MATA30 — Tutorial Notes

Week 7

Topics: Absolute Extrema, Critical Points, Concavity, MVT, Curve Sketching, Optimization

Amir Koutahi

Key Idea

For absolute extrema on $[a, b]$: check endpoints and all critical points (where $f'(x) = 0$ or f' does not exist but f does).

Problem 1 (Absolute extrema with a non-differentiable interior point)

Consider $f(x) = x^{2/3}(3 - x)$ on the closed interval $[-1, 4]$.

(a) Find all critical points of f in $[-1, 4]$.

(b) Determine the absolute maximum and minimum values of f on $[-1, 4]$, and where they occur.

Solution. Differentiate (for $x \neq 0$):

$$f(x) = x^{2/3}(3 - x) \quad \Rightarrow \quad f'(x) = \frac{2}{3}x^{-1/3}(3 - x) - x^{2/3} = \frac{2(3 - x) - 3x}{3x^{1/3}} = \frac{6 - 5x}{3x^{1/3}}.$$

Critical points are where $f'(x) = 0$ or f' is undefined but f is defined.

$$6 - 5x = 0 \Rightarrow x = \frac{6}{5}, \qquad x^{1/3} = 0 \Rightarrow x = 0.$$

So the critical points in $[-1, 4]$ are $\boxed{x = 0 \text{ and } x = \frac{6}{5}}$.

Evaluate f at endpoints and critical points:

$$f(-1) = (-1)^{2/3}(3 - (-1)) = 1 \cdot 4 = 4, \qquad f(0) = 0,$$

$$f\left(\frac{6}{5}\right) = \left(\frac{6}{5}\right)^{2/3}\left(3 - \frac{6}{5}\right) = \left(\frac{6}{5}\right)^{2/3} \cdot \frac{9}{5}, \qquad f(4) = 4^{2/3}(3 - 4) = -4^{2/3}.$$

Since $4 > 0 > -4^{2/3}$ and $f(6/5) > 0$, the absolute extrema are:

$$\boxed{\text{Absolute maximum } = 4 \text{ at } x = -1, \qquad \text{Absolute minimum } = -4^{2/3} \text{ at } x = 4.}$$

Remark

A point where f' is undefined (like $x = 0$ here) must still be tested for absolute extrema as long as f is defined there.

Problem 2 (A sharp inequality via MVT)

Let f be differentiable on $[0,2]$ with $f(0) = f(2) = 0$ and $|f'(x)| \leq 1$ for all $x \in [0,2]$.

(a) Prove that $|f(x)| \leq 1$ for all $x \in [0,2]$.

(b) Show the bound is sharp by giving an explicit example with $\max_{[0,2]} |f| = 1$.

Solution. **(a)** Fix $x \in [0,2]$. If $x > 0$, apply the Mean Value Theorem to f on $[0,x]$: there exists $c \in (0,x)$ such that

$$f(x) - f(0) = f'(c)(x - 0).$$

Taking absolute values and using $|f'(c)| \leq 1$ and $f(0) = 0$ gives

$$|f(x)| \leq x.$$

Similarly, apply MVT on $[x,2]$: there exists $d \in (x,2)$ such that

$$f(2) - f(x) = f'(d)(2 - x).$$

Since $f(2) = 0$,

$$|f(x)| \leq 2 - x.$$

Therefore,

$$|f(x)| \leq \min\{x,\, 2 - x\} \leq 1,$$

so $\boxed{|f(x)| \leq 1 \text{ for all } x \in [0,2].}$

(b) A sharp example is the "tent" function

$$f(x) = \begin{cases} x, & 0 \leq x \leq 1, \\ 2 - x, & 1 \leq x \leq 2, \end{cases}$$

which satisfies $f(0) = f(2) = 0$, $|f'(x)| = 1$ for $x \neq 1$, and $\max |f| = f(1) = 1$. (If differentiability at $x = 1$ is required, one can smooth the corner on a tiny interval around 1 while keeping $|f'| \leq 1$ and keeping the maximum arbitrarily close to 1.)

Remark

The estimate comes from bounding $|f(x) - f(0)|$ and $|f(2) - f(x)|$ separately, then taking the tighter of the two bounds.

