MATA30 — Tutorial Notes

Week 8: Linearization, MVT/IVT, Rolle, L'Hôpital, Indeterminate Forms

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Problem 1 (Linearization: exponential–polynomial)**.** Let $f(x) = (x^2 + 1)e^x$.

(a) Find the linearization $L(x)$ at $x = 0$.

(b) Use it to approximate $f(0.03)$.

Solution

Compute $f(0) = 1$. Also

$$f'(x) = (2x)e^x + (x^2 + 1)e^x = (x^2 + 2x + 1)e^x,$$

so $f'(0) = 1$. Hence

$$L(x) = f(0) + f'(0)(x - 0) = 1 + x,$$

and

$$f(0.03) \approx L(0.03) = 1.03.$$

Remark

Near the expansion point, linearization is often accurate even when the original function looks complicated.

Problem 2 (Linearization: implicit difficulty hidden)**.** Let $g(x) = \sqrt{1 + \sin x}$.

(a) Find the linearization at $x = 0$.

(b) Approximate $g(0.02)$.

Solution

We have $g(0) = \sqrt{1 + 0} = 1$. Using chain rule,

$$g'(x) = \frac{1}{2}(1 + \sin x)^{-1/2} \cos x,$$

so $g'(0) = \frac{1}{2}$. Thus

$$L(x) = 1 + \frac{1}{2}x.$$

Therefore

$$g(0.02) \approx 1 + \frac{1}{2}(0.02) = 1.01.$$

Key Idea

Always compute the derivative at the expansion point *after* simplifying at that point; it prevents messy algebra.

Problem 3 (MVT: tight bounds)**.** Suppose f is differentiable on $[1, 5]$ and satisfies

$$2 \le f'(x) \le 5 \quad \text{for all } x \in [1, 5],$$

with $f(1) = 4$. Find the tightest possible bounds for $f(5)$.

Solution

By MVT, $f(5) - f(1) = f'(c)(5 - 1) = 4f'(c)$ for some $c \in (1, 5)$. Hence

$$8 \le f(5) - 4 \le 20 \quad \Rightarrow \quad \boxed{12 \le f(5) \le 24.}$$

Remark

The bounds are sharp: take $f'(x) \equiv 2$ or $f'(x) \equiv 5$ (linear functions).

Problem 4 (Rolle's Theorem: force a point with zero derivative)**.** Let f be continuous on $[-1, 2]$ and differentiable on $(-1, 2)$. Suppose $f(-1) = f(2)$. Show that there exists $c \in (-1, 2)$ such that $f'(c) = 0$, and explain *why* the differentiability assumption matters.

Solution

Since f is continuous on $[-1, 2]$, differentiable on $(-1, 2)$, and $f(-1) = f(2)$, Rolle's Theorem applies:

$$\exists\, c \in (-1, 2) \text{ such that } f'(c) = 0.$$

The differentiability assumption matters because Rolle can fail if f has a cusp/corner: the function could still satisfy $f(-1) = f(2)$ but have no point where the derivative is 0 (e.g., a "V"-shape translated to fit the endpoints).

Remark

Rolle/MVT are *differentiability theorems*, not just continuity theorems.

Problem 5 (IVT + uniqueness using monotonicity)**.** Show that the equation

$$e^x = x + 2$$

has exactly one real solution.

Solution

Define $h(x) = e^x - (x + 2)$. Then h is continuous everywhere.

$$h(0) = 1 - 2 = -1 < 0, \qquad h(2) = e^2 - 4 > 0.$$

By IVT, at least one root exists in $(0, 2)$.

For uniqueness, compute

$$h'(x) = e^x - 1.$$

Then $h'(x) > 0$ for all $x > 0$, so h is strictly increasing on $(0, \infty)$. Since the root lies in $(0, 2) \subset (0, \infty)$, it must be unique.

$$\boxed{\text{Exactly one real solution.}}$$

Key Idea

Existence: IVT. Uniqueness: show the function is strictly monotone on an interval containing the root.

Problem 6 (A "Rolle contradiction" uniqueness proof)**.** Show that the equation

$$2x - 1 = \sin x$$

has exactly one real solution.

Solution

Let $p(x) = 2x - 1 - \sin x$. Then p is continuous and differentiable for all x. Existence: $p(0) = -1 < 0$ and $p(\pi) = 2\pi - 1 > 0$, so by IVT a root exists in $(0, \pi)$.

