MATA30 — Tutorial Notes

Week 3

Topics: Limits, Limit Laws, Algebraic Tricks, Squeeze Theorem, Asymptotes

Amir Koutahi

Key Idea

When a limit is indeterminate, your first instinct should be:

- Factor/cancel (for 0/0),
- Divide by the highest power (for ∞/∞),
- Rationalize (for $\infty - \infty$ or $\sqrt{\cdot}$ expressions),
- Squeeze (when something oscillates or is bounded),
- Use standard trig limits (e.g. $\lim_{x\to 0} \frac{\sin x}{x} = 1$).

Hard Practice Problems (with Solutions)

Problem 1 (Leading terms + absolute value)

Evaluate the limits:

(a) $\displaystyle\lim_{x\to\infty} \frac{\sqrt{7x^2 - 4x + 9}}{5x + 1}$

(b) $\displaystyle\lim_{x\to-\infty} \frac{\sqrt{7x^2 - 4x + 9}}{5x + 1}$

Solution.

(a) Divide top and bottom by x (and assume $x > 0$ for $x \to \infty$):

$$\lim_{x\to\infty} \frac{\sqrt{7x^2 - 4x + 9}}{5x + 1} = \lim_{x\to\infty} \frac{\sqrt{x^2\left(7 - \frac{4}{x} + \frac{9}{x^2}\right)}}{x\left(5 + \frac{1}{x}\right)} = \lim_{x\to\infty} \frac{|x|\sqrt{7 - \frac{4}{x} + \frac{9}{x^2}}}{x\left(5 + \frac{1}{x}\right)}.$$

Since $x > 0$ eventually, $|x| = x$, so

$$= \lim_{x\to\infty} \frac{\sqrt{7 - \frac{4}{x} + \frac{9}{x^2}}}{5 + \frac{1}{x}} = \frac{\sqrt{7}}{5}.$$

(b) Now $x \to -\infty$, so $|x| = -x$ eventually:

$$\lim_{x\to-\infty} \frac{|x|\sqrt{7 - \frac{4}{x} + \frac{9}{x^2}}}{x\left(5 + \frac{1}{x}\right)} = \lim_{x\to-\infty} \frac{-x\sqrt{7 - \frac{4}{x} + \frac{9}{x^2}}}{x\left(5 + \frac{1}{x}\right)} = -\frac{\sqrt{7}}{5}.$$

$$\boxed{(a)\ \frac{\sqrt{7}}{5} \qquad (b)\ -\frac{\sqrt{7}}{5}}$$

Problem 2 (High power trig limit)

Evaluate

$$\lim_{x\to 0} \frac{\sin^7(3x)}{x^7}.$$

Solution. Rewrite as

$$\frac{\sin^7(3x)}{x^7} = 3^7 \left(\frac{\sin(3x)}{3x}\right)^7.$$

Then $\lim_{x\to 0} \frac{\sin(3x)}{3x} = 1$, so

$$\boxed{\lim_{x\to 0} \frac{\sin^7(3x)}{x^7} = 3^7 = 2187.}$$

Problem 3 (Mixed trig and algebra; requires factor trick)

Evaluate

$$\lim_{x\to 0} \frac{\sin(5x) - 5x\cos(5x)}{x^3}.$$

Solution. Factor out $5x$:

$$\sin(5x) - 5x\cos(5x) = 5x\left(\frac{\sin(5x)}{5x} - \cos(5x)\right).$$

So

$$\frac{\sin(5x) - 5x\cos(5x)}{x^3} = 5 \cdot \frac{\frac{\sin(5x)}{5x} - \cos(5x)}{x^2}.$$

Now use the identity $\cos(5x) = 1 - 2\sin^2\left(\frac{5x}{2}\right)$ and write

$$\frac{\sin(5x)}{5x} - \cos(5x) = \left(\frac{\sin(5x)}{5x} - 1\right) + (1 - \cos(5x)).$$

We know (from standard limits) that as $u \to 0$,

$$\frac{\sin u}{u} - 1 \sim -\frac{u^2}{6}, \qquad 1 - \cos u \sim \frac{u^2}{2}.$$

Applying with $u = 5x$ gives

$$\frac{\sin(5x)}{5x} - 1 \sim -\frac{(5x)^2}{6} = -\frac{25}{6}x^2, \qquad 1 - \cos(5x) \sim \frac{(5x)^2}{2} = \frac{25}{2}x^2.$$

So their sum is

$$\frac{\sin(5x)}{5x} - \cos(5x) \sim \left(-\frac{25}{6} + \frac{25}{2}\right)x^2 = \frac{25}{3}x^2.$$

Therefore

$$5 \cdot \frac{\frac{\sin(5x)}{5x} - \cos(5x)}{x^2} \to 5 \cdot \frac{25}{3} = \frac{125}{3}.$$

$$\boxed{\lim_{x\to 0} \frac{\sin(5x) - 5x\cos(5x)}{x^3} = \frac{125}{3}.}$$

Remark

This is a "hard Week 3" style limit: you reduce it until everything is built from standard small-angle facts.