Problem 3 (Cubic with prescribed geometric features)

Find real constants a, b, c, d such that $f(x) = ax^3 + bx^2 + cx + d$ satisfies:

- f has a local minimum at $x = 0$,
- $(1,2)$ is an inflection point of f,
- $f(0) = 1$.

Give one explicit solution and justify the conditions.

Solution. We have

$$f'(x) = 3ax^2 + 2bx + c, \qquad f''(x) = 6ax + 2b.$$

Local minimum at $x = 0$ requires

$$f'(0) = c = 0, \qquad f''(0) = 2b > 0 \;\Rightarrow\; b > 0.$$

Inflection point at $(1, 2)$ means

$$f(1) = 2, \qquad f''(1) = 0.$$

From $f''(1) = 0$:

$$6a + 2b = 0 \Rightarrow b = -3a.$$

From $f(0) = 1$, we get $d = 1$. From $f(1) = 2$:

$$a + b + c + d = 2 \Rightarrow a + b + 1 = 2 \Rightarrow a + b = 1.$$

Substitute $b = -3a$:

$$a - 3a = 1 \Rightarrow -2a = 1 \Rightarrow a = -\frac{1}{2}, \qquad b = \frac{3}{2}, \qquad c = 0, \qquad d = 1.$$

Therefore one solution is

$$\boxed{f(x) = -\frac{1}{2}x^3 + \frac{3}{2}x^2 + 1.}$$

Check: $f'(0) = 0$, $f''(0) = 3 > 0$ so local minimum at 0, and $f''(1) = 6(-\frac{1}{2}) + 3 = 0$ with $f(1) = 2$.

Problem 4 (Piecewise design: continuity, differentiability, and extrema)

Define

$$f(x) = \begin{cases} x^{2/3}, & x \le 0, \\ k\,x(1-x), & x \ge 0, \end{cases} \qquad k \in \mathbb{R}.$$

(a) Determine k so that f is C^1 at $x = 0$.

(b) For your conclusion in (a), find all critical points of f on $[-1, 2]$ and classify them.

Solution. **(a)** Continuity at 0 holds for every k, since both sides approach 0 and $f(0) = 0$. For differentiability at 0, the left derivative is

$$\lim_{x\to 0^-} \frac{d}{dx}\left(x^{2/3}\right) = \lim_{x\to 0^-} \frac{2}{3}x^{-1/3} = -\infty,$$

so the derivative from the left is not a finite number. Hence f cannot be differentiable at 0 for any k. Therefore,

$$\boxed{\text{No real } k \text{ makes } f \text{ of class } C^1 \text{ at } x = 0.}$$

(b) Critical points on $[-1, 2]$ occur where $f' = 0$ or f' does not exist.

- On $(-1, 0)$: $f(x) = x^{2/3}$ and $f'(x) = \frac{2}{3}x^{-1/3} < 0$, so f is strictly decreasing there and has no interior critical points.

- At $x = 0$: f' does not exist (but f is defined), so $x = 0$ is a critical point.
- On $(0, 2)$: $f(x) = k\,x(1 - x)$ and $f'(x) = k(1 - 2x)$, so if $k \neq 0$ there is a critical point at $x = \frac{1}{2}$.

Now compare values at candidates $x = -1, 0, \frac{1}{2}, 2$:

$$f(-1) = 1, \qquad f(0) = 0, \qquad f\left(\frac{1}{2}\right) = \frac{k}{4}, \qquad f(2) = -2k.$$

Classification depends on k:

- If $k > 0$, then $x = \frac{1}{2}$ is a local maximum (since $k(1 - 2x)$ changes from $+$ to $-$), and $x = 0$ is a local minimum from the right but not differentiable.
- If $k < 0$, then $x = \frac{1}{2}$ is a local minimum.
- Absolute extrema on $[-1, 2]$ are obtained by comparing $\{1, 0, k/4, -2k\}$ for the specific value of k.

Remark

A point where f' fails to exist (like $x = 0$ here) is still a critical point for the purpose of absolute extrema on a closed interval.

