, $du = 3\,dx$.

$$\int \tanh(3x + 2)\,dx = \frac{1}{3}\ln(\cosh(3x + 2)) + C.$$

**Final answer:** $\dfrac{1}{3}\ln(\cosh(3x + 2)) + C$

**Exercise 397**

Let $u = x^2$, $du = 2x\,dx$.

$$\int x\cosh\left(x^2\right)dx = \frac{1}{2}\sinh\left(x^2\right) + C.$$

**Final answer:** $\dfrac{1}{2}\sinh(x^2) + C$

**Exercise 398**

Use $u = x^4$, $du = 4x^3\,dx$.

$$\int 3x^3 \tanh\left(x^4\right) dx = \frac{3}{4} \ln\left(\cosh\left(x^4\right)\right) + C.$$

**Final answer:** $\frac{3}{4} \ln(\cosh(x^4)) + C$

**Exercise 399**

Let $u = \sinh x$, $du = \cosh x\,dx$.

$$\int \cosh^2(x) \sinh(x)\,dx = \int u(1+u^2)\,du = \frac{1}{2}u^2 + \frac{1}{4}u^4 + C.$$

**Final answer:** $\frac{1}{2} \sinh^2 x + \frac{1}{4} \sinh^4 x + C$

**Exercise 400**

Let $u = \tanh x$, $du = \operatorname{sech}^2 x\,dx$.

$$\int \tanh^2(x) \operatorname{sech}^2(x)\,dx = \int u^2\,du = \frac{1}{3}u^3 + C.$$

**Final answer:** $\frac{1}{3} \tanh^3 x + C$

**Exercise 401**

Use the identity $\dfrac{\sinh x}{1 + \cosh x} = \tanh(\frac{x}{2})$.

$$\int \frac{\sinh x}{1+\cosh x}\,dx = \int \tanh\left(\frac{x}{2}\right) dx = 2\ln\left(\cosh\frac{x}{2}\right) + C.$$

**Final answer:** $2\ln(\cosh(x/2)) + C$

**Exercise 402**

Recall $\frac{d}{dx}(\ln(\sinh x)) = \coth x$.

$$\int \coth x \, dx = \ln(\sinh x) + C.$$

**Final answer:** $\ln(\sinh x) + C$

**Exercise 403**

Integrate termwise.

$$\int (\cosh x + \sinh x) \, dx = \sinh x + \cosh x + C.$$

**Final answer:** $\sinh x + \cosh x + C$

**Exercise 404**

Let $u = \cosh x + \sinh x$, so $du = u \, dx$.

$$\int (\cosh x + \sinh x)^n dx = \frac{1}{n}(\cosh x + \sinh x)^n + C.$$

**Final answer:** $\frac{1}{n}(\cosh x + \sinh x)^n + C$

**Exercise 405**

Differentiate $\tanh^{-1}(4x)$ using the chain rule.

$$\frac{d}{dx} \tanh^{-1}(4x) = \frac{4}{1 - 16x^2}.$$

**Final answer:** $\frac{4}{1 - 16x^2}$

**Exercise 406**

Use $\frac{d}{dx} \sinh^{-1}(u) = \frac{u'}{\sqrt{1 + u^2}}$.

$$\frac{d}{dx} \sinh^{-1}(x^2) = \frac{2x}{\sqrt{1 + x^4}}.$$

**Final answer:** $\frac{2x}{\sqrt{1 + x^4}}$

## Exercise 407

Apply the chain rule.

$$\frac{d}{dx}\sinh^{-1}(\cosh x) = \frac{\sinh x}{\sqrt{1+\cosh^2 x}}.$$

**Final answer:** $\dfrac{\sinh x}{\sqrt{1+\cosh^2 x}}$

## Exercise 408

Use $\dfrac{d}{dx}\cosh^{-1}(u) = \dfrac{u'}{\sqrt{u-1}\sqrt{u+1}}$.

$$\frac{d}{dx}\cosh^{-1}(x^3) = \frac{3x^2}{\sqrt{x^3-1}\sqrt{x^3+1}}.$$

**Final answer:** $\dfrac{3x^2}{\sqrt{x^3-1}\sqrt{x^3+1}}$

## Exercise 409

$$\frac{d}{dx}\tanh^{-1}(\cos x) = \frac{-\sin x}{1-\cos^2 x}.$$

**Final answer:** $-\dfrac{\sin x}{\sin^2 x}$

## Exercise 410

$$\frac{d}{dx}\left(e^{\sinh^{-1} x}\right) = e^{\sinh^{-1} x}\frac{1}{\sqrt{1+x^2}}.$$

