MATA30 — Tutorial Notes

Week 10: Definite Integrals, FTC, and Substitution

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Problem 1 (Definite Integrals and Geometry)**.** Suppose f is continuous on $[0,4]$ and

$$\int_0^4 (1+f(x))\,dx = 7, \qquad \int_0^2 \left(3f(x) - \sqrt{4-x^2}\right) dx = 1-\pi.$$

Find $\displaystyle\int_2^4 f(x)\,dx$.

Solution

From linearity,

$$\int_0^4 (1+f(x))\,dx = 4 + \int_0^4 f(x)\,dx = 7,$$

so

$$\int_0^4 f(x)\,dx = 3.$$

Let

$$A = \int_0^2 f(x)\,dx.$$

Then

$$3A - \int_0^2 \sqrt{4-x^2}\,dx = 1-\pi.$$

The integral

$$\int_0^2 \sqrt{4-x^2}\,dx$$

represents the area of a quarter circle of radius 2, hence equals π.

Thus

$$3A - \pi = 1-\pi \quad \Rightarrow \quad A = \frac{1}{3}.$$

Finally,

$$\int_2^4 f(x)\,dx = \int_0^4 f(x)\,dx - \int_0^2 f(x)\,dx = 3 - \frac{1}{3} = \boxed{\frac{8}{3}}.$$

Problem 2 (FTC with Variable Limits)**.** Define

$$F(x) = \int_{x^2}^{\sqrt{x}} \frac{t^2+1}{t^4+1}\,dt, \qquad x>0.$$

Find $F'(x)$.

Remark

When both limits of integration depend on x, differentiate using

$$\frac{d}{dx}\int_{u(x)}^{v(x)} f(t)\,dt = f(v(x))v'(x) - f(u(x))u'(x).$$

Always apply the chain rule to each limit separately.

Solution

Let $f(t) = \dfrac{t^2+1}{t^4+1}$. Using the Fundamental Theorem of Calculus with the chain rule,

$$F'(x) = f(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}} - f(x^2) \cdot 2x.$$

Compute:

$$f(\sqrt{x}) = \frac{x+1}{x^2+1}, \qquad f(x^2) = \frac{x^4+1}{x^8+1}.$$

Therefore,

$$\boxed{F'(x) = \frac{x+1}{2\sqrt{x}(x^2+1)} - \frac{2x(x^4+1)}{x^8+1}.}$$

Problem 3 (Concavity of an Integral-Defined Function)**.** Let

$$f(x) = \int_0^x (t^3 - 3t)e^{t^2}\,dt.$$

(a) Find $f'(x)$ and $f''(x)$.

(b) Determine where f is increasing or decreasing.

(c) Find all inflection points.

Solution

(a) By the FTC,

$$f'(x) = (x^3 - 3x)e^{x^2}.$$

Differentiate using the product rule:

$$f''(x) = (3x^2-3)e^{x^2} + (x^3-3x)(2x)e^{x^2} = (2x^4 - 3x^2 - 3)e^{x^2}.$$

(b) Since $e^{x^2} > 0$, the sign of $f'(x)$ is the sign of

$$x(x^2-3).$$

Thus f is decreasing on $(-\infty, -\sqrt{3})$ and $(0, \sqrt{3})$, and increasing on $(-\sqrt{3}, 0)$ and $(\sqrt{3}, \infty)$.

(c) Solve

$$2x^4 - 3x^2 - 3 = 0.$$

Let $u = x^2$:

$$2u^2 - 3u - 3 = 0 \quad \Rightarrow \quad u = \frac{3 + \sqrt{33}}{4}.$$

Hence inflection points occur at

$$\boxed{x = \pm\sqrt{\frac{3 + \sqrt{33}}{4}}.}$$

Problem 4 (Definite Integral via Substitution)**.** Evaluate

$$\int_0^2 x^3 \sqrt{x^2 + 1}\, dx.$$

Solution

Let $u = x^2 + 1$, so $du = 2x\, dx$ and $x^3 dx = (u - 1)\frac{1}{2}\, du$. Change limits: $u(0) = 1$, $u(2) = 5$.

$$\int_0^2 x^3 \sqrt{x^2 + 1}\, dx = \frac{1}{2} \int_1^5 (u - 1) u^{1/2}\, du.$$

Integrate:

$$= \frac{1}{2} \left[\frac{2}{5} u^{5/2} - \frac{2}{3} u^{3/2} \right]_1^5 = \boxed{\frac{50\sqrt{5} + 2}{15}}.$$

Problem 5 (Trigonometric Substitution)**.** Evaluate

$$\int_0^{\pi/2} \sin^5 x \cos^3 x\, dx.$$

Solution

Write $\cos^3 x = \cos^2 x \cos x = (1 - \sin^2 x)\cos x$. Let $u = \sin x$, $du = \cos x\, dx$:

$$\int_0^1 (u^5 - u^7)\, du = \left[\frac{u^6}{6} - \frac{u^8}{8} \right]_0^1 = \boxed{\frac{1}{24}}.$$

Problem 6 (Rational Substitution)**.** Evaluate

$$\int_0^1 \frac{2x + 3}{x^2 + 3x + 2}\, dx.$$

Solution

Let $u = x^2 + 3x + 2$, so $du = (2x + 3)\, dx$. Change limits: $u(0) = 2$, $u(1) = 6$.

$$\int_0^1 \frac{2x + 3}{x^2 + 3x + 2}\, dx = \int_2^6 \frac{1}{u}\, du = \ln 6 - \ln 2 = \boxed{\ln 3}.$$