MATA30 — Tutorial Notes

Week 6

Topics: Implicit Differentiation, Inverses, Logarithmic Differentiation, Related Rates, Inverse Trig

Amir Koutahi

Key Idea

For implicit differentiation: differentiate *every* term with respect to x, then collect all y' terms on one side and factor.

Practice Problems (with Solutions)

Problem 1 (Chain rule with a composition $f(g(x))$)

Suppose f and g are differentiable and satisfy

$$f(g(x)) = e^x \qquad \text{and} \qquad f'(x) = \frac{1}{1+x^2}.$$

Find $g'(x)$ *in terms of* x *only.*

Solution. Differentiate $f(g(x)) = e^x$:

$$f'(g(x))\, g'(x) = e^x.$$

Using the given formula for f',

$$f'(g(x)) = \frac{1}{1+[g(x)]^2}.$$

So

$$\frac{g'(x)}{1+[g(x)]^2} = e^x \quad \Rightarrow \quad g'(x) = e^x\big(1+[g(x)]^2\big).$$

Now use the fact that $f'(x) = \frac{1}{1+x^2}$ matches the derivative of $\arctan x$. So we may take $f(x) = \arctan x + C$ for some constant C. Then

$$f(g(x)) = \arctan(g(x)) + C = e^x \quad \Rightarrow \quad \arctan(g(x)) = e^x - C \quad \Rightarrow \quad g(x) = \tan(e^x - C).$$

Therefore

$$g'(x) = \sec^2(e^x - C)\cdot e^x = e^x\big(1+\tan^2(e^x - C)\big) = e^x\big(1+[g(x)]^2\big).$$

But $f(g(x)) = e^x$ also gives $e^x - C = \arctan(g(x))$, so

$$g(x) = \tan(e^x - C).$$

If we take the natural choice $C = 0$ (i.e. $f = \arctan$), then

$$\boxed{g(x) = \tan(e^x)} \qquad \Rightarrow \qquad \boxed{g'(x) = e^x \sec^2(e^x)}.$$

Remark

The condition "in terms of x only" forces you to identify f (up to a constant) from $f'(x)$.

Problem 2 (Implicit differentiation)

Find $\dfrac{dy}{dx}$ if

$$x^3y^2 = \sin(y) + x^2.$$

Solution. Differentiate both sides:

$$\frac{d}{dx}(x^3y^2) = \frac{d}{dx}(\sin y) + \frac{d}{dx}(x^2).$$

Left side (product rule, treating $y = y(x)$):

$$3x^2y^2 + x^3 \cdot 2y\, y'.$$

Right side:

$$\cos(y)\, y' + 2x.$$

Collect y' terms:

$$3x^2y^2 + 2x^3y\, y' = \cos(y)\, y' + 2x$$
$$(2x^3y - \cos y)\, y' = 2x - 3x^2y^2.$$

Thus

$$\boxed{y' = \frac{2x - 3x^2y^2}{2x^3y - \cos y}.}$$

Problem 3 (Horizontal tangents)

Find points on the curve

$$y^2 + xy = 3x + 5$$

where the tangent line is horizontal.

Solution. Differentiate implicitly:

$$2y\, y' + (x\, y' + y) = 3.$$

Group:

$$(2y + x)\, y' = 3 - y.$$

Horizontal tangent means $y' = 0$, so $3 - y = 0 \Rightarrow y = 3$. Substitute into the original equation:

$$9 + 3x = 3x + 5 \Rightarrow 9 = 5,$$

which is impossible. Therefore there are *no* points on the curve with a horizontal tangent.

$$\boxed{\text{No horizontal tangents.}}$$

Remark

If $y' = 0$ forces a y-value that does not satisfy the original equation, then the curve has no horizontal tangents.

Problem 4 (Tangent line to an implicit log equation)

Find the equation of the tangent line to

$$\ln(xy) = 1 + x$$

at $\left(e, e^{-1}\right)$.