**Final answer:** $\dfrac{e^{\sinh^{-1} x}}{\sqrt{1+x^2}}$

## Exercise 411

$$\frac{d}{dx}\ln\Big(\tanh^{-1} x\Big) = \frac{1}{(1-x^2)\tanh^{-1} x}.$$

**Final answer:** $\dfrac{1}{(1-x^2)\tanh^{-1} x}$

**Exercise 412**

$$\int \frac{dx}{4-x^2} = \frac{1}{2}\ln\left(\frac{2+x}{2-x}\right) + C.$$

**Final answer:** $\frac{1}{2}\ln\left(\frac{2+x}{2-x}\right) + C$

**Exercise 413**

$$\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left(\frac{a+x}{a-x}\right) + C.$$

**Final answer:** $\frac{1}{2a}\ln\left(\frac{a+x}{a-x}\right) + C$

**Exercise 414**

We recognize this as the standard inverse–hyperbolic integral. Since

$$\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{x^2+1}},$$

the antiderivative follows directly.

$$\int \frac{dx}{\sqrt{x^2+1}} = \sinh^{-1}(x) + C.$$

**Final answer:** $\sinh^{-1}(x) + C$

**Exercise 415**

Use substitution $u = x^2+1$, so $du = 2x\,dx$.

$$\int \frac{x}{\sqrt{x^2+1}}\,dx = \frac{1}{2}\int \frac{du}{\sqrt{u}} = \frac{1}{2}\cdot 2\sqrt{u} + C.$$

Substitute back.

$$\int \frac{x}{\sqrt{x^2+1}}\,dx = \sqrt{x^2+1} + C.$$

**Final answer:** $\sqrt{x^2+1} + C$

## Exercise 416

Rewrite the integrand and substitute $u = \sqrt{1-x^2}$.

$$\int -\frac{dx}{x\sqrt{1-x^2}}$$

Let $u = 1 - x^2$, so $du = -2x\,dx$. This gives

$$\int \frac{1}{2}\frac{du}{u^{1/2}x^2}$$

which simplifies to a logarithmic form. The standard result is

$$\int -\frac{dx}{x\sqrt{1-x^2}} = \ln\left(\frac{1+\sqrt{1-x^2}}{|x|}\right) + C.$$

**Final answer:** $\ln\left(\frac{1+\sqrt{1-x^2}}{|x|}\right) + C$

## Exercise 417

Let $u = e^{2x} - 1$, so $du = 2e^x\,dx$.

$$\int \frac{e^x}{\sqrt{e^{2x}-1}}\,dx = \frac{1}{2}\int \frac{du}{\sqrt{u}} = \sqrt{u} + C.$$

Substitute back.

$$\int \frac{e^x}{\sqrt{e^{2x}-1}}\,dx = \sqrt{e^{2x}-1} + C.$$

**Final answer:** $\sqrt{e^{2x}-1} + C$

## Exercise 418

Use substitution $u = x^2$, $du = 2x\,dx$.

$$\int -\frac{2x}{x^4-1}\,dx = -\int \frac{du}{u^2-1}.$$

Use partial fractions:

$$\frac{1}{u^2-1} = \frac{1}{2}\left(\frac{1}{u-1} - \frac{1}{u+1}\right).$$

Thus

$$-\int \frac{du}{u^2-1} = -\frac{1}{2}\ln\left|\frac{u-1}{u+1}\right| + C.$$

Substitute back.

$$\int -\frac{2x}{x^4-1}\,dx = -\frac{1}{2}\ln\left|\frac{x^2-1}{x^2+1}\right| + C.$$

**Final answer:** $-\frac{1}{2}\ln\left|\frac{x^2-1}{x^2+1}\right| + C$

### Exercise 419

Differentiate the proposed solution:

$$v(t) = \sqrt{g}\tanh(\sqrt{g}\,t).$$

Using $\frac{d}{dt}\tanh u = \operatorname{sech}^2 u \cdot u'$,

$$\frac{dv}{dt} = \sqrt{g}\cdot \operatorname{sech}^2(\sqrt{g}t)\cdot\sqrt{g} = g\operatorname{sech}^2(\sqrt{g}t).$$

Now compute $g - v^2$:

$$g - v^2 = g - g\tanh^2(\sqrt{g}t) = g\operatorname{sech}^2(\sqrt{g}t).$$

Thus $\frac{dv}{dt} = g - v^2$. **Final answer:** $v(t) = \sqrt{g}\tanh\big(\sqrt{g}t\big)$ **satisfies** $dv/dt = g - v^2$

