MATA30 — Tutorial Notes

Week 11: U-Substitution Integrals

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Key Idea

Look for an "inner" expression whose derivative appears (up to a constant factor). Then substitute $u =$ (inner expression), rewrite the integral in u, integrate, and substitute back.

Problem 1. Evaluate:

$$\int \frac{x^3}{\sqrt{1+x^4}}\,dx.$$

Solution

Let $u = 1 + x^4$, so $du = 4x^3\,dx$.

$$\int \frac{x^3}{\sqrt{1+x^4}}\,dx = \frac{1}{4}\int u^{-1/2}\,du = \frac{1}{2}u^{1/2} + C = \boxed{\frac{1}{2}\sqrt{1+x^4} + C}.$$

Problem 2. Evaluate:

$$\int \frac{\ln^4(3x+2)}{3x+2}\,dx.$$

Solution

Let $u = \ln(3x+2)$, so $du = \dfrac{3}{3x+2}\,dx$.

$$\int \frac{\ln^4(3x+2)}{3x+2}\,dx = \frac{1}{3}\int u^4\,du = \frac{u^5}{15} + C = \boxed{\frac{1}{15}\ln^5(3x+2) + C}.$$

Problem 3. Evaluate:

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx.$$

Solution

Let $u = \sqrt{x}$, so $x = u^2$ and $dx = 2u\,du$.

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = \int \frac{e^u}{u}(2u)\,du = 2\int e^u\,du = \boxed{2e^{\sqrt{x}} + C}.$$

Problem 4. Evaluate:

$$\int \frac{x^2}{(1+x^3)^{7/3}}\,dx.$$

Solution

Let $u = 1 + x^3$, so $du = 3x^2\,dx$.

$$\int \frac{x^2}{(1+x^3)^{7/3}}\,dx = \frac{1}{3}\int u^{-7/3}\,du = \frac{1}{3}\cdot\frac{u^{-4/3}}{-4/3} + C = -\frac{1}{4}u^{-4/3} + C.$$

$$\boxed{-\frac{1}{4(1+x^3)^{4/3}} + C}.$$

Problem 5. Evaluate:

$$\int \frac{\sin(x)\,\cos(3\cos^2(x))}{\sqrt{1+\sin^2(x)}}\,dx.$$

Solution

A natural first step is $u = \cos x$ (since $du = -\sin x\,dx$). Then

$$\sqrt{1+\sin^2 x} = \sqrt{1+(1-u^2)} = \sqrt{2-u^2},$$

so the integral becomes

$$-\int \frac{\cos(3u^2)}{\sqrt{2-u^2}}\,du.$$

This integral does *not* simplify to an elementary antiderivative via standard u-substitution methods.

Remark

With the integrand exactly as written, u-substitution reduces it to a non-elementary form. If you intended a pure u-substitution exercise, one factor is likely missing (typically a $\cos x$ or a different trig composition).

Problem 6. Evaluate:

$$\int \frac{\sqrt{\ln(\ln(x))}}{x\,\ln(x)}\,dx.$$

Solution

Let $u = \ln x$, so $du = \frac{1}{x}\,dx$:

$$\int \frac{\sqrt{\ln(\ln x)}}{x\ln x}\,dx = \int \frac{\sqrt{\ln u}}{u}\,du.$$

Let $v = \ln u$, so $dv = \frac{1}{u}\,du$:

$$\int v^{1/2}\,dv = \frac{2}{3}v^{3/2} + C = \boxed{\frac{2}{3}\left[\ln(\ln x)\right]^{3/2} + C}.$$

Problem 7. Evaluate:

$$\int \frac{x^5}{(4+x^6)^2}\,dx.$$

Solution

Let $u = 4 + x^6$, so $du = 6x^5\,dx$.

$$\int \frac{x^5}{(4+x^6)^2}\,dx = \frac{1}{6}\int u^{-2}\,du = -\frac{1}{6u} + C = \boxed{-\frac{1}{6(4+x^6)} + C}.$$

Problem 8. Evaluate:

$$\int \frac{\cos(x)\,e^{\sin(x)}}{\sqrt{5+e^{\sin(x)}}}\,dx.$$

Solution

Let $u = e^{\sin x}$, so $du = e^{\sin x}\cos x\,dx$.

$$\int \frac{\cos x\,e^{\sin x}}{\sqrt{5+e^{\sin x}}}\,dx = \int \frac{du}{\sqrt{5+u}} = 2\sqrt{5+u} + C = \boxed{2\sqrt{5+e^{\sin x}} + C}.$$

Problem 9. Evaluate:

$$\int \frac{x}{(1+x^2)^3\sqrt{4+\ln(1+x^2)}}\,dx.$$

Solution

Let $u = 1 + x^2$, so $du = 2x\,dx$:

$$\int \frac{x}{(1+x^2)^3\sqrt{4+\ln(1+x^2)}}\,dx = \frac{1}{2}\int \frac{u^{-3}}{\sqrt{4+\ln u}}\,du.$$

Let $v = 4 + \ln u$, so $dv = \dfrac{1}{u}\,du$ and $du = u\,dv$:

$$\frac{1}{2}\int \frac{u^{-3}}{\sqrt{v}}\,du = \frac{1}{2}\int \frac{u^{-3}}{\sqrt{v}}(u\,dv) = \frac{1}{2}\int \frac{u^{-2}}{\sqrt{v}}\,dv.$$

Since $u = e^{v-4}$, we have $u^{-2} = e^{-2(v-4)} = e^8 e^{-2v}$. Hence

$$\frac{1}{2}\int \frac{u^{-2}}{\sqrt{v}}\,dv = \frac{e^8}{2}\int v^{-1/2}e^{-2v}\,dv.$$

This antiderivative is non-elementary; it can be written using the complementary error function erfc:

$$\int v^{-1/2}e^{-2v}\,dv = -\sqrt{\frac{\pi}{2}}\,\operatorname{erfc}\bigl(\sqrt{2v}\bigr) + C.$$

Therefore,

$$\boxed{\int \frac{x}{(1+x^2)^3\sqrt{4+\ln(1+x^2)}}\,dx = -\frac{e^8}{2}\sqrt{\frac{\pi}{2}}\,\operatorname{erfc}\Bigl(\sqrt{2\bigl(4+\ln(1+x^2)\bigr)}\Bigr) + C.}$$

Remark

Up to substitution, the integral becomes $\int v^{-1/2}e^{-2v}\,dv$, which is a standard special-function form.

Problem 10. Evaluate:

$$\int \frac{6x^2 - 4x + 7}{(2x^3 - 2x^2 + 7x - 5)^{5/2}}\,dx.$$

Solution

Let $u = 2x^3 - 2x^2 + 7x - 5$, so $du = (6x^2 - 4x + 7)\,dx$.

$$\int \frac{6x^2 - 4x + 7}{(2x^3 - 2x^2 + 7x - 5)^{5/2}}\,dx = \int u^{-5/2}\,du = -\frac{2}{3}u^{-3/2} + C.$$

$$\boxed{-\frac{2}{3\left(2x^3 - 2x^2 + 7x - 5\right)^{3/2}} + C}.$$