Solution. Differentiate:

$$\frac{d}{dx}\ln(xy) = \frac{1}{xy}\cdot\frac{d}{dx}(xy) = \frac{1}{xy}(y + xy').$$

Right side derivative is 1. So

$$\frac{y + xy'}{xy} = 1 \quad \Rightarrow \quad y + xy' = xy \quad \Rightarrow \quad xy' = y(x-1) \quad \Rightarrow \quad y' = \frac{y(x-1)}{x}.$$

At $(x, y) = (e, e^{-1})$,

$$m = y' = \frac{e^{-1}(e-1)}{e} = \frac{e-1}{e^2}.$$

So the tangent line is

$$\boxed{y - e^{-1} = \frac{e-1}{e^2}(x - e)}.$$

Problem 5 (Derivative of an inverse at a point)

If $f(x) = x^2 + \ln x$ (domain $x > 0$), find $(f^{-1})'(2)$.

Solution. Use the inverse derivative formula:

$$(f^{-1})'(2) = \frac{1}{f'(a)} \quad \text{where } a = f^{-1}(2) \text{ satisfies } f(a) = 2.$$

So $a > 0$ solves

$$a^2 + \ln a = 2.$$

There is no simple closed form, so we approximate. Since

$$f(1.3) = 1.69 + \ln(1.3) \approx 1.69 + 0.262 = 1.952,$$

$$f(1.32) = 1.7424 + \ln(1.32) \approx 1.7424 + 0.2788 = 2.0212,$$

the solution is near $a \approx 1.315$.

Now

$$f'(x) = 2x + \frac{1}{x}.$$

Thus

$$(f^{-1})'(2) \approx \frac{1}{2(1.315) + 1/1.315} = \frac{1}{2.63 + 0.760} \approx \frac{1}{3.390} \approx 0.295.$$

$$\boxed{(f^{-1})'(2) \approx 0.295.}$$

Remark

Because $f'(x) > 0$ for $x > 0$, f is strictly increasing, so the equation $f(a) = 2$ has a unique solution a.

Problem 6 (Second derivative with chain rule)

Let $h(x) = e^{x^2+3x}$. Compute $h''(x)$.

Solution. First derivative:

$$h'(x) = e^{x^2+3x}(2x+3).$$

Second derivative (product rule):

$$h''(x) = e^{x^2+3x}(2x+3)^2 + e^{x^2+3x} \cdot 2.$$

So

$$\boxed{h''(x) = e^{x^2+3x}\big((2x+3)^2 + 2\big).}$$

Problem 7 (Logarithmic differentiation)

Find $\dfrac{dy}{dx}$ for $x^y = y^x$ (assume $x > 0$, $y > 0$).

Solution. Take ln of both sides:

$$\ln(x^y) = \ln(y^x) \quad \Rightarrow \quad y \ln x = x \ln y.$$

Differentiate:

$$y' \ln x + y \cdot \frac{1}{x} = \ln y + x \cdot \frac{1}{y}\, y'.$$

Collect y' terms:

$$y' \left(\ln x - \frac{x}{y} \right) = \ln y - \frac{y}{x}.$$

Hence

$$\boxed{\frac{dy}{dx} = \frac{\ln y - \frac{y}{x}}{\ln x - \frac{x}{y}}.}$$

Problem 8 (Related rates: sphere)

A spherical balloon is inflated so that the volume increases at 10 cm^3/s. Find the rate of change of its radius when $r = 5$ cm.

