MATA30 — Tutorial Notes

Week 1: Absolute Values & Rational Inequalities

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Problem 1. Solve for all real x:

$$|x^2 - 3x + 2| \le x.$$

Solution

Factor the expression inside the absolute value:

$$x^2 - 3x + 2 = (x-1)(x-2).$$

> **Key Idea**
>
> When solving inequalities involving absolute values, always split into cases based on the sign of the expression inside $|\cdot|$.

Case 1: $x^2 - 3x + 2 \ge 0$

Then $|x^2 - 3x + 2| = x^2 - 3x + 2$, and

$$x^2 - 3x + 2 \le x \quad \Longleftrightarrow \quad x^2 - 4x + 2 \le 0.$$

The discriminant is $\Delta = 16 - 8 = 8$, giving roots

$$x = 2 \pm \sqrt{2}.$$

Hence,

$$2 - \sqrt{2} \le x \le 2 + \sqrt{2}.$$

Since $x^2 - 3x + 2 \ge 0 \iff x \le 1$ or $x \ge 2$, the intersection yields

$$x \in [\,2,\ 2 + \sqrt{2}\,].$$

Case 2: $x^2 - 3x + 2 < 0$

Then $|x^2 - 3x + 2| = -x^2 + 3x - 2$, and

$$-x^2 + 3x - 2 \le x \quad \Longleftrightarrow \quad x^2 - 2x + 2 \ge 0.$$

Since $(x-1)^2 + 1 > 0$ for all x, this inequality always holds. Combining with $1 < x < 2$, all such x are solutions.

$$\boxed{x \in (1,2)\ \cup\ [2,\ 2+\sqrt{2}]}$$

> **Remark**
>
> Always check the domain restriction coming from each case. Solving the algebraic inequality alone is not sufficient.

Problem 2. Solve for all real x:

$$|x - 3| + |x + 1| = 6.$$

Solution

We split into cases based on the critical points $x = -1$ and $x = 3$.

Case 1: $x \geq 3$

$$(x-3) + (x+1) = 2x - 2 = 6 \quad \Longrightarrow \quad x = 4.$$

Case 2: $-1 \leq x < 3$

$$(3-x) + (x+1) = 4 \neq 6,$$

so there is no solution.

Case 3: $x < -1$

$$(3-x) + (-x-1) = 2 - 2x = 6 \quad \Longrightarrow \quad x = -2.$$

$$\boxed{x = -2 \text{ or } x = 4}$$

Remark

Between consecutive critical points of absolute values, the expression becomes linear. This guarantees at most one solution per interval.

Problem 3. Find polynomials P, Q with minimal total degree such that

$$\frac{P(x)}{Q(x)} \geq 0$$

has solution set

$$[-3, -1) \cup (0, 2] \cup [4, \infty).$$

Solution

To match the required sign pattern:

- Closed endpoints $-3, 2, 4$ require zeros of P.
- Open endpoints $-1, 0$ require zeros of Q.

Choose

$$P(x) = (x+3)(x-2)(x-4), \qquad Q(x) = x(x+1).$$

Testing signs across critical points confirms that

$$\frac{P(x)}{Q(x)} \geq 0 \quad \text{exactly on} \quad [-3, -1) \cup (0, 2] \cup [4, \infty).$$

$$\boxed{P(x) = (x+3)(x-2)(x-4), \quad Q(x) = x(x+1)}$$

Remark

Minimality follows since three included endpoints force $\deg P \geq 3$, and two excluded points force $\deg Q \geq 2$.