Uniqueness: Suppose there are two roots $a < b$. Then $p(a) = p(b) = 0$, so by Rolle's Theorem there exists $c \in (a, b)$ with $p'(c) = 0$. But

$$p'(x) = 2 - \cos x \geq 2 - 1 = 1 > 0,$$

so $p'(x) \neq 0$ for all x, contradiction. Hence at most one root.

$$\boxed{\text{Exactly one real solution.}}$$

Remark

This is one of the cleanest "IVT + Rolle" templates you'll reuse all term.

Problem 7 (L'Hôpital: controlled use)**.** Evaluate

$$\lim_{x \to 0} \frac{\ln(1 + 2x) - 2x}{x^2}.$$

Solution

Direct substitution gives $0/0$. Apply L'Hôpital once:

$$\lim_{x \to 0} \frac{\frac{2}{1+2x} - 2}{2x} = \lim_{x \to 0} \frac{\frac{2-2(1+2x)}{1+2x}}{2x} = \lim_{x \to 0} \frac{-4x}{(1 + 2x)2x} = \lim_{x \to 0} \frac{-2}{1 + 2x} = -2.$$

$$\boxed{-2}$$

Key Idea

Before using L'Hôpital repeatedly, simplify after the first derivative — many limits collapse immediately.

Problem 8 (L'Hôpital twice: classic trig-log)**.** Evaluate

$$\lim_{x \to 0} \frac{\ln(\cos x)}{x^2}.$$

Solution

This is 0/0. Apply L'Hôpital:

$$\lim_{x \to 0} \frac{-\tan x}{2x}.$$

Still 0/0. Apply L'Hôpital again:

$$\lim_{x \to 0} \frac{-\sec^2 x}{2} = -\frac{1}{2}.$$

$$\boxed{-\frac{1}{2}}$$

Remark

This is a good benchmark limit: $\ln(\cos x) \sim -\frac{x^2}{2}$ for small x.

Problem 9 (Indeterminate exponential form ∞^0)**.** Compute

$$\lim_{x \to \infty} x^{1/\ln x}.$$

Solution

Let $L = \lim_{x \to \infty} x^{1/\ln x}$. Take logs:

$$\ln L = \lim_{x \to \infty} \frac{\ln x}{\ln x} = 1.$$

Therefore $L = e^1 = e$.

$$\boxed{e}$$

Key Idea

For indeterminate powers $f(x)^{g(x)}$, take logs first: $\ln L = \lim g(x) \ln f(x)$.

Problem 10 (Harder combined expression: cancellation + L'Hôpital)**.** Evaluate

$$\lim_{x \to 1} \left(\frac{x}{x-1} - \frac{1}{\ln x} - \frac{1}{2} \right).$$

Solution

First combine the two main terms:

$$\frac{x}{x-1} - \frac{1}{\ln x} = \frac{x \ln x - (x-1)}{(x-1)\ln x}.$$

At $x \to 1$ this is 0/0. Apply L'Hôpital to the fraction: Differentiate numerator and denominator:

$$\text{Num}' = \ln x + 1 - 1 = \ln x, \qquad \text{Den}' = (x-1) \cdot \frac{1}{x} + \ln x \cdot 1 = \frac{x-1}{x} + \ln x.$$

So the limit becomes

$$\lim_{x \to 1} \frac{\ln x}{\frac{x-1}{x} + \ln x}.$$

Still 0/0. Apply L'Hôpital again:

$$\lim_{x\to 1} \frac{1/x}{\left(\frac{x-(x-1)}{x^2}\right) + 1/x} = \lim_{x\to 1} \frac{1/x}{\frac{1}{x^2} + \frac{1}{x}} = \lim_{x\to 1} \frac{1}{\frac{1}{x} + 1} = \frac{1}{2}.$$

Thus

$$\lim_{x\to 1} \left(\frac{x}{x-1} - \frac{1}{\ln x} \right) = \frac{1}{2},$$

so

$$\boxed{\lim_{x\to 1} \left(\frac{x}{x-1} - \frac{1}{\ln x} - \frac{1}{2} \right) = 0.}$$

Remark

Subtracting the correct constant removes the finite "limit offset" and reveals the true cancellation.