MATA30 — Tutorial Notes

Week 9: Exponential Models and Optimization

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> **Key Idea**
>
> For optimization: (1) model the quantity to optimize as a single-variable function, (2) identify the correct domain, (3) test critical points *and* endpoints.

Problem 1 (Population Growth with a Rate Change). A city had population 1.2 million in 1990 and 1.8 million in 2000. Assume exponential growth up to 2000. From 2000 onward, the growth rate is reduced by 20%.

   (a) Determine an exponential model $P(t)$ (in millions) for t years after 1990, valid for $0 \le t \le 10$.

   (b) Write a piecewise model for $t > 10$ reflecting the reduced growth rate.

   (c) Estimate the population in 2025, and compute the percentage error if the actual population is 3.25 million.

Solution

(a) Let $P(t) = 1.2e^{kt}$ for $0 \le t \le 10$. Using $P(10) = 1.8$:

$$1.8 = 1.2e^{10k} \quad \Rightarrow \quad e^{10k} = 1.5 \quad \Rightarrow \quad k = \frac{\ln(1.5)}{10}.$$

So

$$\boxed{P(t) = 1.2e^{(\ln(1.5)/10)\,t}, \quad 0 \le t \le 10.}$$

(b) The new rate is $k_2 = 0.8k = \dfrac{0.8\ln(1.5)}{10}$. For $t > 10$,

$$P(t) = P(10)e^{k_2(t-10)} = 1.8e^{k_2(t-10)}.$$

Thus

$$\boxed{P(t) = \begin{cases} 1.2e^{(\ln(1.5)/10)\,t}, & 0 \le t \le 10, \\ 1.8e^{(0.8\ln(1.5)/10)\,(t-10)}, & t > 10. \end{cases}}$$

(c) Year 2025 corresponds to $t = 35$. Then

$$P(35) = 1.8e^{k_2(25)}.$$

Numerically, $k_2 \approx 0.03244$, so

$$P(35) \approx 1.8e^{0.811} \approx 4.05 \text{ million.}$$

Percentage error (relative to actual 3.25):

$$\boxed{\text{error} \approx \frac{4.05 - 3.25}{3.25} \times 100\% \approx 24.6\%.}$$

Problem 2 (Open-Top Box with Different Material Costs). An open-top box with square base must have volume $V = 50{,}000$ cm^3. The base material costs twice as much per cm^2 as the side material.

(a) Write the total cost $C(x)$ in terms of base side length x.

(b) Find the dimensions minimizing $C(x)$.

(c) Compare with the minimum-surface-area open-top box of the same volume.

Solution

Let base side length be x and height be h. Volume constraint: $x^2 h = 50{,}000 \Rightarrow h = \dfrac{50{,}000}{x^2}$.

Cost $\propto$ (base area)$\times 2$ + (side area)$\times 1$:

$$C(x) = 2x^2 + 4xh = 2x^2 + 4x \cdot \frac{50{,}000}{x^2} = 2x^2 + \frac{200{,}000}{x}.$$

Differentiate:
$$C'(x) = 4x - \frac{200{,}000}{x^2}.$$

Set $C'(x) = 0$:
$$4x^3 = 200{,}000 \Rightarrow x^3 = 50{,}000 \Rightarrow \boxed{x = \sqrt[3]{50{,}000} \approx 36.84 \text{ cm.}}$$

Then
$$h = \frac{50{,}000}{x^2} \approx 36.84 \text{ cm.}$$

So the minimizer satisfies $\boxed{h = x}$.

Problem 3 (Maximum-Area Rectangle Under a Parabola). For $x \geq 0$, consider $y = 9 - x^2$. A rectangle is inscribed under the curve with its base on the x-axis from 0 to x, and its top-right corner on the curve.

(a) Write the area $A(x)$.

(b) Find the x maximizing the area and compute the maximum area.

(c) What fraction of the area under the curve from $x = 0$ to $x = 3$ does this rectangle occupy?

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Solution

Height is $9 - x^2$ and width is x, so
$$A(x) = x(9 - x^2) = 9x - x^3.$$

Differentiate:
$$A'(x) = 9 - 3x^2 = 0 \Rightarrow x^2 = 3 \Rightarrow \boxed{x = \sqrt{3}}.$$

Maximum area:
$$A(\sqrt{3}) = 9\sqrt{3} - 3\sqrt{3} = \boxed{6\sqrt{3}}.$$

Total area under $y = 9 - x^2$ from 0 to 3:
$$\int_0^3 (9 - x^2)\, dx = \left(9x - \frac{x^3}{3}\right)_0^3 = 27 - 9 = 18.$$

Fraction:
$$\boxed{\frac{6\sqrt{3}}{18} = \frac{\sqrt{3}}{3} \approx 0.577.}$$

Problem 4 (Exponential Decay and Dosing Interval). A medication concentration follows
$$C(t) = C_0 e^{-kt}, \qquad C_0 = 50 \text{ mg/L}, \quad k = 0.4 \text{ h}^{-1}.$$
A dose is administered when the concentration falls to 10 mg/L.

   (a) Find the dosing interval T.

   (b) If a daily cost is proportional to the number of doses per day times the average concentration over each cycle, express the daily cost (up to a constant factor) as a function of T, and state what the model predicts about minimizing cost under this setup.

Solution

(a) Solve $10 = 50e^{-0.4T}$:
$$e^{-0.4T} = 0.2 \Rightarrow -0.4T = \ln(0.2) = -\ln 5 \Rightarrow \boxed{T = \frac{\ln 5}{0.4} \approx 4.02 \text{ h.}}$$

(b) Number of doses per day is $N(T) = \dfrac{24}{T}$. Average concentration over one cycle is
$$\overline{C}(T) = \frac{1}{T} \int_0^T 50 e^{-0.4t}\, dt = \frac{50}{0.4T}\left(1 - e^{-0.4T}\right).$$

Thus, up to a constant factor,
$$C_d(T) \propto N(T)\overline{C}(T) = \frac{24}{T} \cdot \frac{50}{0.4T}\left(1 - e^{-0.4T}\right) \propto \frac{1 - e^{-0.4T}}{T^2}.$$

For $T > 0$, this decreases as T increases, so the model predicts lower cost for larger T. Under the rule "dose when $C(t) = 10$", the interval is fixed by part (a).

> **Remark**
>
> If a problem includes both an optimization objective and a dosing/threshold rule, check whether the rule fully determines T. If it does, there is no free parameter left to optimize.