Solution.

$$V = \frac{4}{3}\pi r^3 \quad \Rightarrow \quad \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

So

$$\frac{dr}{dt} = \frac{1}{4\pi r^2}\frac{dV}{dt} = \frac{10}{4\pi(5)^2} = \frac{10}{100\pi} = \frac{1}{10\pi}.$$

$$\boxed{\frac{dr}{dt} = \frac{1}{10\pi}\ \text{cm/s.}}$$

Problem 9 (Logarithmic differentiation)

Use logarithmic differentiation to find $f'(x)$ if

$$f(x) = \frac{(x^2+1)^3\sqrt{1-x}}{e^{2x}}.$$

Solution. Take ln:

$$\ln f = 3\ln(x^2+1) + \frac{1}{2}\ln(1-x) - 2x.$$

Differentiate:

$$\frac{f'}{f} = 3\cdot\frac{2x}{x^2+1} + \frac{1}{2}\cdot\frac{-1}{1-x} - 2 = \frac{6x}{x^2+1} - \frac{1}{2(1-x)} - 2.$$

Therefore

$$\boxed{f'(x) = f(x)\left(\frac{6x}{x^2+1} - \frac{1}{2(1-x)} - 2\right).}$$

Problem 10 (Related rates: two cars)

At noon, a car is 40 km west of an intersection and travels east at 80 km/h, while another car is 30 km south of the intersection moving north at 60 km/h. How fast is the distance between the cars changing at 2:00 PM?

Solution. Let t be hours after noon. Put the intersection at the origin.

$$A(t) = (-40+80t,\ 0), \qquad B(t) = (0,\ -30+60t).$$

So the relative displacement is

$$(\Delta x, \Delta y) = (40-80t,\ -30+60t),$$

and distance

$$z(t) = \sqrt{(40-80t)^2 + (-30+60t)^2}.$$

Differentiate z^2:

$$2z\frac{dz}{dt} = 2(40-80t)(-80) + 2(-30+60t)(60).$$

At $t=2$:

$$\Delta x = 40 - 160 = -120, \qquad \Delta y = -30 + 120 = 90,$$

$$z = \sqrt{(-120)^2 + 90^2} = \sqrt{14400 + 8100} = \sqrt{22500} = 150.$$

Then

$$2(150)\,\frac{dz}{dt} = 2(-120)(-80) + 2(90)(60) = 19200 + 10800 = 30000,$$

so

$$\frac{dz}{dt} = \frac{30000}{300} = 100.$$

$$\boxed{\frac{dz}{dt} = 100 \text{ km/h at 2:00 PM.}}$$

Remark

A quick sanity check: at $t = 2$, the displacement is a 3–4–5 triangle scaled by 30 (since $90 : 120 : 150$), which often makes the arithmetic cleaner.

Problem 11 (Rewrite without trig/inverse trig)

Rewrite $\cos(2\tan^{-1} x)$ so that no trig or inverse trig functions appear.

Solution. Let $\theta = \tan^{-1} x$, so $\tan\theta = x$. Using the identity

$$\cos(2\theta) = \frac{1 - \tan^2\theta}{1 + \tan^2\theta},$$

we get

$$\cos\big(2\tan^{-1} x\big) = \frac{1 - x^2}{1 + x^2}.$$

$$\boxed{\cos(2\tan^{-1} x) = \frac{1 - x^2}{1 + x^2}.}$$

Problem 12 (Rewrite without trig/inverse trig)

Simplify $\tan(2\sin^{-1} x)$ so that no trig or inverse trig functions appear.

Solution. Let $\theta = \sin^{-1} x$, so $\sin\theta = x$ and (on the principal range) $\cos\theta = \sqrt{1 - x^2}$. Then

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{x}{\sqrt{1 - x^2}}.$$

Use

$$\tan(2\theta) = \frac{2\tan\theta}{1 - \tan^2\theta}.$$

Compute:

$$\tan(2\theta) = \frac{2\frac{x}{\sqrt{1-x^2}}}{1 - \frac{x^2}{1-x^2}} = \frac{2\frac{x}{\sqrt{1-x^2}}}{\frac{1-2x^2}{1-x^2}} = \frac{2x\sqrt{1 - x^2}}{1 - 2x^2}.$$

Thus

$$\boxed{\tan(2\sin^{-1} x) = \frac{2x\sqrt{1-x^2}}{1-2x^2}.}$$

Remark

Your final expression can legitimately include $\sqrt{1-x^2}$ because that is algebraic (no trig left). The principal range of arcsin is what fixes $\cos\theta \geq 0$.