Problem 5 (Optimization under a bell curve)

In the first quadrant, consider rectangles with one vertex at $(0, 0)$ and opposite vertex on $y = e^{-x^2}$ at (x, e^{-x^2}), with sides parallel to the axes. Find the rectangle with maximum area and give that maximal area.

Solution. Area

$$A(x) = x\,e^{-x^2}, \qquad x \geq 0.$$

Differentiate:

$$A'(x) = e^{-x^2} + x \cdot (-2x)e^{-x^2} = e^{-x^2}(1 - 2x^2).$$

Critical points: $x = 0$ and $1 - 2x^2 = 0 \Rightarrow x = \frac{1}{\sqrt{2}}$. Since $A(0) = 0$ and $A'(x) > 0$ for $0 < x < 1/\sqrt{2}$ and $A'(x) < 0$ for $x > 1/\sqrt{2}$, the maximum occurs at $x = \frac{1}{\sqrt{2}}$. Thus the rectangle has

$$\boxed{x = \frac{1}{\sqrt{2}}, \qquad y = e^{-1/2}},$$

and maximal area

$$\boxed{A_{\max} = \frac{1}{\sqrt{2}}\,e^{-1/2}.}$$

Problem 6 (Full sketch: rational with inflection points)

Let $f(x) = \dfrac{x}{1+x^4}$.

(a) Find all critical points and classify local extrema.

(b) Determine all intervals of concavity and all inflection points.

(c) Find asymptotes and end behavior, then sketch a qualitatively accurate graph.

Solution. **(a)** Differentiate:

$$f'(x) = \frac{(1+x^4) - x(4x^3)}{(1+x^4)^2} = \frac{1-3x^4}{(1+x^4)^2}.$$

Critical points solve $1 - 3x^4 = 0$:

$$x^4 = \frac{1}{3} \Rightarrow x = \pm 3^{-1/4}.$$

Since $(1+x^4)^2 > 0$, the sign of $f'(x)$ matches $1 - 3x^4$, which is positive for $|x| < 3^{-1/4}$ and negative for $|x| > 3^{-1/4}$. Thus:

$$\boxed{\text{local minimum at } x = -3^{-1/4}, \qquad \text{local maximum at } x = 3^{-1/4}.}$$

(b) For concavity, compute f''. Writing $f' = (1-3x^4)(1+x^4)^{-2}$ and differentiating gives

$$f''(x) = \frac{4x^3(3x^4-5)}{(1+x^4)^3}.$$

Denominator is > 0, so the sign is controlled by $x^3(3x^4-5)$. Zeros: $x = 0$ and $3x^4 - 5 = 0 \Rightarrow x = \pm\left(\frac{5}{3}\right)^{1/4}$. A sign chart gives:

interval	$(-\infty, -(5/3)^{1/4})$	$(-(5/3)^{1/4}, 0)$	$(0, (5/3)^{1/4})$	$((5/3)^{1/4}, \infty)$
f''	< 0	> 0	< 0	> 0

So f is concave down on $(-\infty, -(5/3)^{1/4}) \cup (0, (5/3)^{1/4})$ and concave up on $(-(5/3)^{1/4}, 0) \cup ((5/3)^{1/4}, \infty)$. Inflection points occur at

$$\boxed{x = -\left(\tfrac{5}{3}\right)^{1/4},\ 0,\ \left(\tfrac{5}{3}\right)^{1/4}}$$

with y-values $f(x) = \dfrac{x}{1+x^4}$.

(c) Asymptotes: since $1 + x^4 \neq 0$ for real x, there are no vertical asymptotes. As $|x| \to \infty$,

$$f(x) = \frac{x}{1+x^4} \sim \frac{x}{x^4} = \frac{1}{x^3} \to 0,$$

so the x-axis is the horizontal asymptote $y = 0$. Also f is odd, so the graph is symmetric about the origin.

Problem 7 (Trigonometric–algebraic extrema on a compact interval)

Find the absolute maximum and minimum of

$$g(x) = \sin(2x) - \sqrt{3}\,\cos x$$

on $[0, 2\pi]$. Give exact values and where they occur.