Problem 4 (Absolute value near a point)

Compute

$$\lim_{x\to -3} \frac{|x+3|}{x+3}.$$

Solution. For $x > -3$, $x+3 > 0$, so $|x+3| = x+3$ and the ratio is 1. For $x < -3$, $x+3 < 0$, so $|x+3| = -(x+3)$ and the ratio is -1. One-sided limits differ, so the two-sided limit does not exist.

$$\boxed{\text{DNE (left limit } = -1,\ \text{right limit } = 1).}$$

Problem 5 (Parameter limit-existence with a removable term)

Let

$$f(x) = \begin{cases} \dfrac{x^2-1}{x-1} + k(x-1), & x \neq 1, \\ c, & x = 1. \end{cases}$$

(a) For which k does $\lim_{x\to 1} f(x)$ exist?

(b) Choose c (in terms of k) so that f is continuous at $x = 1$.

Solution. For $x \neq 1$,

$$\frac{x^2-1}{x-1} = \frac{(x-1)(x+1)}{x-1} = x+1.$$

So

$$f(x) = x + 1 + k(x-1) \quad (x \neq 1).$$

Then the limit exists for *all* k since this is a polynomial expression near 1:

$$\lim_{x\to 1} f(x) = 1 + 1 + k(0) = 2.$$

Continuity requires $c = \lim_{x\to 1} f(x) = 2$.

$$\boxed{(a)\text{ all } k \in \mathbb{R} \qquad (b)\ c = 2.}$$

Problem 6 (Squeeze with oscillation and a power)

Evaluate

$$\lim_{x\to\infty} \frac{\sin(x^2)}{\sqrt{x}}.$$

Solution. We know $-1 \le \sin(x^2) \le 1$, so dividing by $\sqrt{x}$ (positive for $x \to \infty$):

$$-\frac{1}{\sqrt{x}} \le \frac{\sin(x^2)}{\sqrt{x}} \le \frac{1}{\sqrt{x}}.$$

Both bounds go to 0 as $x \to \infty$, so by Squeeze Theorem the limit is 0.

$$\boxed{0.}$$

Problem 7 (Squeeze at a point with different blow-up rates)

Suppose for all real $x \neq 2$,

$$\frac{1}{(x-2)^6+4} \;\le\; f(x) \;\le\; \frac{1}{3(x-2)^2+4}.$$

Determine $\lim_{x\to 2} f(x)$.

Solution. Compute the limits of the bounds:

$$\lim_{x\to 2} \frac{1}{(x-2)^6+4} = \frac{1}{0+4} = \frac{1}{4}, \qquad \lim_{x\to 2} \frac{1}{3(x-2)^2+4} = \frac{1}{0+4} = \frac{1}{4}.$$

Since the bounds agree, Squeeze Theorem gives

$$\boxed{\lim_{x\to 2} f(x) = \frac{1}{4}.}$$

Problem 8 (Vertical asymptote vs removable discontinuity)

Let

$$F(x) = \frac{x^2-6x+9}{x^2-9}.$$

(a) Find all discontinuities.

(b) Identify which are vertical asymptotes and which are removable, and compute the corresponding limit(s).

Solution. Factor:

$$x^2-6x+9 = (x-3)^2, \qquad x^2-9 = (x-3)(x+3).$$

So for $x \neq \pm 3$,

$$F(x) = \frac{(x-3)^2}{(x-3)(x+3)} = \frac{x-3}{x+3}.$$

Discontinuities at $x = 3$ and $x = -3$.

At $x = 3$:

$$\lim_{x\to 3} F(x) = \lim_{x\to 3} \frac{x-3}{x+3} = \frac{0}{6} = 0,$$

so $x = 3$ is removable.