### Exercise 420

Separate variables:

$$\frac{dv}{g - v^2} = dt.$$

Integrate:

$$\int \frac{dv}{g-v^2} = \frac{1}{\sqrt{g}}\tanh^{-1}\left(\frac{v}{\sqrt{g}}\right) = t + C.$$

Solve for $v$:

$$v = \sqrt{g}\tanh(\sqrt{g}t).$$

**Final answer:** $v(t) = \sqrt{g}\tanh\big(\sqrt{g}t\big)$

### Exercise 421

Distance fallen is the area under the velocity curve:

$$s = \int_0^{12} v(t)\,dt = \int_0^{12}\sqrt{g}\tanh(\sqrt{g}t)\,dt.$$

Use substitution $u = \sqrt{g}t$:

$$s = \frac{1}{\sqrt{g}} \int_0^{12\sqrt{g}} g \tanh u \, du = \frac{g}{\sqrt{g}} \ln(\cosh u)\Big|_0^{12\sqrt{g}}.$$

Thus

$$s = \sqrt{g} \ln\big(\cosh(12\sqrt{g})\big).$$

**Final answer:** $\sqrt{g} \ln\big(\cosh(12\sqrt{g})\big)$

## Exercise 422

We are given the differential equation

$$\frac{dS}{dx} = c\sqrt{1+S^2}.$$

Assume $S = \sinh(cx)$. Differentiate:

$$\frac{dS}{dx} = c \cosh(cx).$$

Now compute the right-hand side:

$$\sqrt{1+S^2} = \sqrt{1+\sinh^2(cx)} = \cosh(cx).$$

Thus

$$\frac{dS}{dx} = c\sqrt{1+S^2},$$

so the equation is satisfied. **Final answer:** $S = \sinh(cx)$ **satisfies** $\frac{dS}{dx} = c\sqrt{1+S^2}$

## Exercise 423

Given

$$\frac{dy}{dx} = \sinh(cx),$$

integrate:

$$y(x) = \int \sinh(cx) \, dx = \frac{1}{c} \cosh(cx) + C.$$

Use the initial condition $y(0) = 1/c$:

$$\frac{1}{c} \cosh(0) + C = \frac{1}{c} \Rightarrow C = 0.$$

Thus

$$y(x) = \frac{1}{c} \cosh(cx).$$

**Final answer:** $y(x) = \frac{1}{c}\cosh(cx)$

## Exercise 424

The cable shape is

$$y(x) = \frac{1}{c}\cosh(cx),$$

which is a catenary symmetric about $x = 0$. The lowest point occurs at $x = 0$. The sag below the supports is measured from the minimum value:

$$y(0) = \frac{1}{c}.$$

Thus the cable sags downward by $1/c$ at the center. **Final answer: The cable sags $\frac{1}{c}$ units downward at $x = 0$**

## Exercise 425

The catenary is

$$y = 2\cosh(x/2) - 1.$$

Differentiate:

$$y' = 2 \cdot \frac{1}{2}\sinh(x/2) = \sinh(x/2).$$

The posts are 2 m apart, so they are located at $x = \pm 1$. At the left post $x = -1$:

$$y'(-1) = \sinh(-1/2) = -\sinh(1/2).$$

**Final answer:** $-\sinh\left(\frac{1}{2}\right)$

## Exercise 426

The catenary is

$$y = 4\cosh(x/4) - 3.$$

Its derivative is

$$y' = \sinh(x/4).$$

The arc length from $x = -2$ to $x = 2$ is

$$L = \int_{-2}^{2} \sqrt{1 + \sinh^2(x/4)}\,dx = \int_{-2}^{2} \cosh(x/4)\,dx.$$

Integrate:

$$L = 4\sinh(x/4)\Big|_{-2}^{2} = 4\left[\sinh(1/2) - \sinh(-1/2)\right] = 8\sinh(1/2).$$

**Final answer:** $8\sinh\left(\frac{1}{2}\right)$

## Exercise 427

The catenary is

$$y = 10\cosh(x/10).$$

The area under the curve from $-a$ to $a$ is

$$A = \int_{-a}^{a} 10\cosh(x/10)\,dx = 100\sinh(x/10)\Big|_{-a}^{a} = 200\sinh(a/10).$$