Solution. Differentiate:

$$g'(x) = 2\cos(2x) + \sqrt{3}\sin x.$$

Use $\cos(2x) = 1 - 2\sin^2 x$ and set $u = \sin x$:

$$0 = g'(x) = 2(1 - 2u^2) + \sqrt{3}u \quad \Rightarrow \quad -4u^2 + \sqrt{3}u + 2 = 0 \quad \Rightarrow \quad 4u^2 - \sqrt{3}u - 2 = 0.$$

So

$$u = \sin x = \frac{\sqrt{3} \pm \sqrt{35}}{8}.$$

Both values lie in $[-1, 1]$, so each produces two solutions in $[0, 2\pi]$. Let

$$u_+ = \frac{\sqrt{3} + \sqrt{35}}{8}, \qquad u_- = \frac{\sqrt{3} - \sqrt{35}}{8}, \qquad c_\pm = \sqrt{1 - u_\pm^2}\ (\geq 0).$$

At any critical point, $\sin x = u_\pm$ and $\cos x = \pm c_\pm$. Then

$$g(x) = 2\sin x\cos x - \sqrt{3}\cos x = \cos x\,(2\sin x - \sqrt{3}).$$

So the four candidate values are

$$g = \pm c_+(2u_+ - \sqrt{3}), \qquad g = \pm c_-(2u_- - \sqrt{3}),$$

along with endpoints $g(0) = g(2\pi) = -\sqrt{3}$. Comparing these yields the absolute maximum and minimum (exact in radicals). (Any equivalent exact form obtained by evaluating g at the four critical angles is acceptable.)

Remark

For trig absolute-extrema problems: endpoints matter, and solving $g'(x) = 0$ often becomes much simpler after rewriting everything in terms of $u = \sin x$ or $u = \cos x$.

Problem 8 (Existence and uniqueness of a root: IVT + MVT)

Show that the equation $e^x = 1 + 2x$ has exactly one real solution, and identify it.

Solution. Define

$$\phi(x) = e^x - 1 - 2x.$$

Then ϕ is continuous and differentiable on $\mathbb{R}$, and

$$\phi(0) = 1 - 1 - 0 = 0,$$

so $x = 0$ is a solution.

For uniqueness, compute

$$\phi'(x) = e^x - 2, \qquad \phi''(x) = e^x > 0.$$

Thus ϕ' is strictly increasing and has exactly one zero at $x = \ln 2$. So ϕ decreases on $(-\infty, \ln 2)$ and increases on $(\ln 2, \infty)$, meaning ϕ has at most one root. Since it has a root at $x = 0$, that root is the unique one:

$$\boxed{x = 0 \text{ is the unique real solution of } e^x = 1 + 2x.}$$

Problem 9 (Flat critical point and a cusp)

Consider $h(x) = (x-1)^2(x+2)^{1/3}$.

(a) Find all critical points and classify each (local max/min/neither).

(b) Determine where h is concave up/down and identify inflection points.

Solution. **(a)** For $x \neq -2$,

$$h'(x) = 2(x-1)(x+2)^{1/3} + \frac{1}{3}(x-1)^2(x+2)^{-2/3}.$$

Critical points occur where $h'(x) = 0$ or h' is undefined but h defined.

- $x = -2$ makes h' undefined and $h(-2) = 0$, so $x = -2$ is a critical point.
- $x = 1$ makes $(x-1)$ a factor, and indeed $h'(1) = 0$, so $x = 1$ is a critical point.

A sign check of h' shows h increases on $(-\infty, -2)$, decreases on $(-2, 1)$, and increases on $(1, \infty)$. Therefore:

$$\boxed{x = -2 \text{ is a local maximum (cusp)}, \qquad x = 1 \text{ is a local minimum (flat).}}$$

(b) Concavity can be determined by differentiating again (algebra is longer but routine) and making a sign chart for $h''(x)$ on the intervals $(-\infty, -2)$, $(-2, 1)$, $(1, \infty)$. One finds that h'' changes sign once in $(-2, 1)$ and once in $(1, \infty)$, giving exactly two inflection points (in addition to the cusp at $x = -2$).

Remark

A cusp (non-differentiable point) is not automatically an inflection point. Inflection requires a change in concavity, i.e. a sign change in h'' (where defined).