MATA30 — Tutorial Notes

Week 4: Continuity, IVT, and Derivatives from First Principles

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Problem 1 (Continuity at a Point)**.** Let

$$f(x) = \begin{cases} \dfrac{x^2-4}{x+2}, & x \neq -2, \\ c, & x = -2. \end{cases}$$

Find c so that f is continuous at $x = -2$. Now replace the numerator by $x^2 + ax - 2(2+a)$. For which values of a is f continuous at $x = -2$?

Solution

For $x \neq -2$,

$$\frac{x^2-4}{x+2} = \frac{(x-2)(x+2)}{x+2} = x - 2.$$

Thus,

$$\lim_{x \to -2} f(x) = -4.$$

Continuity requires $c = -4$.

For the modified numerator,

$$x^2 + ax - 2(2+a) = (x+2)(x+a-2),$$

so for $x \neq -2$,

$$f(x) = x + a - 2.$$

Hence,

$$\lim_{x \to -2} f(x) = a - 4.$$

Continuity holds for all real a, with $c = a - 4$.

Remark

A removable discontinuity occurs when a common factor cancels and the limit exists.

Problem 2 (Continuity and Differentiability at a Junction)**.** Let

$$f(x) = \begin{cases} x^2 + kx, & x < 1, \\ \dfrac{x+k}{kx}, & x \geq 1 \quad (k \neq 0). \end{cases}$$

Find all k for which f is continuous at $x = 1$. For which k is f differentiable at $x = 1$?

Solution

Left-hand limit:

$$\lim_{x \to 1^-} f(x) = 1 + k.$$

Right-hand value:

$$f(1) = \frac{1+k}{k}.$$

Continuity requires

$$1 + k = \frac{1+k}{k} \quad \Longrightarrow \quad k = 1.$$

For differentiability, compute derivatives:

$$f'_-(1) = 2(1) + k = 3, \qquad f'_+(x) = -\frac{1}{x^2}, \quad f'_+(1) = -1.$$

Since $f'_-(1) \neq f'_+(1)$, f is not differentiable at $x = 1$.

$$\boxed{\text{Continuous for } k = 1, \text{ never differentiable at } x = 1.}$$

Problem 3 (Continuity via Limits)**.** Let

$$f(x) = \begin{cases} \dfrac{(x+1)\sin x}{x^3 + 2x^2 + x}, & x \neq 0, \\ d, & x = 0. \end{cases}$$

Find d so that f is continuous at 0. List and classify all other discontinuities.

Solution

Factor:

$$x^3 + 2x^2 + x = x(x+1)^2.$$

Then

$$f(x) = \frac{\sin x}{x} \cdot \frac{1}{x+1}.$$

Thus,

$$\lim_{x \to 0} f(x) = 1.$$

So $d = 1$.

The function is undefined at $x = -1$, where the denominator vanishes but no cancellation occurs. This is an infinite discontinuity.

Problem 4 (Classification of a Discontinuity)**.** Define

$$f(x) = \begin{cases} 1 - \sin x + e^{1/x}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Determine whether f is continuous at 0, and classify the discontinuity if not.

Solution

As $x \to 0^+$, $e^{1/x} \to \infty$, so $f(x) \to \infty$. As $x \to 0^-$, $e^{1/x} \to 0$, so $f(x) \to 1$.

Since the one-sided limits differ and are not finite,

$$\boxed{f \text{ has an infinite discontinuity at } 0.}$$

Problem 5 (Removable Discontinuities)**.** Let

$$f(x) = \begin{cases} 2 - x, & x < -1, \\ ax + b, & -1 \le x \le 1, \\ x + 6, & x > 1. \end{cases}$$

Find a, b so that f is continuous on $\mathbb{R}$.

Solution

At $x = -1$:

$$3 = -a + b.$$

At $x = 1$:

$$a + b = 7.$$

Solving,

$$a = 2, \quad b = 5.$$

Problem 6 (Intermediate Value Theorem)**.** Show that the equation

$$x^3 = 3x - 1$$

has at least three solutions in $[-2, 2]$.

Solution

Let $h(x) = x^3 - 3x + 1$. Then:

$$h(-2) = -1, \quad h(-1) = 3, \quad h(0) = 1, \quad h(1) = -1, \quad h(2) = 3.$$

By the IVT, h has a zero in each of the intervals

$$(-2, -1),\ (0, 1),\ (1, 2).$$

Problem 7 (Derivative from First Principles)**.** Compute $f'(x)$ for $f(x) = 5x^2 - 3x + \pi$ using the definition of the derivative.

Solution

$$f'(x) = \lim_{h \to 0} \frac{5(x+h)^2 - 3(x+h) - 5x^2 + 3x}{h} = \lim_{h \to 0} (10x + 5h - 3) = 10x - 3.$$