MATA30 — Summary

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1 Preliminaries: Functions and Review (Week 1)

Elementary Functions

The course assumes fluency with:

- Polynomial and rational functions
- Trigonometric functions $(\sin, \cos, \tan)$
- Exponential functions a^x, e^x
- Logarithmic functions $\ln x, \log_a x$

Definition 1. A function f maps each element of its domain to exactly one real number.

Inverse Functions

Definition 2. A function f has an inverse f^{-1} if and only if it is one-to-one.

Key Idea

To find an inverse: swap x and y, then solve for y.

Remark

Many functions require domain restriction to become invertible (e.g. $\sin x$).

2 Limits and Limit Laws (Week 2)

Definition 3. We say $\lim_{x \to a} f(x) = L$ if $f(x)$ approaches L as x approaches a.

Limit Laws

If limits exist,

$$\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x).$$

Algebraic Techniques

- Factoring and cancellation
- Rationalization
- Piecewise analysis

Key Idea

Never cancel terms unless they are factors.

3 Advanced Limits (Week 3)

Squeeze Theorem

Theorem 1 (Squeeze Theorem)**.** If $g(x) \le f(x) \le h(x)$ and

$$\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L,$$

then $\lim_{x\to a} f(x) = L$.

Infinite Limits and Asymptotes

Definition 4. If $f(x)$ grows without bound as $x \to a$, we write

$$\lim_{x\to a} f(x) = \pm\infty.$$

Remark

Vertical asymptotes correspond to infinite limits.

Limits at Infinity

Horizontal asymptotes describe end behavior.

Key Idea

For rational functions, compare degrees of numerator and denominator.

4 Continuity and IVT (Week 4)

Definition 5. A function is continuous at $x = a$ if:

1. $f(a)$ exists,
2. $\lim_{x\to a} f(x)$ exists,
3. they are equal.

Theorem 2 (Intermediate Value Theorem)**.** If f is continuous on $[a, b]$, then it takes every value between $f(a)$ and $f(b)$.

Remark

IVT guarantees existence, not location.

5 Derivatives: Definitions and Meaning (Week 4)

Definition 6. The derivative of f at a is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Interpretations

- Slope of the tangent line
- Instantaneous rate of change
- Velocity when position is given

Key Idea

Differentiability implies continuity.

6 Differentiation Rules (Week 5)

Basic Rules

$$\frac{d}{dx}(x^n) = nx^{n-1} \quad \frac{d}{dx}(e^x) = e^x$$

Product, Quotient, Chain Rules

$$(uv)' = u'v + uv'$$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Remark

The chain rule is the most common source of errors.

7 Implicit Differentiation and Applications (Week 6)

Implicit Differentiation

Differentiate both sides treating y as a function of x.

Related Rates

- Identify variables
- Differentiate with respect to time
- Substitute known values last

Key Idea

Units help catch mistakes in related rates.

8 Applications of Derivatives (Week 7)

Critical Points and Extrema

Definition 7. A critical point occurs where $f'(x) = 0$ or undefined.

Theorem 3 (Extreme Value Theorem)**.** A continuous function on a closed interval attains a maximum and minimum.

Concavity

$$f''(x) > 0 \Rightarrow \text{concave up}$$

Remark

Second derivative describes curvature, not slope.

9 Mean Value Theorem and L'Hôpital (Week 8)

Theorem 4 (Mean Value Theorem)**.** If f is continuous on $[a, b]$ and differentiable on (a, b), then

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

for some $c \in (a, b)$.

L'Hôpital's Rule

If $\frac{0}{0}$ or $\frac{\infty}{\infty}$,

$$\lim \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)}$$

Key Idea

Only apply L'Hôpital after confirming an indeterminate form.

10 Exponential Models and Optimization (Week 9)

Growth and Decay

$$P(t) = P_0 e^{kt}$$

Optimization

- Write objective function
- Find critical points
- Test endpoints

Remark

Optimization problems test modeling more than calculus.

11 Integration and Area (Weeks 10–12)

Antiderivatives

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C$$

Definite Integrals

Definition 8.

$$\int_a^b f(x)\,dx$$

represents signed area.

Theorem 5 (Fundamental Theorem of Calculus)**.**

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where $F' = f$.

Substitution Rule

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du$$

Area Between Curves

$$\int_a^b (\text{top} - \text{bottom})\,dx$$

Key Idea

Always sketch curves before integrating areas.

End of MATA30 Summary