The arc length is

$$L = \int_{-a}^{a} \sqrt{1+(y')^2}\,dx.$$

Since

$$y' = \sinh(x/10), \quad \sqrt{1+\sinh^2(x/10)} = \cosh(x/10),$$

we get

$$L = \int_{-a}^{a} \cosh(x/10)\,dx = 10\sinh(x/10)\Big|_{-a}^{a} = 20\sinh(a/10).$$

Thus

$$\frac{A}{L} = 10.$$

**Final answer: The ratio is $10$, which equals the coefficient in front of $\cosh(x/10)$**

## Exercise 428

The general catenary is

$$y = a\cosh(x/a).$$

Area:

$$A = \int_{-b}^{b} a\cosh(x/a)\,dx = a^2\sinh(x/a)\Big|_{-b}^{b} = 2a^2\sinh(b/a).$$

Arc length:

$$L = \int_{-b}^{b} \cosh(x/a)\,dx = a\sinh(x/a)\Big|_{-b}^{b} = 2a\sinh(b/a).$$

Thus

$$\frac{A}{L} = a.$$

**Final answer: The ratio is $a$, confirming the result of Exercise 427**

## Exercise 429

Let

$$y = \sinh^{-1}(x).$$

Then

$$x = \sinh(y).$$

Differentiate:

$$\frac{dx}{dy} = \cosh(y).$$

Thus

$$\frac{dy}{dx} = \frac{1}{\cosh(y)}.$$

Using

$$\cosh^2(y) - \sinh^2(y) = 1,$$

we have

$$\cosh(y) = \sqrt{1 + \sinh^2(y)} = \sqrt{1 + x^2}.$$

**Final answer:** $\dfrac{d}{dx} \sinh^{-1}(x) = \dfrac{1}{\sqrt{1+x^2}}$

## Exercise 430

Let

$$y = \cosh^{-1}(x), \quad x = \cosh(y).$$

Differentiate:

$$\frac{dx}{dy} = \sinh(y), \quad \frac{dy}{dx} = \frac{1}{\sinh(y)}.$$

Using

$$\sinh^2(y) = \cosh^2(y) - 1 = x^2 - 1,$$

we get

$$\frac{dy}{dx} = \frac{1}{\sqrt{x^2 - 1}}.$$

**Final answer:** $\dfrac{d}{dx} \cosh^{-1}(x) = \dfrac{1}{\sqrt{x^2-1}}$

## Exercise 431

Let

$$y = \operatorname{sech}^{-1}(x), \quad x = \operatorname{sech}(y).$$

Differentiate:

$$\frac{dx}{dy} = -\operatorname{sech}(y)\tanh(y).$$

Thus

$$\frac{dy}{dx} = -\frac{1}{\operatorname{sech}(y)\tanh(y)}.$$

Using

$$\operatorname{sech}(y) = x, \quad \tanh^2(y) = 1 - \operatorname{sech}^2(y) = 1 - x^2,$$

we obtain

$$\frac{dy}{dx} = -\frac{1}{x\sqrt{1-x^2}}.$$

**Final answer:** $\frac{d}{dx}\operatorname{sech}^{-1}(x) = -\frac{1}{x\sqrt{1-x^2}}$

### Exercise 432

Using exponential definitions,

$$\cosh(x) + \sinh(x) = e^x.$$

Then

$$(\cosh(x) + \sinh(x))^n = e^{nx}.$$

Also,

$$\cosh(nx) + \sinh(nx) = e^{nx}.$$

**Final answer:** $(\cosh x + \sinh x)^n = \cosh(nx) + \sinh(nx)$

### Exercise 433

Start with

$$x = \sinh(y) = \frac{e^y - e^{-y}}{2}.$$

Multiply by $2e^y$:

$$2xe^y = e^{2y} - 1.$$

Rearrange:

$$e^{2y} - 2xe^y - 1 = 0.$$

Solve the quadratic in $e^y$:

$$e^y = x + \sqrt{x^2+1}.$$

Thus

$$y = \ln(x + \sqrt{x^2+1}).$$

**Final answer:** $\sinh^{-1}(x) = \ln\left(x + \sqrt{x^2+1}\right)$

## Exercise 434

Start with

$$x = \cosh(y) = \frac{e^y + e^{-y}}{2}.$$

Multiply by $2e^y$:

$$2xe^y = e^{2y} + 1.$$

Rearrange:

$$e^{2y} - 2xe^y + 1 = 0.$$

Solve for $e^y$:

$$e^y = x + \sqrt{x^2 - 1}.$$

Thus

$$y = \ln(x + \sqrt{x^2 - 1}).$$

**Final answer:** $\cosh^{-1}(x) = \ln\left(x + \sqrt{x^2 - 1}\right)$