At $x = -3$:

$$\lim_{x\to -3} F(x) = \lim_{x\to -3} \frac{x-3}{x+3},$$

denominator $\to 0$ but numerator $\to -6 \neq 0$, so $x = -3$ is a vertical asymptote. One-sided behavior: as $x \to -3^+$, $x + 3 \to 0^+$ and $x - 3 \to -6$, so $F(x) \to -\infty$. As $x \to -3^-$, $x + 3 \to 0^-$ and $x - 3 \to -6$, so $F(x) \to +\infty$.

$$\boxed{\text{Discontinuities at } x = \pm 3;\ \text{removable at } 3,\ \text{vertical asymptote at } -3.}$$

Problem 9 ($\infty - \infty$ rationalization)

Compute

$$\lim_{x\to\infty} \left(\sqrt{4x^2 + 7x} - 2x\right).$$

Solution. Rationalize:

$$\sqrt{4x^2 + 7x} - 2x = \frac{(4x^2 + 7x) - (2x)^2}{\sqrt{4x^2 + 7x} + 2x} = \frac{7x}{\sqrt{4x^2 + 7x} + 2x}.$$

Divide numerator and denominator by x (with $x > 0$):

$$= \frac{7}{\sqrt{4 + \frac{7}{x}} + 2} \xrightarrow[x\to\infty]{} \frac{7}{2+2} = \frac{7}{4}.$$

$$\boxed{\frac{7}{4}.}$$

Problem 10 (Horizontal asymptotes: different at $\pm\infty$)

Find the horizontal asymptote(s) of

$$g(x) = x - \sqrt{x^2 + 6x + 5}.$$

Solution. Consider $x \to \infty$ and $x \to -\infty$ separately.

<u>As $x \to \infty$:</u> rationalize:

$$x - \sqrt{x^2 + 6x + 5} = \frac{x^2 - (x^2 + 6x + 5)}{x + \sqrt{x^2 + 6x + 5}} = \frac{-6x - 5}{x + \sqrt{x^2 + 6x + 5}}.$$

Divide top and bottom by x (positive):

$$= \frac{-6 - \frac{5}{x}}{1 + \sqrt{1 + \frac{6}{x} + \frac{5}{x^2}}} \to \frac{-6}{1+1} = -3.$$

So $y = -3$ is a horizontal asymptote as $x \to \infty$.

As $x \to -\infty$: again

$$\frac{-6x - 5}{x + \sqrt{x^2 + 6x + 5}}.$$

Now $\sqrt{x^2 + 6x + 5} = |x|\sqrt{1 + \frac{6}{x} + \frac{5}{x^2}}$ and for $x \to -\infty$, $|x| = -x$. So the denominator behaves like

$$x + |x| \cdot (\cdots) \approx x + (-x) \cdot 1 = 0,$$

so we must be careful: use the original form with $|x|$ explicitly:

$$g(x) = x - |x|\sqrt{1 + \frac{6}{x} + \frac{5}{x^2}}.$$

For $x \to -\infty$, $|x| = -x$, hence

$$g(x) = x - (-x)\sqrt{1 + \frac{6}{x} + \frac{5}{x^2}} = x + x\sqrt{1 + \frac{6}{x} + \frac{5}{x^2}}.$$

Factor x:

$$g(x) = x\left(1 + \sqrt{1 + \frac{6}{x} + \frac{5}{x^2}}\right).$$

As $x \to -\infty$, the bracket $\to 1 + 1 = 2$, so $g(x) \sim 2x \to -\infty$. Thus there is *no* horizontal asymptote as $x \to -\infty$.

$$\boxed{\text{Horizontal asymptote: } y = -3 \text{ (only as } x \to \infty).}$$

Remark

Many expressions with $\sqrt{x^2 + \cdots}$ behave differently at $+\infty$ and $-\infty$ because $\sqrt{x^2} = |x|$.

Problem 11 (Concept check: asymptotes and crossing)

Decide whether each statement is always true. If not, give a counterexample.

(a) If f has a vertical asymptote at $x = a$, then $f(a)$ must be undefined.

(b) A function can never cross its horizontal asymptote.

Solution.

(a) Not always true. Example:

$$f(x) = \begin{cases} \frac{1}{x}, & x \neq 0, \\ 7, & x = 0. \end{cases}$$

Then $x = 0$ is a vertical asymptote (since $\lim_{x \to 0} 1/x = \pm\infty$) but $f(0)$ is defined.

(b) Not always true. Example: $f(x) = \dfrac{\sin x}{x}$ for $x \neq 0$ and $f(0) = 1$. It has horizontal asymptote $y = 0$ as $x \to \infty$, and it crosses $y = 0$ at $x = n\pi$ (for integers $n \neq 0$).

$$\boxed{\text{Both statements are \textit{not} always true.}}$$