

# MATA30 — Tutorial Notes

Weeks 5: Implicit Differentiation, Inverses, Logarithms, Related Rates

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**Problem 1** (Derivative of an Inverse Function). Suppose  $f$  and  $g$  are differentiable inverse functions satisfying

$$f'(x) = 1 + [f(x)]^2, \quad f(g(x)) = x.$$

Determine  $g'(x)$ .

## Solution

Using the inverse function derivative identity,

$$g'(x) = \frac{1}{f'(g(x))}.$$

Since  $f(g(x)) = x$ ,

$$f'(g(x)) = 1 + x^2,$$

and therefore

$$g'(x) = \frac{1}{1 + x^2}.$$

## Remark

The derivative of the inverse depends only on the outer variable because  $f(g(x)) = x$ .

**Problem 2** (Implicit Differentiation). The curve

$$x^2y^3 = 17\sqrt{y} + x + 1$$

defines  $y$  implicitly as a function of  $x$ . Find  $y'$  and evaluate it at points where  $y = 1$ .

## Solution

Differentiate implicitly:

$$2xy^3 + 3x^2y^2y' - \frac{17}{2}y^{-1/2}y' - 1 = 0.$$

Solving for  $y'$  gives

$$y' = \frac{1 - 2xy^3}{3x^2y^2 - \frac{17}{2}y^{-1/2}}.$$

At  $y = 1$ , the original equation yields

$$x^2 - x - 16 = 0 \quad \Rightarrow \quad x = \pm 4.$$

Thus,

$$y' = \begin{cases} -\frac{14}{79}, & x = 4, \\ \frac{18}{79}, & x = -4. \end{cases}$$

**Problem 3** (Perpendicular Tangents). On the curve

$$xy = 3y + x,$$

find all points where the tangent line is perpendicular to  $y = 3x + 1$ .

**Solution**

Differentiate implicitly:

$$y + xy' = 3y' + 1 \quad \Rightarrow \quad y' = \frac{1 - y}{x - 3}.$$

Perpendicular slopes satisfy  $y' = -\frac{1}{3}$ , so

$$\frac{1 - y}{x - 3} = -\frac{1}{3} \quad \Rightarrow \quad x = 3y.$$

Substitute into the curve equation:

$$3y^2 = 6y \quad \Rightarrow \quad y = 0 \text{ or } 2.$$

Hence the points are

$$\boxed{(0, 0) \text{ and } (6, 2)}.$$

**Problem 4** (Implicit Differentiation and Inverses). Let

$$\tan(x + y) = x + 1.$$

Find  $\frac{dy}{dx}$  and evaluate it at  $(0, \frac{\pi}{4})$ .

**Solution**

Differentiate implicitly:

$$\sec^2(x + y)(1 + y') = 1 \quad \Rightarrow \quad y' = \cos^2(x + y) - 1 = -\sin^2(x + y).$$

At  $(0, \frac{\pi}{4})$ ,

$$\boxed{y' = -\frac{1}{2}}.$$

**Remark**

When a relation defines  $y$  locally as a function of  $x$ , the slope of the inverse curve is the reciprocal.

**Problem 5** (Logarithmic Differentiation). Let

$$g(x) = e^{e^{x^2+1}}.$$

Compute  $g'(x)$  and simplify  $\frac{g'(x)}{g(x)}$ .

**Solution**

Differentiate using the chain rule:

$$g'(x) = e^{e^{x^2+1}} \cdot e^{x^2+1} \cdot 2x.$$

Thus,

$$\boxed{\frac{g'(x)}{g(x)} = 2x e^{x^2+1}}.$$

**Problem 6** (Derivative of an Inverse at a Point). Let

$$f(x) = \ln(x + 1) + 2x + 1.$$

Find  $(f^{-1})'(1)$ .

**Solution**

We find  $a$  such that  $f(a) = 1$ :

$$\ln(a + 1) + 2a = 0 \quad \Rightarrow \quad a = 0.$$

Compute

$$f'(x) = \frac{1}{x + 1} + 2, \quad f'(0) = 3.$$

Therefore,

$$\boxed{(f^{-1})'(1) = \frac{1}{3}}.$$

**Problem 7** (Related Rates). A cylindrical tank of radius 4 m is filled at a rate of

$$\frac{dV}{dt} = 2 \text{ m}^3/\text{min}.$$

Find  $\frac{dh}{dt}$ .

**Solution**

The volume is

$$V = 16\pi h.$$

Differentiate:

$$\frac{dV}{dt} = 16\pi \frac{dh}{dt}.$$

Thus,

$$\boxed{\frac{dh}{dt} = \frac{1}{8\pi} \text{ m/min.}}$$

**Problem 8** (Rate of Change of Distance). Two ships move so that one travels east at 7 km/h and the other north at 4 km/h. Initially, the eastbound ship is 10 km west of the other. Find the rate of change of the distance between them after  $t$  hours.

**Solution**

Coordinates:

$$A(t) = (-10 + 7t, 0), \quad B(t) = (0, 4t).$$

Distance:

$$d(t) = \sqrt{(10 - 7t)^2 + (4t)^2}.$$

Differentiate:

$$\boxed{\frac{dd}{dt} = \frac{-70 + 65t}{\sqrt{100 - 140t + 65t^2}}}.$$