MATA30 — Tutorial Notes

Week 2: Inverse Trigonometric Functions

Amir Koutahi

Problem 1. Evaluate each expression exactly. Answers must be real numbers, and branch choices must be justified.

1. $\sin\bigl(2\arccos(\frac{3}{5})\bigr)$
2. $\cos\bigl(2\arctan(-\frac{7}{24})\bigr)$
3. $\arctan\bigl(\sin(\arccos(\frac{4}{5}))\bigr)$
4. $\arctan\bigl(\tan(\frac{23\pi}{6})\bigr)$
5. $\arcsin\bigl(\sin(\frac{17\pi}{4})\bigr)$

Solution

Let $\theta = \arccos(\frac{3}{5})$, so $\theta \in [0, \pi]$, $\cos\theta = \frac{3}{5}$, and $\sin\theta = \frac{4}{5}$.

1. $$\sin(2\theta) = 2\sin\theta\cos\theta = 2 \cdot \frac{4}{5} \cdot \frac{3}{5} = \boxed{\frac{24}{25}}.$$

2. Let $\phi = \arctan(-\frac{7}{24})$ with $\phi \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Then
$$\cos(2\phi) = \frac{1 - \tan^2\phi}{1 + \tan^2\phi} = \frac{1 - \frac{49}{576}}{1 + \frac{49}{576}} = \boxed{\frac{527}{625}}.$$

3. $$\sin(\arccos(\tfrac{4}{5})) = \sqrt{1 - \left(\tfrac{4}{5}\right)^2} = \frac{3}{5},$$
so
$$\boxed{\arctan(\tfrac{3}{5})}.$$

4. $$\tan\left(\frac{23\pi}{6}\right) = \tan\left(\frac{11\pi}{6}\right) = -\frac{1}{\sqrt{3}}.$$
Since arctan returns values in $(-\frac{\pi}{2}, \frac{\pi}{2})$,
$$\boxed{-\frac{\pi}{6}}.$$

5. $$\sin\left(\frac{17\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}.$$
Since arcsin returns $[-\frac{\pi}{2}, \frac{\pi}{2}]$,
$$\boxed{\frac{\pi}{4}}.$$

Remark

Inverse trigonometric functions do *not* undo trigonometric functions unless the input lies in the principal branch.

Problem 2. Define

$$h(x) = \arccos\Big(\sin(\arctan x)\Big).$$

1. Rewrite $h(x)$ without trigonometric functions.
2. Determine the range of h.

Solution

Let $\theta = \arctan x$ with $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Then

$$\sin\theta = \frac{x}{\sqrt{1+x^2}}.$$

Hence

$$\boxed{h(x) = \arccos\left(\frac{x}{\sqrt{1+x^2}}\right)}.$$

Since $\frac{x}{\sqrt{1+x^2}} \in (-1, 1)$ for all x,

$$\boxed{\mathrm{Range}(h) = (0, \pi)}.$$

Remark

The range of an inverse trig function is determined entirely by its principal branch, not by the input expression.

Problem 3. Let

$$f(x) = \arccos(2x-1), \qquad g(x) = \arcsin(3x-2).$$

Find the domains of $F = g \circ f$ and $G = f \circ g$.

Solution

Base domains:

$$\mathrm{Dom}(f) = [0, 1], \qquad \mathrm{Dom}(g) = \left[\tfrac{1}{3}, 1\right].$$

For F: require $x \in [0, 1]$ and $f(x) \in [\frac{1}{3}, 1]$, giving

$$\boxed{\mathrm{Dom}(F) = \left[\frac{1+\cos(1)}{2}, \frac{1+\cos(\frac{1}{3})}{2}\right]}.$$

For G: require $x \in [\frac{1}{3}, 1]$ and $3x - 2 \in [0, 1]$, yielding

$$\boxed{\mathrm{Dom}(G) = \left[\tfrac{2}{3}, 1\right]}.$$

Problem 4. Analyze

$$\lim_{x\to 0} \frac{\sqrt{x+a} - \sqrt{a}}{x^b}$$

for real parameters a and b.

Solution

Rationalizing,

$$\frac{\sqrt{x+a}-\sqrt{a}}{x^b} = \frac{x^{1-b}}{\sqrt{x+a}+\sqrt{a}}.$$

- If $a > 0$:

$$\boxed{\begin{cases} 0, & b < 1, \\ \dfrac{1}{2\sqrt{a}}, & b = 1, \\ \pm\infty, & b > 1. \end{cases}}$$

- If $a = 0$: the limit is one-sided. It equals 0 if $b < \frac{1}{2}$, equals 1 if $b = \frac{1}{2}$, and diverges if $b > \frac{1}{2}$.
- If $a < 0$: no real limit exists near $x = 0$.

Remark

Always check whether a limit is two-sided. Square roots frequently restrict the domain.

Problem 5. Evaluate

$$\lim_{x\to 0} \frac{\arcsin(2x) - 2\arctan(x)}{x^3}.$$

Solution

Using Taylor expansions:

$$\arcsin t = t + \frac{t^3}{6} + O(t^5), \qquad \arctan t = t - \frac{t^3}{3} + O(t^5),$$

with $t = 2x$,

$$\arcsin(2x) - 2\arctan x = 2x^3 + O(x^5).$$

Thus,

$$\boxed{2}.$$